From *The Physics of Nanostructures*, J.H. Davies and A.R. Long, Eds., Proceedings of the 38th Scottish Universities Summer School in Physics. IOP Publishing LTD, 1992.

Theory of Coherent Quantum Transport

A D Stone

Yale University
New Haven, Connecticut, U.S.A.

1 Introduction

The importance of phase-coherence effects in normal electron transport has been appreciated for some time through the study of localisation, and particularly weak localisation, in macroscopic conductors (Anderson 1958, Lee and Ramakrishnan 1985a, Bergmann 1984). Nonetheless, the dramatic improvement in fabrication of ultrasmall (mesoscopic) conducting devices has increased interest in this subject because of the experimental accessibility of novel phenomena associated with quantum interference in such systems, and the potential applicability of these phenomena to new microelectronic devices in the (admittedly distant) future.

Quantum coherence in the diffusive limit

If we exclude from discussion phase-coherence phenomena directly related to interaction (*e.g.* superconductivity, or the Coulomb suppression in the electronic density of states), we are now able to identify three distinct classes of quantum coherence effects in conductors in the *diffusive* limit. Here 'diffusive' means that the *elastic* mean free path, l, is smaller than the dimensions of the device.

First there are the *weak localisation* (WL) effects on the average conductance, known since the work of Abrahams *et al.* (1979), which arise due to the coherent back-scattering of diffusing electrons in the presence of time-reversal symmetry. Because of this coherent back-scattering the *average* low temperature conductance of a film or wire of arbitrary size was shown to be sensitive to a weak magnetic field or weak spin-orbit scattering (Figure 1). The sensitivity to magnetic field was understood theoretically (Altshuler *et al.* 1980, 1981) to be a manifestation of the Aharonov-Bohm effect for multiply-scattered electrons, with the average over disorder leading to an effective doubling of

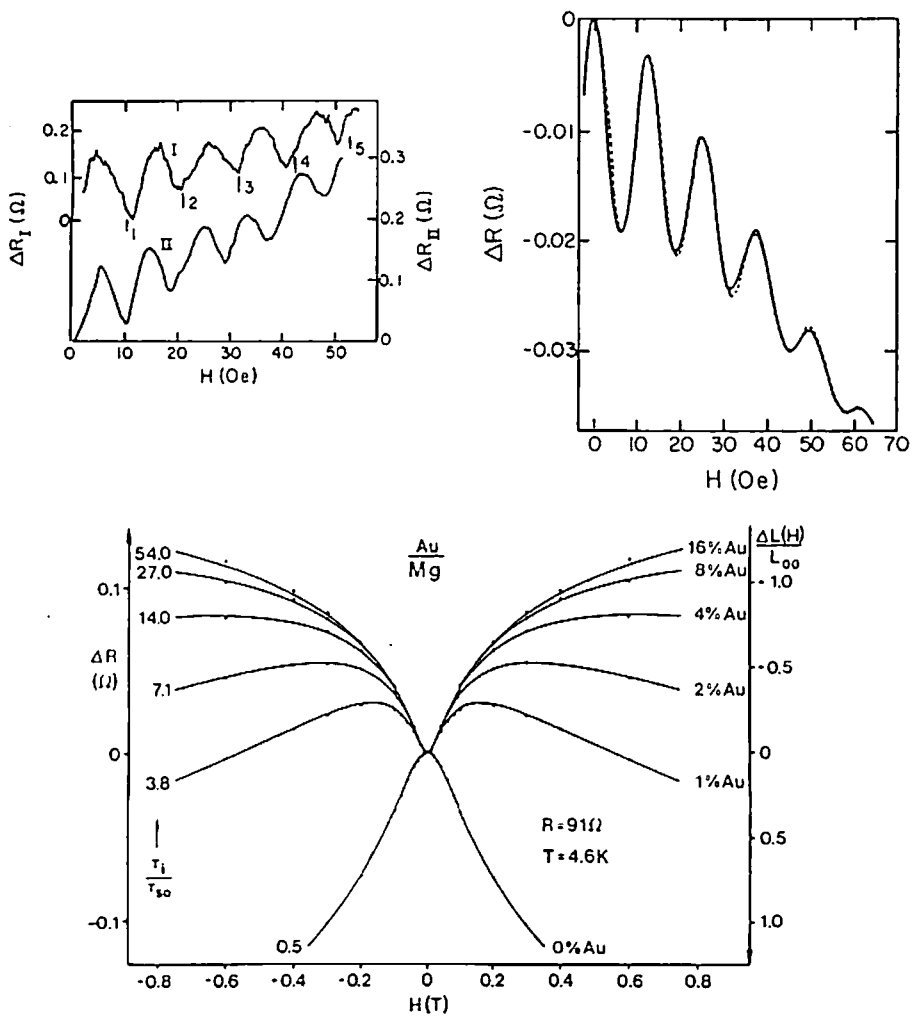

Figure 1: *Normal metal Aharonov-Bohm effect with period h/2e in magnesium (top left) and lithium (top right) cylinders, from Aronov and Sharvin (1987). Weak localisation magnetoresistance of a magnesium film coated with gold to enhance spin-orbit scattering (bottom), from Bergmann (1984). Note the reversal of sign due to spin-orbit scattering.*

the electron charge ($e \rightarrow 2e$). We shall review this briefly below. Recently it has been shown (Mathur and Stone 1991) that the sensitivity to weak spin-orbit scattering is a manifestation of the Aharonov-Casher effect in which an electric field couples to the phase of the wavefunction by its influence on the spin magnetic moment (Aharonov and Casher 1984).

In both cases the WL effects arise from the influence of dynamically negligible per-

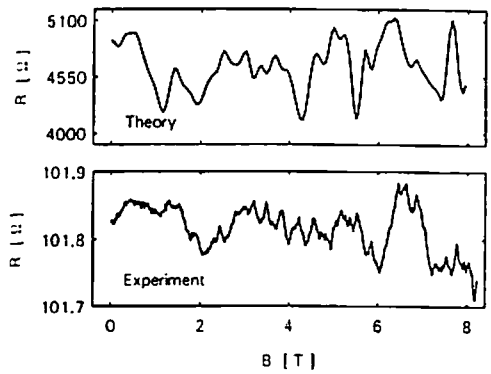

Figure 2: *Universal conductance fluctuations in a simulation of the Schrödinger equation (top) and in a gold-palladium wire of length* 790 nm *and width* 50 nm *(bottom); from Stone (1985).*

turbations on the phase of the wavefunctions, and hence are interference effects in the purest sense. The weak localisation effects depend on dimensionality and on the inelastic scattering length, $l_{\rm in}$, but not on the the size of the sample. They are a manifestation of time-reversal symmetry, and are completely suppressed by a moderate magnetic field. Because they are corrections to the average conductance they are generic: two samples of the same material will show the same WL effect.

The second class of quantum coherence effects in disordered conductors are the sample-specific variations in the transport properties of mesoscopic devices at low temperature (Washburn and Webb 1986; see Figure 2). These are described statistically by the theory of universal conductance fluctuations (UCF) (Altshuler 1985, Lee and Stone 1985, Stone 1985, Altshuler and Shklovskii 1986, Lee *et al.* 1987). In this theory the conductance, g, of a phase-coherent device is shown to be sensitive to small changes in magnetic field, Fermi energy or impurity configuration. The maximum degree of sensitivity is given by the condition

$$\delta g \approx \frac{e^2}{h}, \tag{1}$$

independent of $\langle g \rangle$. The scale of variation of the control parameter needed to achieve this maximal fluctuation is simply that needed to alter the action (phase) along a typical diffusive path by order unity (see, for example, Stone and Imry 1986). Hence the conductance fluctuations and related effects show a sensitivity to weak magnetic field and spin-orbit scattering as do the WL effects. This sensitivity appears in two ways. First, in a given sample the resistance oscillates on a small field scale, typically around 100 G at 1 K (Figure 2); second, the magnitude of the variance of g depends on the time-reversal and spin symmetry of the scattering processes. The breaking of either symmetry leads to universal reduction factors for the variance of g (Altshuler and Shklovskii 1986, Stone 1989), but not (as in WL) to a complete suppression of the effect. Again the effects of a magnetic field arise completely from coupling to the phase of the wavefunction, and not from dynamical effects such as Landau quantisation, which was completely neglected in the standard theory. In fact studies of semiconductor

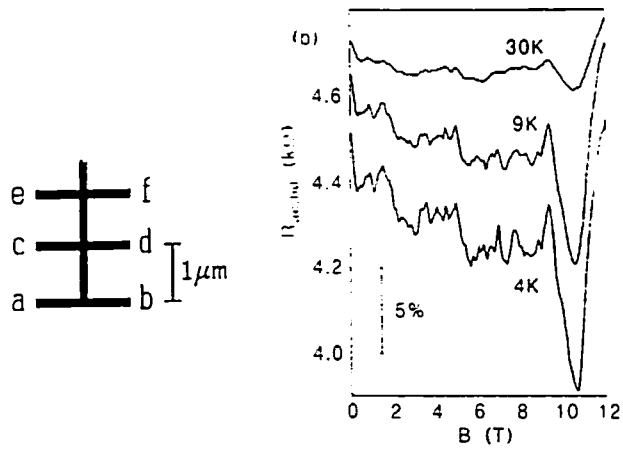

Figure 3: *Magnetoresistance traces showing co-existence of Shubnikov–de Haas oscillations with* UCF*; from Geim et al. (1991).*

microstructures show that UCF effects coexist with the Shubnikov–de Haas oscillations arising from Landau-level quantisation (Figure 3) and even in the transition region between quantum Hall steps. Below we shall show how to generalise UCF theory to the high-field limit within the self-consistent Born approximation.

In general, UCF effects depend on the ratio of the inelastic length to the sample dimensions and are not observable in macroscopic samples. However they do not depend on time-reversal or any other symmetry for their existence and are robust against magnetic field

A third class of phenomena which has recently received a great deal of theoretical attention relate to quantum coherence effects in the *thermodynamic* properties of mesoscopic samples, such as the persistent current or orbital magnetic response. The sample-to-sample fluctuations of these quantities are conceptually quite similar to UCF, but it has recently been predicted theoretically that there should be an *average* effect as well due to the constraint of fixed particle number in such isolated mesoscopic systems (Altshuler *et al.* 1991). This average effect is unlike UCF in that it should be observable in large *arrays* of isolated mesoscopic elements (*e.g.* quantum dots), but would also be unlike WL in that the amplitude will decay with the ratio of the size of individual elements to the inelastic mean free path. Although average persistent currents have been observed experimentally by Levy *et al.* (1990), it is unclear whether their origin is this new type of quantum coherence effect. Nonetheless I believe that the prediction of such effects opens a new and fascinating area for future work.

Quantum coherence in the ballistic limit

All of the above interference phenomena involve diffusing electrons and have their mathematical origin in the properties of the disorder-averaged two-particle Green function, and in particular in the dominance of the contributions in the cooperon or diffuson channels at long distances and low temperatures. A fourth class of phenomena at present

outside of this type of description are coherent effects in *ballistic* transport, where the motion may not be diffusive and there is no obvious ensemble over which to average. Initially it was assumed that such devices would show rather elementary wave-guide interference effects, equivalent to microwave interferometers. However we have recently shown that fluctuating interference phenomena very similar to UCF occur in ballistic systems at low temperatures if the sample geometry is sufficiently complex so that the classical scattering from such a confining potential is chaotic (Jalabert *et al.* 1990); thus it remains an open question whether such simple interference effects are achievable in ballistic conductors.

In addition to *interference* effects, there are other novel transport phenomena in the ballistic limit which are essentially classical in origin. These include the quenching of the Hall resistance or four-probe bend resistance (see, for example, Beenakker and van Houten 1989), as well as effects related to mode quantisation such as the quantised conductance of a point contact (van Houten *et al.* 1990).

Finally one must mention the quantised Hall effect (Prange and Girvin 1987). Although not a phenomenon that requires phase coherence across the sample, we shall see below that the occurence of Hall quantisation and vanishing longitudinal resistance in a two-dimensional electron gas (2DEG) can be understood quite simply by the Landauer-Büttiker approach commonly used in studying phase-coherent transport. Moreover the stability of this phenomenon against disorder, and the occurence of plateaus, can only be understood by invoking localisation effects related to phase coherence. Unfortunately in this case a detailed physical picture has not yet been developed comparable with the theory of weak localisation.

Formalism for quantum transport

Since all the phenomena discussed above are directly related to interference of multiply-scattered electron waves, they are not described by the conventional approach to transport in solids based on the Boltzmann equation. Instead they require a formalism in which quantum coherence may be incorporated from the beginning. Linear response theory (Kubo 1957) has traditionally been employed to treat such situations, although often with some cost to physical intuition due to its complexity. An alternative approach which has proved extremely useful in the study of mesoscopic transport was pioneered by Landauer over thirty years ago (Landauer 1957, 1970). Using a counting argument and the Einstein relation, he related the resistance of a 1D conductor to the quantum transmission probability for the conductor (treated as a single composite scattering center); in principle the exact transmission coefficient will contain all of the interference effects associated with multiple scattering of the electrons. Landauer's original work did not consider in detail the nature of the resistance measurement, but experimental progress in the study of mesoscopic conductors rapidly uncovered the important role of the contacts and the measuring geometry in determining the results of transport measurements (Benoit *et al.* 1986, Skocpol *et al.* 1986). For example, a qualitative difference was found between two-probe and four-probe measurements of resistance with respect to their symmetry under reversing the magnetic field (Stone and Szafer 1988). This discovery motivated a crucial generalisation of the original Landauer approach, due to Büttiker (1986), in which all the measurement probes are treated on

an equal footing and the correct reciprocity relations for multi-probe measurements in magnetic fields arise naturally. It was subsequently shown that Büttiker's multi-probe formula could be derived from a version of Kubo linear response theory (Stone and Szafer 1988, Baranger and Stone 1989). Although the Green function formulation is often more convenient for microscopic calculations, the Landauer-Büttiker (LB) formulation in terms of transmission coefficients has often proven useful and more congenial to physical intuition. We review briefly the physical assumptions of this approach and its derivation from linear response theory below.

2 Landauer-Büttiker approach to transport

2.1 Counting argument

The basic physical idea of the LB approach is to consider the sample whose resistance is being measured as a single phase-coherent unit attached to perfect reservoirs serving as current source and sink and as voltage probes. In an ideal *two-probe* measurement the sample is attached between two perfect reservoirs with electrochemical potentials, μ_1 and $\mu_2 = \mu_1 + eV$ respectively, where V is the applied voltage; these reservoirs serve both as current source and sink *and* as voltage terminals. In the energy interval eV between μ_2 and μ_1, electrons are injected into right-going states emerging from reservoir 1, but none are injected into left-going states emerging from reservoir 2. Thus there is a net right-going current proportional to the number of states in the interval $\mu_2 - \mu_1$, given by

$$I = e\sum_{i}^{N_c} v_i \frac{dn_i}{d\varepsilon} eV \sum_{j}^{N_c} T_{ij} = \left(\frac{e^2}{h}\sum_{i,j}^{N_c} T_{ij}\right) V, \tag{2}$$

where N_c is the number of propagating channels including spin *in the sample*, v_i is the the longitudinal velocity for the ith momentum channel at the Fermi surface, T_{ij} is the transmission probability from j to i, and we have used the fact that $dn_i/d\varepsilon = 1/hv_i$ for a quasi-1D density of states. Equation (2) yields an expression for the two-probe conductance in terms of the total transmission coefficient (normalised to N_c),

$$g = \frac{e^2}{h}\sum_{i,j}^{N_c} T_{ij} \equiv \frac{e^2}{h} T. \tag{3}$$

Many experimental measurements are not made in a two-probe configuration, but in a multi-probe configuration in which current and voltage probes are different. Büttiker (1986) generalised the above argument to calculate the current in a phase-coherent system connected to N_L reservoirs, where any two can serve as current source and sink, and a voltage can be applied (or induced) between any two. The currents and voltages on all the leads are now related by a matrix whose elements, g_{mn}, are known as conductance coefficients. Büttiker showed that an argument exactly analogous to that leading to Equation (3) yields the result

$$I_m = \sum_{n}^{N_L} g_{mn} V_n = \frac{e^2}{h}\sum_{n}^{N_L} (T_{mn} - N_c\delta_{mn}) V_n\,, \tag{4}$$

where I_m is the total current into lead m, V_n is the voltage applied at lead n, and T_{mn} is the total transmission coefficient (or reflection coefficient for the case $m = n$) at the Fermi energy for electrons injected at lead n to be collected at lead m (summed over all channel indices, which have been suppressed for clarity). In the case $N_L = 2$ this formula reduces exactly to Equation (3); for four or more probes it can be inverted to yield the Hall resistance if the T_{mn} are known. This formula is very appealing because it provides a simple Fermi-surface expression for the Hall resistance, R_H, which is valid in an arbitrary magnetic field. This property is noteworthy in view of the fact that the Hall conductance in the Kubo formulation is commonly expressed in terms of *all* the states below the Fermi surface. In fact until recently the conditions under which the LB approach (which is based on the above physical argument, and not on a derivation from an underlying Hamiltonian) and the Kubo approach were equivalent were unclear. Hence a derivation of the LB formulas from linear response theory became of interest.

2.2 Derivation from linear response theory

Derivations of Equation (3) for the two-probe conductance from linear response theory began with the work of Economou and Soukoulis (1981) and Fisher and Lee (1981), although the interpretation in terms of two-probe measurements was not understood until later (Imry 1986). Derivations of this type were generalised to the multi-probe case by Stone and Szafer (1988) for zero magnetic field, and by Baranger and Stone (1989) for arbitrary magnetic field. Here we sketch only the simpler version of the derivation for multi-probe systems for $B = 0$. One considers a non-interacting electron system under the influence of an arbitrary potential $V(\mathrm{r})$ in a finite region of space, defined to be the 'sample'. Electrons can flow out of the sample to infinity along N_L strips or bars of finite width which are translationally invariant in the longitudinal direction ('perfect leads'). The infinite perfect leads serve to make the spectrum continuous in energy, with eigenstates in the form of a wave approaching the sample from infinity in a single mode and a given lead and transmitted and reflected waves leaving the sample in all the leads and modes (the so-called scattering-wave states). Since the system is infinite a d.c. current can flow through the sample in response to a potential difference imposed between two edges of the sample. The perfect leads are assumed to be equipotentials out to infinity and unaffected by the current flow; these assumptions combined with the lack of back-scattering in the leads allows them to function like the perfect reservoirs in the LB argument.

Since we are treating our system as approximately non-interacting, the many-body eigenstates are Slater determinants of single-particle wavefunctions. The expectation value of any single-body operator $\hat{\mathcal{O}}_1$ evolves according to the time-dependent Schrödinger equation, and can be expressed as $\langle \mathcal{O}_1 \rangle = \mathrm{Tr}\{\rho(t)\hat{\mathcal{O}}_1\}$, where $\rho(t)$ is the *single-particle* density matrix satisfying the equation of motion $i\hbar(d\rho/dt) = [H, \rho]$. The unperturbed system is described by

$$\rho_0 = \int d\alpha \, f(\varepsilon_\alpha) \, |\psi_\alpha\rangle \, \langle\psi_\alpha| \,, \tag{5}$$

where $f(\varepsilon_\alpha)$ is the Fermi function, $|\psi_\alpha\rangle$ are the exact scattering-wave states of the equilibrium system with energy ε_α, and we have written an integral over α to emphasise that the energies are continuous.

We assume that the system is perturbed by an external scalar potential oscillating with frequency ω, which is turned on adiabatically from $t = -\infty$; the potential is arbitrary in the sample and approaches a different constant value in each lead. By solving the equation of motion for ρ to obtain the correction to ρ_0 to linear order in the perturbation, and then taking its trace with the current operator in the limit $\omega \to 0$, one obtains

$$\langle \mathrm{J}(\mathrm{r}) \rangle = \int d\mathrm{r}' \, \underline{\sigma}(\mathrm{r}, \mathrm{r}') \cdot \mathrm{E}(\mathrm{r}'), \tag{6}$$

where the local Kubo conductivity tensor is given by

$$\underline{\sigma}(\mathrm{r}, \mathrm{r}') = -\hbar \int d\alpha \, d\beta \left[f'(\varepsilon_\alpha) \pi \delta(\varepsilon_{\beta\alpha}) + i \frac{f_{\beta\alpha}}{\varepsilon_{\beta\alpha}} P\left(\frac{1}{\varepsilon_{\beta\alpha}}\right) \right] \mathrm{J}_{\beta\alpha}(\mathrm{r}) \, \mathrm{J}_{\alpha\beta}(\mathrm{r}'). \tag{7}$$

In Equation (7) $\varepsilon_{\beta\alpha} = \varepsilon_\beta - \varepsilon_\alpha$, $f' = \partial f / \partial \varepsilon$, $f_{\beta\alpha} = f(\varepsilon_\beta) - f(\varepsilon_\alpha)$ and P denotes the principal value of the integral. $\mathrm{J}_{\beta\alpha}(\mathrm{r})$ is the matrix element of the current operator between exact eigenstates,

$$\mathrm{J}_{\beta\alpha}(\mathrm{r}) = \frac{-ie\hbar}{2m} \left[\psi^*_\beta(\mathrm{r}) D \psi_\alpha(\mathrm{r}) - \psi_\alpha(\mathrm{r}) D^* \psi^*_\beta(\mathrm{r}) \right], \tag{8}$$

where $D = [\nabla - (ie/\hbar c)\mathrm{A}(\mathrm{r})]$ is the gauge-invariant derivative. It is convenient to introduce a double-sided derivative $\overleftrightarrow{D}$ defined such that

$$\mathrm{J}_{\beta\alpha}(\mathrm{r}) = \frac{-ie\hbar}{2m} \left[\psi^*_\beta(\mathrm{r}) \overleftrightarrow{D} \psi_\alpha(\mathrm{r}) \right].$$

Note that the δ-function term in Equation (7) only involves states at the Fermi surface as $T \to 0$, whereas the principal value term involves a sum over all states. Under time-reversal symmetry, $\mathcal{T}$, the matrix elements of current satisfy $\mathcal{T}[J_{\beta\alpha}(\mathrm{r}, B)] = J^*_{\alpha\beta}(\mathrm{r}, -B)$. By applying time-reversal symmetry and interchanging indices α, β in Equation (7) we see that the principal value term is anti-symmetric in magnetic field whereas the δ-function term is symmetric. At $B = 0$ this means that the principal value term must vanish.

The current I_m through probe m is obtained by integrating $\langle \mathrm{J}(\mathrm{r}) \rangle$ over a cross-section of lead m. One can then express $\mathrm{E}(\mathrm{r})$ as $\nabla\phi$, and use the divergence theorem to express I_m in terms only of the voltage at the boundaries as in Equation (4) [current conservation must be used in this step to eliminate additional terms involving volume integrals (Baranger and Stone 1989)]. One obtains the intuitive result

$$g_{mn} = \int dS_m \int dS_n \, \hat{\mathrm{m}} \cdot \underline{\sigma}(\mathrm{r}, \mathrm{r}') \cdot \hat{\mathrm{n}}, \tag{9}$$

where $\hat{\mathrm{m}}$ and $\hat{\mathrm{n}}$ are unit vectors normal to the cross-section. The task now is to show that this expression with $\underline{\sigma}$ given by Equation (7) is equivalent to $g_{mn} = (e^2/h) T_{mn}$. We only consider here the case of zero magnetic field in which the principal value term of Equation (7) vanishes. In this case one can immediately perform the integration over energy and take the limit $T \to 0$ in Equation (7) so that the integral over α, β becomes a sum over the discrete states at ε_r of the form

$$\frac{1}{2h} \sum_{\alpha,\beta} \int dS_m \, \hat{\mathrm{m}} \cdot \mathrm{J}_{\beta\alpha}(\mathrm{r}) \int dS_n \, \hat{\mathrm{n}} \cdot \mathrm{J}_{\alpha\beta}(\mathrm{r}'), \tag{10}$$

where we have included in Equation (10) the appropriate normalisation factor $(2\pi\hbar)^{-2}$ arising from the integrations over energy.

Thus we need to evaluate the current matrix elements of the exact eigenstates $\mathrm{J}_{\beta\alpha}(\mathrm{r})$ integrated over the cross-sections of leads m, n. Since their energy is now fixed at ε_F, states α, β are specified by a mode index a, b and a lead index p, q denoting the lead and mode which contains an incoming wave from infinity. Assume that $p, q \neq m, n$ for simplicity. In lead m, $\psi_\alpha(\mathrm{r}) = \sum_c t_{cm,ap}\, \phi_c^+(\mathrm{r})$, and $\psi_\beta^*(\mathrm{r}) = \sum_d t_{dm,bq}^*\, \phi_d^{+*}(\mathrm{r})$, where $t_{cm,a\bar{p}}$ is the transmission amplitude for an incident wave in lead p and mode a to scatter to lead m and mode c, and $\phi_c^+(\mathrm{r})$ are the wavefunctions of the infinite perfect leads, consisting of a longitudinal plane wave traveling away from the sample multiplied by the transverse wavefunction for mode c; since the leads are translationally invariant we can always choose the longitudinal part to be a plane wave; the transverse part need not be specified. For $B = 0$ the transverse wavefunctions are orthogonal and we have $\int dS_m\, \hat{\mathrm{m}} \cdot J_{cd} = e\, \delta_{cd}$ (where we have normalised these states to unit flux); for $B \neq 0$ the same identity holds, but one needs to use current conservation to obtain it as the transverse wavefunctions are not orthogonal (Baranger and Stone 1989). Using this one finds

$$\int dS_m\, \hat{\mathrm{m}} \cdot \mathrm{J}_{\beta\alpha}(\mathrm{r}) = \sum_c t_{cm,ap}\, t_{cm,bq}^*. \tag{11}$$

The summation over α, β in Equation (10) will now be equivalent to summing Equation (11) over modes a, b and leads p, q except for the case $p = q = m$, which simply brings in an additional Kronecker delta term due to the incoming wave with unit flux. Hence we can express the conductance coefficient g_{mn} entirely in terms of summations over the transmission and reflection amplitudes for waves incident in all the leads and all the modes. This expression is then easily simplified using the unitarity relations for the scattering matrix to yield

$$g_{mn} = \frac{e^2}{2h}\,(T_{mn} + T_{nm}) = \frac{e^2}{h} T_{mn}. \tag{12}$$

where we have used fact that the scattering matrix is symmetric in the presence of time-reversal symmetry. In the presence of a magnetic field the derivation involving the δ-function term proceeds exactly as above except that $T_{mn} \neq T_{nm}$; however a much more involved argument (Baranger and Stone 1989) shows that the principal value term gives rise to a term $(T_{mn} - T_{nm})/2$ which again yields Büttiker's result when added to the symmetric term. Hence the derivation from linear response theory shows that the LB equations are valid in an arbitrary magnetic field and that one can always think of phase-coherent quantum transport as a scattering problem between electronic states at the Fermi surface.

2.3 Two-probe conductance

The Kubo-type expression for $g_{12} \equiv g$ is simplified in the case of a two-probe measurement because the principal value term is zero even in the presence of a magnetic field. This may be seen by noting that time-reversal symmetry for the current matrix elements applied to Equation (7) implies that $g_{mn}(B) = g_{nm}(-B)$, hence $g_{mm}(B)$ is symmetric in field. But in the two-probe case $g_{12} = -g_{11}$ follows from the requirement

that no current flow in response to zero voltage difference. Hence g is symmetric in field and the principal value term must vanish. In addition the total current flowing through the sample is independent of the cross-sections S_m, S_n in Equation (9) which can be chosen inside the sample as well. Since it is often more convenient in microscopic calculations to express g in terms of integrals over the entire volume, we may integrate Equation (9) over the positions of S_m, S_n and simply divide by L_x^2 (where henceforth we choose the x-axis to lie in the longitudinal direction). It is also convenient to express the exact current matrix elements in terms of a derivative of the advanced and retarded Green functions, $G^{\pm}(E, \mathrm{r}, \mathrm{r}') = \sum_\alpha \psi_\alpha(\mathrm{r})\,\psi_\alpha^*(\mathrm{r}')\,(E - \varepsilon_\alpha \pm i\eta)^{-1}$. Define

$$\Delta G(E, \mathrm{r}, \mathrm{r}') = G^+(\mathrm{r}, \mathrm{r}') - G^-(\mathrm{r}, \mathrm{r}') = -2\pi i \sum_\alpha \psi_\alpha(\mathrm{r})\psi_\alpha^*(\mathrm{r}')\delta(E - \varepsilon_\alpha). \qquad (13)$$

Substitution of this relation into Equations (7) and (9) yields (at $T = 0$)

$$g = -\frac{e^2\hbar^3}{16\pi m^2 L_x^2} \int d\mathrm{r}\, d\mathrm{r}'\, \Delta G(\varepsilon_F, \mathrm{r}, \mathrm{r}') \vec{D}_x^{\,*}\, \vec{D}_{x'}\, \Delta G(\varepsilon_F, \mathrm{r}', \mathrm{r}). \qquad (14)$$

Since we are now integrating over r and r', the double-sided derivatives in Equation (14) are equivalent by integration by parts to twice the velocity operator. Hence Equation (14) can be written in operator form as

$$g = -\frac{e^2\hbar}{4\pi L_x^2} \mathrm{Tr}\left\{v_x\, \Delta G\, v_x\, \Delta G\right\}, \qquad (15)$$

an expression which dates back to the early work of Greenwood (1958) and Kubo (1965) where it is interpreted as the longitudinal conductivity. Thus we see that two-probe conductance is essentially equal to the spatially-averaged symmetric conductivity tensor, whereas the four-probe resistance is a more complicated quantity, which in general receives contributions from the symmetric and anti-symmetric part of the conductivity tensor.

3 Physical consequences of the LB formula

A number of physical consequences can be obtained without detailed calculations based on the LB formula.

3.1 Quantised contact resistance

Since the conductance coefficients are expressed as the fundamental quantum of conductance, e^2/h, multiplied by transmission coefficients, it is easily seen that quantisation of these transmission coefficients leads to quantisation of transport coefficients in units of e^2/h. The simplest example is the two-probe conductance in the ballistic limit. In the absence of scattering in the sample we will have $T_{ij} = \delta_{ij}$ in Equation (3), leading to

$$g = 2N_c \frac{e^2}{h}, \qquad (16)$$

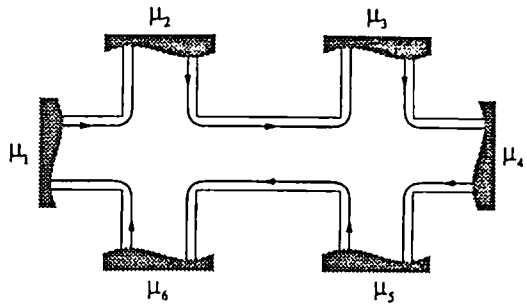

Figure 4: *Standard geometry for Hall effect measurements.*

where I have explicitly introduced a factor of two for spin degeneracy. This is the 'quantised contact resistance' which has now been widely observed in semiconductor point contacts in GaAs heterostructures (van Houten *et al.* 1990). Typically a 2DEG in such systems is divided into two regions by a constriction of variable width created by a split gate. As the width is varied, steps in g of height $2e^2/h$ are observed, separated by reasonably flat plateaus. The experimental observations indicate that the wide regions of the 2DEG behave remarkably like the perfect reservoirs invoked in the LB approach. In particular the existence of fairly sharp steps as a new channel opens is somewhat surprising in a real system where no attempt was made to eliminate impedance mismatch at the interface between the wide and narrow regions. A number of theoretical calculations have now been performed to understand this point (see, for example, Glazman *et al.* 1988, Szafer *et al.* 1989). Although robust with respect to the geometry of the interface, the quantised point contact resistance is quite sensitive to disorder and disappears for constrictions longer than few thousand Ångstroms (Timp *et al.* 1989).

3.2 Quantised Hall effect

A more spectacular and robust example of quantised transport coefficients which can be understood by the LB approach is the integer quantised Hall effect. Here one must of course use the multi-probe formula, Equation (4). The resistance is obtained from this formula by assuming that a current I is fed in from the source probe and withdrawn through the sink, and that the net current in the voltage probes is zero, and inverting Equation (4) to find the voltage difference of interest. Zero current in the voltage probes in steady state is achieved by adjusting their voltages to null any transient current.

Now, consider a two-dimensional electron gas in high magnetic field with the standard Hall geometry as shown in Figure 4. The high field causes the formation of Landau levels with large degeneracy and low velocity throughout the bulk of the 2DEG; however for each bulk Landau level (LL) there exists a current-carrying quantum state at the edge of the sample with (roughly speaking) the the same cyclotron energy and an additional kinetic energy due to its skipping motion along the edge of the sample (see, for example, Streda *et al.* 1987, Büttiker 1988). Assume that the Fermi energy is above the centre of the Nth LL so that a current $N(e/h)\mu_1$ is being carried into the sample by these states from reservoir 1 along the upper edge. There is disorder in the sample which in general can cause scattering; however *if* all the current continues to flow along that

edge and into voltage probe 2 (*i.e.* $T_{21} = N$), then we must have $N(e/h)(\mu_1 - \mu_2) = 0$ in order for the total current into probe 2 to vanish, *i.e.* $\mu_1 = \mu_2 = \mu_{\text{source}}$. By repeating this argument for μ_3 one immediately sees that all voltage probes on this side of the current path will adjust their chemical potentials to be equal to that of the source. The current sink, on the other hand, is maintained at a different chemical potential μ_{sink}, and by the same argument all probes on the lower side of the current path will be at the potential μ_{sink}. Thus we see that a net current $I = N(e/h)(\mu_{\text{source}} - \mu_{\text{sink}})$ can flow from source to sink with no voltage appearing between any two probes on the same side of the current path, *i.e.* the longitudinal resistance R_L vanishes. The Hall resistance is just the ratio of the voltage induced between any two probes separated by the current path, which is $(\mu_{\text{source}} - \mu_{\text{sink}})/e$, divided by the net current, giving $R_H = (h/Ne^2)$. Since N, the number of states at the Fermi level, is equal to the number of Landau levels below the Fermi surface, this formula yields the familiar quantised Hall resistance. This argument establishes that if the transmission coefficients $T_{n,n-1}$ are quantised to N, and if all other transmission coefficients are zero, then one obtains at the same time the quantised Hall resistance and the vanishing longitudinal resistance (which is normally not treated on the same footing by Laughlin's famous argument).

It is easy to see how these quantised transmission values can occur when all the states at the Fermi level are true edge states. Electrons are injected near one edge, and in order to back-scatter they must be removed at the other edge; as long as the cyclotron radius is short compared to the width of the sample, and the disordered potential is weak, it will be unable to scatter them across the sample (Streda *et al.* 1987, Jain and Kivelson 1988, Büttiker 1988). An arbitrary amount of forward scattering between edge states will not break the quantisation, since by unitarity the transmission coefficients must still sum to N. However it must be emphasised that the Fermi level is typically pinned near the center of the LL due to its high density of states in a macroscopic 2DEG of the type typically used for quantum Hall measurements, and the Fermi level resides in the region of edge states only very briefly at the center of a plateau. Hence the existence of plateaus cannot be explained by this simple argument; they arise due to localisation, which prevent states near the center of the LL from carrying current across the sample [see Stone *et al.* (1990) for a detailed discussion of this point].

3.3 Reciprocity symmetry of resistance

Although I derived Hall quantisation from a more detailed physical argument above, it is possible simply to solve Equation (4) in general for a four-probe measurement with the total current I through the current probes and zero current in the voltage probes. The conductance matrix g_{mn} is non-invertible in the whole vector space because its rows all sum to zero, but since the currents considered also always sum to zero it is invertible in the relevant subspace, and it is easy to confirm that voltage *differences* are always uniquely determined (Stone and Szafer 1988). If the current probes are designated 1, 3 and the voltage probes are 2, 4, one finds the resistance

$$R_{13,24} = \frac{h}{e^2}\left[\frac{T_{21}T_{43} - T_{41}T_{23}}{S}\right], \tag{17}$$

where S is any 3×3 sub-determinant of the conductance matrix. If one substitutes into Equation (17) the conditions $T_{nm} = N\delta_{m,n+1}$, considered in our physical argument

above, it is easily seen to yield a quantised Hall resistance $h/(Ne^2)$.

Equation (17) is also useful in establishing the exact time-reversal symmetry of quantum transport measurements (Büttiker 1986). As noted above, the scattering matrix is not only unitary but symmetric in the presence of time-reversal symmetry, hence $g_{mn}(B) = g_{nm}(-B)$. It is then easily seen from Equation (17) that

$$R_{13,24}(-B) = R_{24,13}(B) \tag{18}$$

under field-reversal, *i.e.* time-reversal symmetry connects two *different* measurements in which current and voltage probes are interchanged and the field is reversed. This relation is referred to as *reciprocity symmetry* and was known prior to the work of Büttiker, although not on such a fundamental basis. The familiar symmetry of longitudinal resistance and antisymmetry of Hall resistance measurements under field reversal results not simply from time-reversal symmetry but from additional spatial symmetries of the scattering potential in the sample. Such symmetries exist on average in macroscopic samples, but are essentially always violated in mesoscopic samples at low temperatures where interference effects dominate. It was strikingly shown by Benoit *et al.* (1986) that there is a complete absence of symmetry under field-reversal for a fixed measuring configuration in samples with well-developed UCF effects, but that the reciprocity symmetry of Equation (18) is satisfied within experimental accuracy upon interchanging voltage and current probes.

3.4 Interference effects

The relationship between S-matrix coefficients and transport coefficients makes it clear that the latter will show significant interference effects in phase-coherent samples, even in the presence of static disorder. For example it was shown in the original work of Aharonov and Bohm that the scattering matrix of a doubly-connected system threaded by a flux will oscillate periodically with the enclosed flux due to the coupling between the vector potential and the phase of the quantum wavefunctions. It is straightforward then to show that for a one-dimensional ring-shaped conductor connected to leads one expects Aharonov-Bohm oscillations of order the average conductance even with additional scattering in the ring (Gefen *et al.* 1984). The crucial point is that elastic scattering only alters the phase of the wavefunction in a fixed, sample-specific manner, without averaging over that phase. An additional source of averaging, such as the energy fluctuations associated with inelastic scattering, is needed to eliminate the contributions due to interference. However in mesoscopic samples at low temperatures the inelastic scattering time can be much longer than the diffusive transit time across the sample, and the full sample-specific interference effects will show up in transport measurements. A physical analogy is to a laser beam transmitted through a static medium with randomly-varying dielectric constant; such a beam when detected will create a *speckle pattern* in which the spatial intensity varies rapidly due to the complex interference of the various light paths to a given point.

It is then natural to consider the electronic system using the analogue of ray optics, which is the semi-classical path integral formulation of quantum mechanics. To consider transmission and d.c. conductance we are interested in the Green function at fixed energy, ε_F, instead of at a fixed time, so we employ the formulation of Gutzwiller

(1990) in which the semiclassical (WKB) Green function is expressed as a sum over classical paths from r to r′ at energy E. Since from Equation (14) we are interested in derivatives of the product of two Green functions, we use Gutzwiller's expression and define the electronic intensity as

$$\begin{aligned} I(\mathrm{r},\mathrm{r}') &= G(E,\mathrm{r},\mathrm{r}')\,G(E,\mathrm{r}',\mathrm{r}) \\ &= \frac{1}{\hbar^3}\sum_{p(\mathrm{r},\mathrm{r}')}\sum_{q(\mathrm{r},\mathrm{r}')}\sqrt{D_p}\sqrt{D_q}\exp\left\{\frac{i}{\hbar}\left[S_p(E)-S_q(E)\right]-i\frac{\pi}{2}\mu_{pq}\right\}, \end{aligned} \tag{19}$$

where S_p is the action integral along classical path p at energy E, D_p is a positive amplitude given by the stability of the path with respect to variations in its initial momentum and μ_{pq} is the difference of the Maslov indices of the paths p and q, equal to the number of conjugate points along each trajectory (the details of this formulation will not be relevant for the qualitative discussion here, so we need not go into them in depth). The terms in the double sum for which $p \neq q$ represent the interference of the different paths between the points r and r′.

To get a feeling for how important such interference might be in a disordered but weak potential in which many paths may exist between any two points, assume that there are N_p paths with equal amplitudes (which we take to be unity) and equal Maslov indices, so that

$$I(\mathrm{r},\mathrm{r}') = \sum_{p(\mathrm{r},\mathrm{r}')}\sum_{q(\mathrm{r},\mathrm{r}')}\exp\left\{\frac{i}{\hbar}\left[S_p(E)-S_q(E)\right]\right\}. \tag{20}$$

Assume that the phases of all terms with $p \neq q$ are uncorrelated and vanish upon averaging over disorder. It is then a trivial exercise to show that

$$\frac{\langle(\delta I)^2\rangle}{\langle I\rangle^2} = 1+\frac{1}{N_p}; \tag{21}$$

the local relative intensity fluctuations are order unity. In fact many speckle patterns obey Equation (21) to good accuracy even though the argument was over-simplified, and we expect similarly complex local transmission fluctuations in electronic systems.

As noted above, the particular interference pattern should be sensitive to external parameters which couple to the phase, such as energy or magnetic field, and of course to the particular realisation of the random scattering potential. To estimate the scale of sensitivity to E and B one need only consider the change in phase of a given path as these parameters are varied. Let Λ represent an external parameter upon which the action depends, and assume that the classical path smoothly evolves as a function of this parameter over the scale of interest (so that its identification remains unambiguous). Then

$$\Delta\phi = \frac{1}{\hbar}\left[S_p(\Lambda+\Delta\Lambda)-S_p(\Lambda)\right] \approx \frac{1}{\hbar}\frac{\partial S_p}{\partial\Lambda}(\Delta\Lambda). \tag{22}$$

When $\Lambda = E$, we have from classical mechanics the relation $\partial S_p/\partial\Lambda = T_p$, the time required to traverse the path. The interference pattern will fluctuate when most paths which traverse a sample of length L have changed their phase by order unity, so estimating the correlation scale E_c by $\Delta\phi(\Delta E = E_c) \approx 1$ yields

$$E_c \approx \frac{\hbar}{\langle T_p\rangle}, \tag{23}$$

where $\langle T_p \rangle$ is the mean traversal time. The mean time is of order D/L^2 in the diffusive case, where D is the diffusion constant, so we have

$$(E_c)_{\text{diff}} \approx \frac{\hbar D}{L^2}. \tag{24}$$

When the parameter Λ is a uniform field B perpendicular to the plane of motion, the action depends on the field through a term $(e/c)\int \mathbf{A}\cdot\mathrm{d}\mathbf{l}$ where the line integral is along the path. If we assume that change in field ΔB has a negligible effect on the path, we have

$$\Delta\phi(\Delta B) = \frac{e}{\hbar c}\left\langle \int \Delta\mathbf{A}\cdot\mathrm{d}\mathbf{l} \right\rangle. \tag{25}$$

For a diffusive trajectory which intersects itself many times, the line integral will just give ΔB times the area enclosed, which for a path crossing the sample will be of order the sample area. Hence

$$(B_c)_{\text{diff}} \approx \frac{L^2}{(hc/e)}\Delta B. \tag{26}$$

A slightly more complicated argument along these lines can be used in the diffusive limit to estimate the sensitivity of the interference pattern to changes in the scattering potential (Feng *et al.* 1986). Without reproducing the detailed argument here, let me note that it is clear from the above analysis that a change in the scattering phase shift of order unity anywhere in a given path will alter its phase sufficiently. Since in diffusion a number of scattering event of order L^2 occurs in traversing the sample, a change of this magnitude in the scattering potential anywhere in the sample will be sufficient to alter the phase of a fraction of order unity of all the paths in 2D and 1D. Hence a phase-coherent 2D conductor will be sensitive to the motion of a single impurity in a manner which is independent of the size of the system. This extreme sensitivity to small changes in the sample leads to a new kind of quantum low-frequency noise (Feng *et al.* 1986) which has now been convincingly observed in a number of experiments (see, for example, Birge *et al.* 1989). I will not discuss this aspect of the theory further.

Finally, it is worth noting that these general ideas concerning the sensitivity of interference effects to changing parameters have recently been shown to apply to the ballistic as well as the diffusive regime (Jalabert *et al.* 1990). It is only necessary to evaluate the average of $\partial S/\partial\Lambda$ for the particular classical dynamics relevant to the system of interest.

In addition to the fluctuations arising from the interference of unrelated classical paths between $\mathbf{r}$ and $\mathbf{r}'$ in Equation (19), there are interference effects associated with paths related by time-reversal symmetry, which lead to weak localisation. By a well-known argument which I shall not repeat here (Bergmann 1984), these effects double the intensity $I(\mathbf{r},\mathbf{r})$ in the absence of magnetic field and spin-orbit scattering. The suppression of these effects by a magnetic field gives the distinctive negative magnetoresistance associated with weak localisation. A quantitative theory of weak localisation based on the semiclassical approach has been developed by Chakravarty and Schmid (1986).

Although it is possible to estimate the local fluctuations in intensity and the correlation lengths from elementary considerations using the semiclassical approach, one needs to evaluate the amplitudes in Equation (19), and most importantly to understand the spatial correlations in the intensity, to evaluate the interference effects in the

total transmission coefficient (*i.e.* the conductance). This is a difficult task which has not been accomplished so far using the semiclassical technique. Hence we leave this approach at this point and employ the impurity-averaged Green function technique, which provides a systematic perturbation theory for the quantities of interest.

4 Impurity-average technique in real space

Below I will review the aspects of the impurity-average Green function technique required to develop the theory of universal conductance fluctuations. Almost identical calculations arise in the theory of weak localisation and persistent currents, so the techniques are of wide applicability. I choose to use the less common formulation in real space primarily because we have recently shown (Xiong and Stone 1991) that this approach can be generalised to arbitrary magnetic fields which satisfy $N \gg 1$ (where N is the Landau level index) within the self-consistent Born approximation, whereas the previous theory only applied at fields for which the cyclotron radius was much larger than the elastic mean free path. The condition $r_c \gg l$ is violated at moderate fields in most two-dimensional electron gas systems, so the generalisation of the theory is of some importance.

4.1 Historical background

The impurity-averaging formalism dates back to Edwards (1958), having been proposed and used to derive the Drude conductivity only a few months after Greenwood's paper in which he first derived Equation (15) for the conductivity. At zero magnetic field and weak disorder the technique provides a systematic expansion in the small parameter $(\varepsilon_F \tau/\hbar)^{-1}$, where τ is the elastic mean free time (see, for example, Abrikosov *et al.* 1965). The Drude conductivity can be obtained by treating a particular contribution known as the ladder diagrams. The technique was used in the study of superconductivity during the 60's, and Langer and Neal (1966) discovered that the perturbation theory for the conductivity was formally divergent in dimensions $d \leq 2$, in the sense that there existed contributions of lower order in $(\varepsilon_F \tau)^{-1}$ which depended upon the lower momentum cut-off and scaled with the system size. However it was not until the work of Abrahams *et al.* (1979) that this divergence was shown to indicate the nonexistence of extended states in the infinite system at $T = 0$ in 2D. The divergent contribution studied by Langer and Neal, referred to either as the maximally-crossed graphs or as the *cooperon* contribution because of its role in the study of superconductivity, then became the basis of the theory of weak localisation. The divergence of such contributions is always ultimately cut off by finite temperature effects, and thus only leads to a small correction to g in good 2D conductors at any practical temperature.

In 1985 Altshuler, and Lee and Stone, independently discovered that the ladder or *diffuson* contribution, which was well-behaved for the average conductivity, was divergent in $d \leq 4$ for the variance of g. This led to conductance fluctuations in phase-coherent metals that are anomalously large from the classical point of view, and formed the basis of the theory of universal conductance fluctuations. The theories of WL and UCF only included the effect of a magnetic field through its coupling to the phase of

the wavefunction, as discussed in Section 3.4. A version of the impurity-averaging technique appropriate for a 2DEG in a high field where Landau level (LL) quantisation is important was developed by Ando (1974, 1975) based on the *self-consistent Born approximation* (SCBA). Carra, Chalker and Benedict (1989) showed that this was a systematic expansion in $1/N$ in the limit of a short-ranged potential, where N is the LL index. Again the expansion for the conductivity was shown to contain divergences in terms of lower order, indicating the importance of the localisation effects in such systems which are essential to the quantum Hall effect as discussed above. Hence this theory is also not expected to apply to the infinite system at $T = 0$; nonetheless for systems at finite temperature the perturbation theory appears to have a reasonable range of validity near the center of the LL. For example the SCBA predicts that the height of the peaks in σ_{xx} is linear in the Landau index, N, in the limit of a short-ranged potential, and this prediction is often quantitatively satisfied in 2DEG's (see, for example, Luo *et al.* 1989). The perturbation theory should be even better near the center of the LL in mesoscopic systems which are of course far from the limit of an infinite volume. Below we will show for the first time that the there exists a generalisation of UCF theory to arbitrary magnetic field, valid to order $(\varepsilon_F\tau)^{-1}$ when the cyclotron radius $r_c > l$, and valid to order $1/N$ when $r_c < l$.

4.2 White noise model for average Green function

The basic principles of the impurity-averaging technique are as follows.

1. To express quantities of physical interest in terms of the electronic Green functions for a given configuration of the random impurity potential $V(\mathrm{r})$. If the system is assumed non-interacting, all such quantities will be expressible as products of the advanced and retarded one-particle Green functions (1PGF), $G^{\pm}$, introduced above before Equation (15).

2. To express the Green functions as a perturbation expansion in $V(\mathrm{r})$ using the Dyson equation for the 1PGF.

3. To average the quantities of interest over these realisations; this averaging is equivalent to introducing a special kind of static two-body interaction between electrons which leads to a non-trivial perturbation expansion.

Since our system is non-interacting the 1PGF is just the inverse of the one-body hamiltonian which we take to be $H = H_0 + V(\mathrm{r})$, with

$$H_0 = \frac{1}{2m}\left(\mathrm{P} - \frac{e}{c}\mathrm{A}\right)^2, \tag{27}$$

with m the effective mass and $\mathrm{A}(\mathrm{r})$ the vector potential of a uniform magnetic field perpendicular to the transport direction. Let $G_0^{\pm} = [E - H_0 \pm i\eta]^{-1}$ where the superscript $\pm$ gives the sign of the infinitesimal η which fixes the analytic properties of G, but will be suppressed henceforth except where it is needed to resolve an ambiguity.

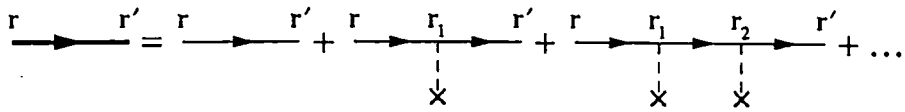

Figure 5: *Dyson equation for the one-particle Green function for a given configuration of impurities, before averaging.*

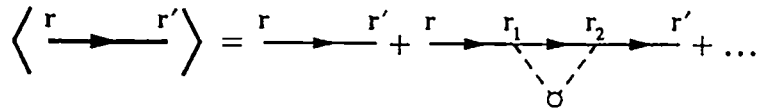

Figure 6: *Dyson equation for the one-particle Green function after averaging; dashed lines connected by a circle denote the effective interaction.*

The operator form of the Dyson equation for $G^{\pm}$ before averaging is

$$\begin{aligned} G(E) &= [E - H_0 - V]^{-1} \\ &= \left[G_0^{-1}(1 - G_0 V)\right]^{-1} \\ &= \left[\sum_{j=0}^{\infty}(G_0 V)^j\right] G_0 \\ &= G_0 + G_0 V G. \end{aligned} \tag{28}$$

This equation is represented by the sequence of diagrams in Figure 5, where the crosses represent interactions with the random potential. Consider the third term in Figure 6, representing the term with $j = 2$ in the third line of Equation (28). Upon averaging this over impurity configurations in a real space representation this takes the form

$$\int dr_1\, dr_2\, G_0(r, r_1)\, G_0(r_1, r_2)\, G_0(r_2, r')\langle V(r_1)V(r_2)\rangle, \tag{29}$$

where we see that the statistical average (denoted either by angle brackets or an overbar below) of the random potential at two points in space enters as an effective elastic two-body interaction (since the energy is unchanged along each line). Diagrammatically we represent this by joining the two crosses to make a single dashed line as shown in Figure 6. Different statistical models for the correlation of moments of the potential are known to give essentially identical results in the low-field limit, so we use the simplest one, the white noise model, which yields a particularly simple theory in the limit of high B. In the WN model

$$\langle V(r)\rangle = 0 \tag{30}$$

$$\langle V(r)V(r')\rangle = c_i u^2 \delta(r - r'). \tag{31}$$

All odd higher moments are zero, and all even higher moments are pairwise decompositions in terms of the second moment; c_i is the number density of impurities and u^2 may be regarded as the mean-squared strength of the scattering potential in Fourier space.

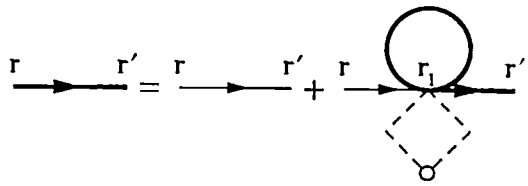

Figure 7: *Diagrammatic representation of the self-consistent equation for the Green function in the self-consistent Born approximation.*

5 Average Green function in SCBA

In the WN model the disorder average is identical to a static two-body interaction for all terms in the perturbation expansion for the disorder-averaged 1PGF, $\overline{G}(\mathrm{r},\mathrm{r}')$. Hence $\overline{G}(\mathrm{r},\mathrm{r}')$ satisfies an integral equation in terms of the proper self-energy insertion of the usual type:

$$\overline{G}(\mathrm{r},\mathrm{r}') = G_0(\mathrm{r},\mathrm{r}') + \int d\mathrm{r}_1\, d\mathrm{r}_2\, G_0(\mathrm{r},\mathrm{r}_1)\,\Sigma(\mathrm{r}_1,\mathrm{r}_2)\,\overline{G}(\mathrm{r}_2,\mathrm{r}'). \tag{32}$$

The self-consistent Born approximation for the self-energy is to approximate Σ as follows:

$$\Sigma(\mathrm{r},\mathrm{r}') \approx c_i u^2\,\overline{G}(\mathrm{r},\mathrm{r}')\,\delta(\mathrm{r}-\mathrm{r}'); \tag{33}$$

this is shown diagramatically in Figure 7. When inserted into Equation (32) the approximation yields the self-consistent equation for $\overline{G}(\mathrm{r},\mathrm{r}')$,

$$\overline{G}(\mathrm{r},\mathrm{r}') = G_0(\mathrm{r},\mathrm{r}') + \int d\mathrm{r}_1\, G_0(\mathrm{r},\mathrm{r}_1)\, c_i u^2\,\overline{G}(\mathrm{r}_1,\mathrm{r}_1)\,\overline{G}(\mathrm{r}_1,\mathrm{r}'). \tag{34}$$

The significance of this equation can be understood by applying $(E - H_0)$ to both sides and using $\langle \mathrm{r}|(E-H_0)G_0|\mathrm{r}'\rangle = \delta(\mathrm{r}-\mathrm{r}')$ to obtain

$$\left[E - c_i u^2\,\overline{G}(\mathrm{r},\mathrm{r}) - H_0\right]\overline{G}(\mathrm{r},\mathrm{r}') = \delta(\mathrm{r}-\mathrm{r}'). \tag{35}$$

Since the system is translationally invariant on average, $\overline{G}(\mathrm{r},\mathrm{r},E)$ is just a complex function of energy, independent of r, and we shall denote this function simply by $\overline{G}(E)$ henceforth. It follows that $\overline{G}(\mathrm{r},\mathrm{r}')$ satisfies exactly the same equation as $G_0(\mathrm{r},\mathrm{r}')$ except that the energy E is replaced by the complex energy $z(E) = E - c_i u^2\,\overline{G}(\mathrm{r},\mathrm{r})$. Hence

$$\overline{G}(\mathrm{r},\mathrm{r}',E) = G_0(\mathrm{r},\mathrm{r}',z). \tag{36}$$

This is the fundamental result of the SCBA for the 1PGF in the WN limit.

From now on we specialise to the two-dimensional case; generalisation to 3D is straightforward. Using Equation (35) we can immediately write an expression for $\overline{G}(\mathrm{r},\mathrm{r}')$ in the SCBA valid for arbitrary field by transcription into the expression for G_0:

$$\overline{G}(\mathrm{r},\mathrm{r}',E) = \sum_{n=0}^{\infty} \frac{P_n(\mathrm{r},\mathrm{r}')}{E - E_n - c_i u^2\,\overline{G}(E)}, \tag{37}$$

where $\overline{G}(E) = \overline{G}(\mathrm{r}, \mathrm{r}, E)$, and $P_n(\mathrm{r}, \mathrm{r}')$ is the projection operator onto the nth LL in real space, and $E_n = (n + \frac{1}{2})\hbar\omega_c$ is the energy of the nth LL. It is possible to calculate P_n exactly in 2D [see the article by Haldane in Prange (1986)] and one finds

$$P_n(\mathrm{r}, \mathrm{r}') = \frac{1}{2\pi\lambda^2} \exp\left(-\frac{R^2}{4\lambda^2}\right) L_n\left(\frac{R^2}{2\lambda^2}\right) \exp\left(\frac{ie}{\hbar c}\int_{\mathrm{r}'}^{\mathrm{r}} \mathbf{A}\cdot \mathrm{dl}\right), \tag{38}$$

where $R = |\mathrm{r} - \mathrm{r}'|$, $\lambda^2 = (\hbar c/eB)$ is the square of the magnetic length and L_n is the nth Laguerre polynomial. The line integral in the phase factor is taken along the straight line from r' to r. Note that $P_n(0) = 1/(2\pi\lambda^2)$, since $L_n(0) = 1$ for all n. Using this fact and setting $\mathrm{r} = \mathrm{r}'$ in Equation (37) we obtain the self-consistent equation for $\overline{G}(E)$,

$$2\pi\lambda^2\overline{G}(E) = \sum_{n=0}^{\infty} \frac{1}{E - E_n - c_i u^2\,\overline{G}(E)}. \tag{39}$$

To obtain $\overline{G}(\mathrm{r}, \mathrm{r}')$ in the limit of high fields, in which the broadening of the LLs by disorder is much less than their spacing $\hbar\omega_c$, we note that in this case $E = \varepsilon_F$ will be pinned near a particular LL with energy $E_N = \left(N + \frac{1}{2}\right)\hbar\omega_c$, and $\overline{G}(E)$ in Equation (37) will be dominated by the term with $n = N$ in the sum:

$$2\pi\lambda^2\overline{G}(0, E \approx E_N) \approx 2\pi\lambda^2\overline{G}_N(E) \approx \frac{1}{E - E_N - c_i u^2\,\overline{G}_N(E)}. \tag{40}$$

Solving this quadratic equation for $G_N(E)$ gives the explicit result

$$G_N^{\pm}(E) = \frac{e^{\pm i\theta(E)}}{2\pi\lambda^2\nu}, \tag{41}$$

where $\nu^2 = c_i u^2/2\pi\lambda^2$ and $\cos\theta(E) = (E - E_N)/2\nu$. Substitution into Equation (37) with terms $n \neq N$ neglected then yields

$$G_N^{\pm}(\mathrm{r}, \mathrm{r}', E) \approx P_N(\mathrm{r}, \mathrm{r}')G_N^{\pm}(E). \tag{42}$$

The spatial range of $P_N(\mathrm{r}, \mathrm{r}')$ is obtained by maximising a polynomial of degree N against the gaussian fall-off with range λ. Just as for the wavefunctions this gives a maximum at the cyclotron radius $r_c = (N + \frac{1}{2})^{1/2}\lambda$. This approximation for $\overline{G}(\mathrm{r}, \mathrm{r}')$ is valid when $r_c \ll l$ and shows explicitly that the range of $\overline{G}(\mathrm{r}, \mathrm{r}')$ is of order r_c and not l in this limit. From the imaginary part of this expression one can obtain the shape of the broadened LL in the WN limit, which is simply a semi-circle of radius $\nu \approx (\hbar^2\omega_c/\tau)^{1/2}$ (Ando *et al.* 1975). In the high field limit the self-consistent calculation of the Green function is crucial because the LL is infinitely degenerate without disorder, and substitution of G_0 on the LHS of Equation (35) would give an infinite spike at $\varepsilon_F = E_N$ and zero elsewhere.

In the zero-field limit the mean density of states is not altered by disorder to leading order, and it is permissible to replace $\overline{G}(E)$ by $G_0(E)$ on the LHS of Equation (35) and neglect the real part of G_0. Since $\mathrm{Im}G_0^{\pm} = \mp\pi\rho_F$, and by definition the elastic scattering rate in Born approximation is $1/\tau = 2\pi c_i u^2 \rho_F$, where ρ_F is the density of states at the Fermi energy, we have

$$\overline{G^{\pm}}(\mathrm{r}, \mathrm{r}', \varepsilon_F, B = 0) = G_0^{\pm}(\mathrm{r}, \mathrm{r}', \varepsilon_F \pm i/2\tau, B = 0). \tag{43}$$

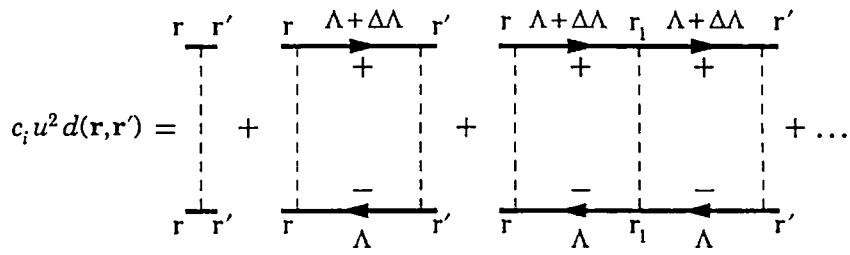

Figure 8: *Class of polarisation diagrams in real space whose sum gives the diffuson contribution.*

The rapidly-varying part of $G_0^{\pm}(\mathrm{r},\mathrm{r}') \sim \exp(\pm ik_F|\mathrm{r}-\mathrm{r}'|)$, where $k_F = (2m\varepsilon_F/\hbar^2)^{1/2}$ is the Fermi wave-vector, so substituting $\varepsilon_F \to \varepsilon_F \pm i/2\tau$ and expanding to leading order in $(\varepsilon_F\tau)^{-1}$ one obtains the familiar result

$$\overline{G^{\pm}}(\mathrm{r},\mathrm{r}',\varepsilon_F,B=0) = G_0^{\pm}(\mathrm{r},\mathrm{r}',\varepsilon_F,B=0)\exp(-|\mathrm{r}-\mathrm{r}'|/2l). \qquad (44)$$

The decay of $\overline{G}$ on a scale l is due to averaging over the random phase-shift of G for a given impurity configuration; it does not mean that the modulus of G decays this rapidly for a given configuration. Nonetheless we see that the average Green function is short-ranged for arbitrary field within the SCBA, with a range given by the shorter of r_c and l which we shall refer to as l_{min}.

5.1 Diffuson correlator

Referring back to Equation (14) one sees that the averages of products of two Green functions are needed in order to calculate the average conductance. This is done by expanding each factor $G(\mathrm{r},\mathrm{r}')$ in a power series in $V(\mathrm{r})$ using Equation (28), and then averaging the product of these two series. Two types of terms result. The first type only involve impurity interactions along each line separately; this corresponds to the factorisation $\overline{GG} = \overline{G}\,\overline{G}$, and all such contributions can be accounted for by making the appropriate self-energy insertion in each line separately in the manner described above. The second type of term arises from the creation of interaction lines that join the two Green functions as shown in Figure 8, and correspond to a non-factorisable contribution. These vertex or polarisation corrections can introduce qualitatively new features because now it is possible to correlate Green functions with different energy, magnetic field and analyticity. These correlators can also be long-ranged in space compared to $\overline{G}\,\overline{G}$, each factor of which must decay over a distance of l_{min}. In particular, the correlator $\overline{G^+G^-}$ can have long-ranged behaviour because of the relation $G^+(\mathrm{r},\mathrm{r}') = [G^-(\mathrm{r}',\mathrm{r})]^*$, which makes it possible for the phase shift at each scattering event to cancel.

This is true for the particular infinite class of diagrams known as the *diffuson* diagrams shown in Figure 8 which contribute to the correlator $\overline{G^+(\mathrm{r},\mathrm{r}')\,G^-(\mathrm{r}',\mathrm{r})}$. Note that in the WN model each impurity interaction line has a single point in space associated with it, so it is tempting to associate a given polarisation diagram (before integrating over intermediate positions) with a particular pair of trajectories in space

which visit the same set of impurity sites in some order. With this interpretation the diffuson contribution is precisely the interference of a given trajectory with itself (complex conjugated) and corresponds to the diagonal approximation discussed in Section 3.4 above. Define the contribution of these diagrams to be $c_i u^2 d(\mathbf{r}, \mathbf{r}')$; then it is clear that d satisfies the integral equation

$$d(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') + \int d\mathbf{r}_1 \, d_0(\mathbf{r}, \mathbf{r}_1) \, d(\mathbf{r}_1, \mathbf{r}'), \tag{45}$$

where

$$d_0(\mathbf{r}, \mathbf{r}') = c_i u^2 \overline{G}^+(\mathbf{r}, \mathbf{r}', \Lambda + \Delta\Lambda) \overline{G}^-(\mathbf{r}', \mathbf{r}, \Lambda), \tag{46}$$

and Λ represent the external parameters such as Fermi energy, frequency, or magnetic field which may be different along each line. Equation (45) is just a geometric series in the operator d_0 and may be solved easily in operator form

$$d = 1 + d_0 d; \quad d = [1 - d_0]^{-1}. \tag{47}$$

Introduce the eigenfunctions, $\chi_j(\mathbf{r})$, and eigenvalues, $1 - \xi_j^2$, of the integral operator d_0 defined by

$$\int d\mathbf{r}' \, d_0(\mathbf{r}, \mathbf{r}') \chi_j(\mathbf{r}') = (1 - \xi_j^2) \chi_j(\mathbf{r}). \tag{48}$$

Then we can write a spectral representation of d,

$$d(\mathbf{r}, \mathbf{r}') = \sum_{j=1}^{\infty} \frac{\chi_j(\mathbf{r}) \chi_j^*(\mathbf{r}')}{\xi_j^2}. \tag{49}$$

First consider the case where $\Delta\Lambda = 0$ so d_0 is given by the real positive function $\left|\overline{G^+}(|\mathbf{r} - \mathbf{r}'|)\right|^2$ whose range is $l_{\min}$. We follow a technique introduced by Altshuler *et al.* (1980) in order to convert the integral equation (45) into a diffusion equation. Since d_0 is short-ranged we can regard the eigenfunction $\chi(\mathbf{r})$ in Equation (45) as slowly-varying and expand it around $\mathbf{r}$:

$$\chi(\mathbf{r}') \approx \chi(\mathbf{r}) + \nabla\chi \cdot (\mathbf{r} - \mathbf{r}') + \tfrac{1}{2} \sum_{a,b} \nabla_a \nabla_b \chi \, (\mathbf{r} - \mathbf{r}')_a (\mathbf{r} - \mathbf{r}')_b + \ldots . \tag{50}$$

Substitution of this expansion into Equation (45) yields a differential equation of the form

$$\left(C_2 \nabla^2 + C_1\right) \chi_j(\mathbf{r}) = (1 - \xi_j^2) \chi_j(\mathbf{r}), \tag{51}$$

where C_2 and C_1 are constants and we have used the fact that d_0 is even in $\mathbf{r} - \mathbf{r}'$ to eliminate terms linear in $\mathbf{r} - \mathbf{r}'$ and simplify the quadratic terms. The constant C_1 is, from Equations (46) and (48),

$$C_1 = c_i u^2 \int d\mathbf{r}' \left|\overline{G^+}(|\mathbf{r} - \mathbf{r}'|)\right|^2. \tag{52}$$

Equation (37) shows that in general the spatial dependence of $\overline{G}(\mathbf{r}, \mathbf{r}')$ comes only from the projection operators $P_n(\mathbf{r}, \mathbf{r}')$, which satisfy

$$\int d\mathbf{r}' P_n(\mathbf{r}, \mathbf{r}') P_m(\mathbf{r}', \mathbf{r}) = \delta_{mn} P_n(\mathbf{r}, \mathbf{r}) = (2\pi\lambda^2)^{-1} \delta_{mn}.$$

Thus one can simplify Equation (52) to

$$C_1 = \frac{c_i u^2}{2\pi\lambda^2} \sum_n^\infty \frac{1}{\left|E - E_n - c_i u^2 \overline{G}^+(E)\right|^2}. \tag{53}$$

If we now use the identity

$$\begin{aligned}
&\sum_n^\infty \frac{1}{\left|E - E_n - c_i u^2 \overline{G}^+(E)\right|^2} \\
&= \frac{1}{c_i u^2 \left[\overline{G}^+(E) - \overline{G}^-(E)\right]} \sum_n^\infty \left[\frac{1}{E - E_n - c_i u^2 \overline{G}^+(E)} - \frac{1}{E - E_n - c_i u^2 \overline{G}^-(E)}\right] \\
&= \frac{1}{c_i u^2 \left[\overline{G}^+(E) - \overline{G}^-(E)\right]} 2\pi\lambda^2 \left[\overline{G}^+(E) - \overline{G}^-(E)\right],
\end{aligned} \tag{54}$$

we find that $C_1 = 1$. In deriving this result we have used the self-consistent Equation (39) relating the Green function $\overline{G}(E)$ and the self-energy $c_i u^2 \overline{G}(E)$, valid for an arbitrary magnetic field. The exact cancellation giving $C_1 = 1$ is a manifestation of a general Ward identity relating the polarisation vertex (here represented just by the prefactor $c_i u^2$) and the Green function that will be valid even in the presence of interactions if a consistent approximation is made for all quantities. We shall see below that $C_1 = 1$ is required for there to be a diffusion pole in $d(\mathrm{r}, \mathrm{r}')$; the robustness of this condition with respect to interactions ensures that diffusive behaviour occurs at long wavelengths even in the presence of interactions as we expect for a Fermi liquid (neglecting localisation effects).

The constant C_2 has the dimensions of length squared so we define $C_2 = l_0^2$, where

$$\begin{aligned}
l_0^2 &= \frac{c_i u^2}{2d} \int d\mathrm{r}' \, (\mathrm{r} - \mathrm{r}')^2 \left|\overline{G^+}(|\mathrm{r} - \mathrm{r}'|)\right|^2 \\
&= \frac{c_i u^2}{2d} \int d\mathrm{R} \, R^2 \left|\overline{G^+}(R)\right|^2.
\end{aligned} \tag{55}$$

From this we see that the length l_0 is just the spatial range of the average Green function. Hence the differential equation that we need to solve to obtain the diffuson with the same external parameters on each line is simply

$$-l_0^2 \nabla^2 \chi_j(\mathrm{r}) = \xi_j^2 \chi_j(\mathrm{r}). \tag{56}$$

It is easily found from Equations (41) and (44) that

$$l_0^2 = \begin{cases} l^2/2, & l \ll r_c \\ r_c^2, & r_c \gg l, \end{cases} \tag{57}$$

consistent with our earlier explicit results for $\overline{G}(\mathrm{r}, \mathrm{r}')$.

Consider a rectangular 2D sample of length L_x and width L_y. The appropriate boundary conditions are that the diffuson must vanish on the surfaces normal to the direction of current flow (which we take to be the x-direction) since no excess density can build up in the leads (reservoirs), and the derivative of the diffuson must vanish

at the walls because the current there (which is proportional to the derivative of the excess density) must vanish. Solving Equation (56) explicitly and substituting into Equation (49) yields

$$d(\mathrm{r},\mathrm{r}') = \sum_{m=1,n=0}^{\infty} \frac{\chi_{mn}(\mathrm{r})\,\chi^{*}_{mn}(\mathrm{r}')}{l_0^2(k_m^2+q_n^2)}, \tag{58}$$

where $q_n = n\pi/L_y$, $k_m = m\pi/L_x$, and the eigenfunctions χ_{mn} are appropriately normalised products of $\sin(k_m x)\cos(q_n y)$. Hence we have the familiar diffusion pole diverging as Q^{-2} for arbitrary magnetic field. If we take the $B=0$ limit $l_0^2 \to D\tau$ (where $D = l^2/2\tau$ is the elastic diffusion constant), and at high field we find $l_0^2 \to \hbar D_N/\nu$, where ν is the LL broadening mentioned above and $D_N = r_c^2\nu/\hbar$ is the diffusion constant for a single LL in SCBA (Ando 1975). Note that the Q^{-2} divergence is automatically cut off by the sample size due to the boundary conditions imposed on Equation (56), which meant that the summation over k_m starts from $m=1$ and not $m=0$.

5.2 Generalised diffuson

For the theory of UCF we need to consider the generalised kernel d_0 of Equation (45) with the factor $\overline{G}^{+}$ taken at shifted values of the energy and magnetic field. The generalisation to finite ΔE is straightforward. The analysis leading to Equation (51) is unchanged, except that the expressions for C_1 and C_2 now contain the generalised kernel. It is easy to see that the correction to C_2 is negligible in the limit of interest, small Q, so we need only consider C_1 explicitly. The calculation of C_1 proceeds as before up to Equation (53) with the only difference that instead of the factor $\left|E - E_n - c_i u^2 \overline{G}^{+}(E)\right|^2$ in the denominator one has the factor $\left[E+\Delta E - E_n - c_i u^2 \overline{G}^{+}(E+\Delta E)\right]\left[E - E_n - c_i u^2 \overline{G}^{-}(E)\right]$ which requires the use of a slight generalisation of the identity used to derive C_1 above. Again using the self-consistent equation for $\overline{G}$ one finds

$$\begin{aligned} C_1(\Delta E) &= \frac{c_i u^2}{2\pi\lambda^2}\sum_n^{\infty} \frac{1}{\left[E+\Delta E - E_n - c_i u^2 \overline{G}^{+}(E+\Delta E)\right]\left[E - E_n - c_i u^2 \overline{G}^{-}(E)\right]} \\ &= \frac{\overline{G}^{+}(E+\Delta E) - \overline{G}^{-}(E)}{\overline{G}^{+}(E+\Delta E) - \overline{G}^{-}(E) - (\Delta E/c_i u^2)} \\ &\approx 1 - i(\Delta E \tau_0/\hbar), \end{aligned} \tag{59}$$

where we defined the generalised scattering time

$$\frac{i\hbar}{\tau_0} = c_i u^2 \left[\overline{G}^{+}(E) - \overline{G}^{-}(E)\right], \tag{60}$$

and we can now ignore the energy difference in $\overline{G}^{+}$ because as usual we are assuming $\Delta E \ll \hbar/\tau_0$. It is easy to check from our above results that $\tau_0 \to \tau$ for $l \ll r_c$, and $\tau_0 \to \hbar/2\nu$ for $r_c \ll l$ and $\varepsilon_F = E_N$.

To generalise the diffusion equation (51) to finite ΔB at arbitrary field we note that each term in Equation (37) for $\overline{G}(\mathrm{r},\mathrm{r}')$ contains an overall phase factor from P_n,

$$\exp\left[\frac{ie}{\hbar c}\int_{\mathrm{r}'}^{\mathrm{r}} \mathbf{A}\cdot\mathrm{dl}\right] = \exp\left[\frac{ie}{\hbar c}\Delta\mathbf{A}(\mathrm{r})\cdot(\mathrm{r}-\mathrm{r}')\right], \tag{61}$$

where we have evaluated the line integral explicitly assuming a uniform field in the z-direction. This phase factor is the same for $\overline{G}^{\pm}(\mathrm{r},\mathrm{r}')$ but of course changes sign when r and r′ are interchanged as they are in the product defining d_0; hence $d_0(\mathrm{r},\mathrm{r}',B+\Delta B)$ involves the difference of this phase (which cancels when $\Delta B = 0$). If we assume that the most rapid variation of d_0 with ΔB comes from this factor then we have

$$\begin{aligned} d_0(\mathrm{r},\mathrm{r}',\Delta B) &= c_i u^2 \overline{G}^{+}(\mathrm{r},\mathrm{r}',B+\Delta B)\overline{G}^{-}(\mathrm{r}',\mathrm{r},B) \\ &= d_0(\mathrm{r},\mathrm{r}')\exp\left[\frac{ie}{\hbar c}\Delta\mathrm{A}(\mathrm{r})\cdot(\mathrm{r}-\mathrm{r}')\right]. \end{aligned} \tag{62}$$

To check that this is indeed the leading dependence on ΔB we recall that at low field the only dependence on B comes through this phase (Section 3), while at high field this phase is of order $\Delta B r_c^2/(\hbar c/e) \approx N\Delta B/B$. Since $N \gg 1$ by assumption, this phase can be of order unity for $\Delta B \ll B$, whereas the denominators in Equation (37) only vary on the scale $\Delta B \approx B$ and may be treated as constant on this scale.

We shall see that the contribution of this phase term to the diffusion equation will become important for $\Delta B r_c^2/(\hbar c/e) \ll 1$, so we can assume that the phase is small in Equation (62), expand it,

$$d_0(\mathrm{r},\mathrm{r}',\Delta B) \approx d_0(\mathrm{r},\mathrm{r}')\left\{1+\frac{ie}{\hbar c}\Delta\mathrm{A}(\mathrm{r})\cdot(\mathrm{r}-\mathrm{r}')+\frac{1}{2}\left[\frac{ie}{\hbar c}\Delta\mathrm{A}(\mathrm{r})\cdot(\mathrm{r}-\mathrm{r}')\right]^2+\ldots\right\}, \tag{63}$$

and multiply this expansion with the Taylor expansion of the $\chi(\mathrm{r})$ in Equation (48). As before we use the symmetry of $d_0(\mathrm{r},\mathrm{r}')$ to eliminate terms odd in $\mathrm{r}-\mathrm{r}'$ and simplify the quadratic terms, and we find unsurprisingly that the effect on the diffusion equation is just to make the minimal substitution, $-i\nabla \to [-i\nabla - (e/\hbar c)\Delta\mathrm{A}(\mathrm{r})]$.

The final generalisation of the equation for the diffuson relevant to UCF concerns the effect of inelastic scattering, which can be taken into account by dressing the Green functions with appropriate interaction corrections to the self-energy and polarisation vertex. Unfortunately there is no general answer to the effect of inelastic processes on the diffuson, since this effect depends on the physical quantity which is being averaged. For the average diffusion constant we noted above that the Ward identity used to obtain $C_1 = 1$ and hence diffusive behaviour was valid in the presence of interactions if the self-energy and vertex are treated consistently. However, when calculating the mesoscopic fluctuations of g and other quantities, the diffuson typically represents the statistical correlation of different *measurements*, and the two Green functions involved cannot be connected by interaction lines. Hence interaction corrections appear in the imaginary part of the self-energy which do not cancel with the undressed vertex, and i/τ_{in} appears as an imaginary energy shift for the diffuson (Lee *et al.* 1987).

Thus the generalisation of Equation (56) valid in an arbitrary magnetic field, with non-zero ΔE, ΔB and τ_{in}, is

$$\left\{l_0^2\left[-i\nabla-\frac{e}{\hbar c}\Delta\mathrm{A}(\mathrm{r})\right]^2+\frac{\tau_0}{\tau_{\mathrm{in}}}-i\frac{\Delta E\tau_0}{\hbar}\right\}\chi_j(\mathrm{r}) = \xi_j^2\chi_j(\mathrm{r}). \tag{64}$$

The generalised diffuson is then given by the spectral representation of Equation (49) with the eigenvalues and eigenfunction obtained from this equation.

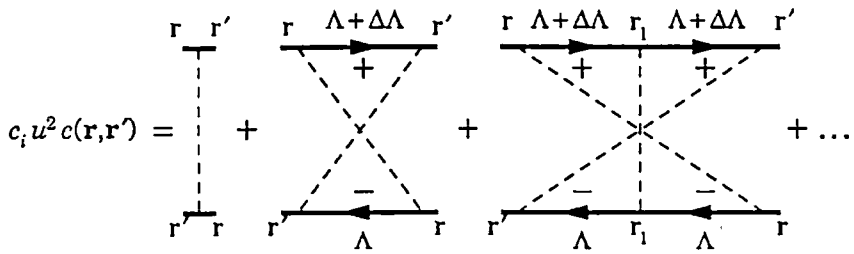

Figure 9: *Class of polarisation diagrams in real space whose sum gives the cooperon contribution.*

6 Cooperon correlator

The other important contribution to the average of the product of two Green functions comes from the *cooperon* diagrams shown in Figure 9. These contribute to $\overline{G^+(\mathbf{r},\mathbf{r}')G^-(\mathbf{r},\mathbf{r}')}$ (note that the second factor is no longer the complex conjugate of the first in general). If we define this contribution as $c_i u^2 c(\mathbf{r},\mathbf{r}')$, it satisfies an integral equation of exactly the same form as the diffuson,

$$c(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}') + \int d\mathbf{r}_1\, c_0(\mathbf{r},\mathbf{r}_1)\, c(\mathbf{r}_1,\mathbf{r}'), \tag{65}$$

where

$$c_0(\mathbf{r},\mathbf{r}') = c_i u^2 \overline{G}^+(\mathbf{r},\mathbf{r}',\Lambda+\Delta\Lambda)\overline{G}^-(\mathbf{r},\mathbf{r}',\Lambda). \tag{66}$$

Equation (65) is again solved in operator form by $c = [1-c_0]^{-1}$ and if we introduce the eigenfunctions $\eta_j(\mathbf{r})$ and eigenvalues $1-\zeta_j^2$ of the integral operator c_0, defined by

$$\int d\mathbf{r}'\, c_0(\mathbf{r},\mathbf{r}')\,\eta_j(\mathbf{r}') = (1-\zeta_j^2)\eta_j(\mathbf{r}), \tag{67}$$

then we can write a spectral representation of c,

$$c(\mathbf{r},\mathbf{r}') = \sum_{j=1}^{\infty} \frac{\eta_j(\mathbf{r})\eta_j^*(\mathbf{r}')}{\zeta_j^2}. \tag{68}$$

Consider $c_0(\mathbf{r},\mathbf{r}',B=0)$. Since we have time-reversal symmetry at $B=0$ the wavefunctions can be chosen real, and it follows from their spectral representation that the Green functions are symmetric in their spatial arguments, $\overline{G}^{\pm}(\mathbf{r},\mathbf{r}') = \overline{G}^{\pm}(\mathbf{r}',\mathbf{r})$. Hence $\overline{G}^-(\mathbf{r},\mathbf{r}')$ *is* the complex conjugate of $\overline{G}^+(\mathbf{r},\mathbf{r}')$, and we have $c_0(\mathbf{r},\mathbf{r}',B=0) = d_0(\mathbf{r},\mathbf{r}',B=0)$; the same conclusion can be reached by noting that $\overline{G}^{\pm}$ is a function only of $|\mathbf{r}-\mathbf{r}'|$ at $B=0$. Thus all the steps used above for d_0 (including those involving a finite energy or frequency difference) go through to yield the differential equation

$$\left(-D\tau\nabla^2 + \frac{\tau}{\tau_{\rm in}} - i\frac{\Delta E\tau}{\hbar}\right)\eta_j(\mathbf{r}) = \zeta_j^2\eta_j(\mathbf{r}), \tag{69}$$

where we have used $\tau_0 = \tau$ since this equation is only valid for $B = 0$ unlike that for the diffuson. This equation will then yield a Q^{-2} pole which does not cancel in the conductivity and gives the weak localisation corrections.

However, as soon as one considers non-zero B the analysis differs importantly from that leading to Equations (56) and (64) for the diffuson. For $B \neq 0$ the phase factors [Equation (61)] in the product $\overline{G}^+(\mathrm{r},\mathrm{r}')\,\overline{G}^-(\mathrm{r},\mathrm{r}')$ do not cancel but rather add to give

$$\begin{aligned} c_0(\mathrm{r},\mathrm{r}',B,\Delta B) &= c_i u^2\,\overline{G}^+(\mathrm{r},\mathrm{r}',B+\Delta B)\,\overline{G}^-(\mathrm{r},\mathrm{r}',B) \\ &= c_0(\mathrm{r},\mathrm{r}',B=0)\exp\left\{\frac{ie}{\hbar c}\left[2\mathrm{A}(\mathrm{r})+\Delta\mathrm{A}(\mathrm{r})\right]\cdot(\mathrm{r}-\mathrm{r}')\right\}. \end{aligned} \tag{70}$$

As before we are interested in points separated by at most the spatial range of c_0 which is approximately equal to l for small B, and we would like to expand this phase factor in the integral equation to yield a differential equation of the diffusion type. However the the typical phase in c_0 even for $\Delta B = 0$ is of order $Bl^2/(\hbar c/2e)$, and is only small compared to unity when $l^2 \ll \lambda^2$. Since $\lambda \approx 1000\,\text{Å}$ at $B = 1\,\text{kG}$, this condition will usually not be satisfied in a 2DEG at fields larger than 1 T and fails for much weaker fields in almost all GaAs heterostructures. It is well known (and we shall see below) that the main effect of the low-temperature divergence in the cooperon channel is removed by even weaker fields such that $Bl_{\text{in}}^2 \approx (\hbar c/2e)$, where $l_{\text{in}}^2 = (D\tau_{\text{in}})$ is the inelastic scattering length. Hence the cooperon contribution is always negligible at the fields where Landau level quantisation becomes important, and there is no high-field ($r_c < l$) analogue of Equations (56) and (64) for the diffuson. However, if we restrict ourselves to fields where $(B+\Delta B)l^2 \ll (\hbar c/2e)$, then the analysis does go through exactly as for the generalised diffuson with $\Delta\mathrm{A} \to 2\mathrm{A} + \Delta\mathrm{A}$ to yield

$$\left\{D\tau\left[-i\nabla - \frac{e}{\hbar c}(2\mathrm{A}+\Delta\mathrm{A})\right]^2 + \frac{\tau}{\tau_{\text{in}}} - i\frac{\Delta E\tau}{\hbar}\right\}\eta_j(\mathrm{r}) = \zeta_j^2\,\eta_j(\mathrm{r}). \tag{71}$$

This is the famous diffusion equation for the cooperon first derived by Altshuler *et al.* (1980), except that usually $\Delta\mathrm{A}$ is taken to be zero and ΔE is replaced by the external frequency of the electric field. Note that when $\Delta\mathrm{A} = 0$ we have a Schrödinger-type equation for a particle of charge $2e$; hence in a doubly-connected geometry with an Aharonov-Bohm flux the solutions are periodic with period $hc/2e$, half the period of the conventional electronic Aharonov-Bohm effect. Thus the weak-localisation correction due to the cooperon oscillates with flux period $hc/2e$ and not hc/e, a dramatic effect first observed by Sharvin and Sharvin (1981).

7 Weak localisation magnetoresistance

Having developed this generalised real-space formulation of impurity-averaged perturbation theory, we now apply it to transport. We review the WL effects before moving on to the more novel UCF effects. Although well known, the WL calculation in real-space will have immediate applications to UCF.

The WL magnetoresistance (neglecting spin effects) is obtained by calculating the impurity-averaged conductance with the cooperon vertex correction. Equation (14)

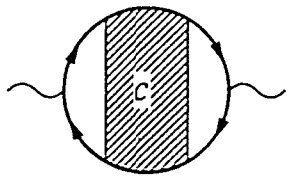

Figure 10: *Conductance diagram with a cooperon polarisation insertion which leads to weak localisation. Wavy lines at the end of the bubble represent the velocity operators.*

for the conductance is the most convenient starting point, and the set of diagrams included is indicated in real-space formulation in Figure 10. Although in principle each of the two Green function lines can correspond to G^+ or G^- due to the factors ΔG in Equation (14), the cooperon divergence only occurs when averaging $\overline{G}^{\pm}\overline{G}^{\mp}$ terms and not $\overline{G}^{\pm}\overline{G}^{\pm}$, so we may drop these from the outset. The two remaining terms from Equation (14) are complex conjugates of each other so we may take only twice the real part of one term. And finally, since the contribution from one term turns out to be real (for $\omega = 0$), we may omit taking the real part for simplicity of notation. With these simplifications the weak localisation magnetoconductance correction $\Delta g_{\rm WL}(B)$ is given by

$$\Delta g_{\rm WL}(B) = \frac{-e^2\hbar}{2\pi L_z^2}\int d\mathrm{r}\, d\mathrm{r}'\, J(\mathrm{r},\mathrm{r}')\, c(\mathrm{r},\mathrm{r}',B), \tag{72}$$

where

$$J(\mathrm{r},\mathrm{r}') = c_i u^2 \int d\mathrm{r}_1\, d\mathrm{r}_2\, v_x(\mathrm{r}_1)\,\overline{G}^-(\mathrm{r}',\mathrm{r}_1)\,\overline{G}^+(\mathrm{r}_1,\mathrm{r})\, v_x(\mathrm{r}_2)\,\overline{G}^-(\mathrm{r}_2,\mathrm{r}), \tag{73}$$

and I have already integrated the double-sided derivatives by parts to replace them by velocity operators acting only to the right.

The factor $J_0(\mathrm{r},\mathrm{r}')$ arises from the external portions of the diagrams and the crucial point here is that it is short-ranged since it only involves factors of $\overline{G}$, and will be small for points r,r' separated by more than l. Thus to leading order in $(k_F l)^{-1}$ we can make the approximation

$$J(\mathrm{r},\mathrm{r}') \approx J_0\delta(\mathrm{r}-\mathrm{r}') \tag{74}$$

with the constant J_0 given by integrating $J(\mathrm{r},\mathrm{r}')$ over both arguments and dividing by the sample area:

$$J_0 = \frac{c_i u^2}{A}\int d\mathrm{r}\, d\mathrm{r}'\, d\mathrm{r}_1\, d\mathrm{r}_2\, v_x(\mathrm{r}_1)\,\overline{G}^-(\mathrm{r}',\mathrm{r}_1)\,\overline{G}^+(\mathrm{r}_1,\mathrm{r})\,\overline{G}^+(\mathrm{r}',\mathrm{r}_2)\, v_x(\mathrm{r}_2)\,\overline{G}^-(\mathrm{r}_2,\mathrm{r}). \tag{75}$$

This expression for J_0 can be put in a more useful general form by noting that the phase factors from the field, $(2ie/\hbar c)\mathbf{A}\cdot(\mathrm{r}-\mathrm{r}')$, will be small for points separated by less than l and can be neglected, so that all factors $\overline{G}$ are symmetric in their spatial arguments. By interchanging spatial arguments in the factors $\overline{G}^+(\mathrm{r}_1,\mathrm{r})$ and $\overline{G}^-(\mathrm{r}',\mathrm{r}_1)$, J_0 can be written as the trace of an operator

$$J_0 = \frac{c_i u^2}{A}\mathrm{Tr}\left\{\overline{G}^+ v_x \overline{G}^-\overline{G}^+ v_x \overline{G}^-\right\} = \frac{c_i u^2}{A}\mathrm{Tr}\left\{\overline{G}^+ v_x \overline{G}^+\overline{G}^- v_x \overline{G}^-\right\}, \tag{76}$$

where in the second step we have used the fact that $\overline{G}^{\pm}$ commute.

This expression can then be simplified using the identity $v_x = (-i/\hbar)[H_0, x]$ for the velocity operator, and the operator equation $\left[E - c_i u^2 \overline{G}^{\pm}(E) - H_0\right] \overline{G} \equiv (Z_0^{\pm})\overline{G} = 1$. Because Z_0 is just a constant shift of H_0 which cancels in the commutator we have

$$v_x = \frac{i}{\hbar}[Z_0^{\pm}, x]. \tag{77}$$

If we substitute this identity with the choice Z_0^+ for the first occurence of v_x in Equation (76), and Z_0^- for the second occurence, we easily obtain

$$\begin{aligned} J_0 &= \frac{-c_i u^2}{A\hbar^2}\mathrm{Tr}\left\{\left(x\overline{G}^+ - \overline{G}^+ x\right)\left(x\overline{G}^- - \overline{G}^- x\right)\right\} \\ &= \frac{c_i u^2}{A\hbar^2}\int d\mathrm{r}\, d\mathrm{r}'\,(x-x')^2\,\overline{G}^+(\mathrm{r},\mathrm{r}')\,\overline{G}^-(\mathrm{r}',\mathrm{r}) \\ &= \frac{c_i u^2}{dA\hbar^2}\int d\mathrm{r}\, d\mathrm{r}'\,(\mathrm{r}-\mathrm{r}')^2\,\overline{G}^+(\mathrm{r},\mathrm{r}')\,\overline{G}^-(\mathrm{r}',\mathrm{r}) \\ &= \frac{2l_0^2}{\hbar^2}, \end{aligned} \tag{78}$$

where we have used the definition of l_0^2, Equation (55). Note that the derivation is completely general once expressed as a trace of this form, and applies to an arbitrary field. For the cooperon contribution to be important, however, we need to consider the weak-field limit as discussed above, in which case $2l_0^2 = l^2$.

Having evaluated J_0 we immediately find from Equations (68) and (72)

$$\begin{aligned} \Delta g_{\mathrm{WL}}(B) &= \frac{-e^2 l^2}{2\pi\hbar L_x^2}\int d\mathrm{r}\, c(\mathrm{r},\mathrm{r},B) \\ &= \frac{-e^2 l^2}{2\pi\hbar L_x^2}\sum_j \frac{1}{\zeta_j^2}. \end{aligned} \tag{79}$$

If one sets $B = \Delta B = 0$ in Equation (71) the solutions are just those discussed for the diffuson after Equation (58) with $\zeta_j^2 = D\tau(Q^2 + l_{\mathrm{in}}^{-2})$, and changing the sum to an integral leads to the familiar logarithmically divergent WL correction,

$$\Delta g_{\mathrm{WL}}(B=0) = \frac{-e^2}{2\pi^2\hbar}\int_L^l \frac{Q\,dQ}{Q^2 + l_{\mathrm{in}}^{-2}}. \tag{80}$$

For $B \neq 0$, $\Delta B = 0$ in 2D the eigenvalues of Equation (71) correspond to Landau levels of a particle with charge $2e$ and mass $\hbar^2/2D\tau$. Expressing the sum over j in terms of these eigenvalues, including the appropriate degeneracy factor per LL, yields the well-known result (Altshuler *et al.* 1980)

$$\Delta g_{\mathrm{WL}}(B) = \frac{-e^2}{4\pi^2\hbar}\sum_{n=0}\frac{1}{n + \frac{1}{2} + \gamma_{\mathrm{in}}}, \tag{81}$$

where $\gamma_{\mathrm{in}} = (\hbar c/4eBl_{\mathrm{in}}^2)$. This expression with the appropriate upper cut-off [determined by the range of validity of the expansion of the phase factor in Equation (70)]

tends correctly to the limit at $B = 0$ and tends to zero for $Bl_{in}^2 \gg \hbar c/e$. This shows that a magnetic field large enough to alter substantially the phase of classical paths within a phase-coherent region is sufficient to destroy the WL effect, which relies on the exact cancelation of phases due to time-reversal symmetry. We need not engage in further analysis of the well-studied WL effects here; the effort paid to them is primarily to emphasise their close affinity to the UCF effects treated below.

8 Universal conductance fluctuations

Having laid the groundwork carefully above, we are now able to derive the microscopic theory of conductance fluctuations in a few lines.

8.1 Definition of correlation function

As discussed in Section (3.4), the elastic transmission coefficient through a multiple-scattering medium is expected to have sizeable fluctuations due to random interference effects, and these interference effects can be modulated by changing external parameters which couple to the phase of the scattered wave. In the case of a mesoscopic solid the two most natural external parameters to consider are the magnetic field and Fermi energy. One needs to determine the scale over which the interference effects vary as a function of these parameters, and the typical amplitude of the fluctuating interference effects. In the conventional low-field theory of UCF Lee and Stone (1985) introduced the *ergodic hypothesis* stating that the amplitude of the fluctuations as a function of external parameters B, ε_F was equal to the variance of g (averaged over impurity configurations) at fixed external parameters. This hypothesis was found to hold quite well numerically for the fluctuations as a function of magnetic field in the low-field limit (Lee and Stone 1985). However in the high-field limit this hypothesis most certainly fails over a sufficiently large interval of field and will have to be modified in a manner to be discussed below. Nonetheless we shall see that typically the scale of the fluctuations is significantly less than the scale over which the ergodic hypothesis breaks down, allowing its use over this interval.

If the ergodic hypothesis holds, the typical amplitude and scale of the sample-specific fluctuations may be obtained from the conductance correlation function

$$F(\Delta E, \Delta B) = \langle \delta g(\varepsilon_F + \Delta E, B + \Delta B)\, \delta g(\varepsilon_F, B) \rangle \,, \tag{82}$$

where $\delta g(\varepsilon_F, B) = g(\varepsilon_F, B) - \overline{g}(\varepsilon_F, B)$. The statistical variance of g is given by $F(0,0) = \langle \delta g^2 \rangle \equiv \mathrm{var}(g)$, which by the ergodic hypothesis is the mean-squared variation as a function of ε_F, B. The decay width of $F(\Delta E, 0)$ gives the correlation range, E_c, of the fluctuations with Fermi energy; similarly the width of $F(0, \Delta B)$ gives the correlation range, B_c, of the fluctuations with magnetic field.

8.2 Evaluation of diagrams

Diagrammatically the correlation function is obtained by considering two conductance bubbles with different values of the external parameters $\Lambda = B$, ε_F (corresponding

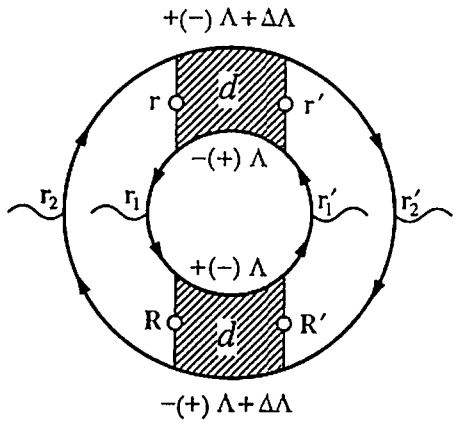

Figure 11: *Diagram representing one of the two diffuson contributions to the conductance correlation function.*

to two different conductance measurements) connected by impurity-averaging lines as shown in Figure 11. All diagrams in which the two bubbles are not connected are contributions to the square of the average conductance and will cancel by definition of var(g). Not surprisingly the leading contribution to F comes from diagrams where the two bubbles are connected by diffuson ladders or cooperon ladders as shown in Figure 11. When B is greater than a few hundred Gauss the cooperon contribution is negligible for the reasons discussed above, whereas for $B = 0$ it gives a contribution identical to that of the diffuson. Its detailed treatment involves a trivial extension of the analysis of the diffuson contribution, and we will not discuss it in detail here. For the diffuson contribution there are only two types of diagrams, one of which is shown in Figure 11. The two types give very similar contributions to F and so we only consider the simpler kind shown in Figure 11, which we denote by F_1.

This diagram consists of two diffuson ladders and two external vertices on each side of the diagram involving two factors $\overline{G}^+$, $\overline{G}^-$ and two velocity vertices. As before, each Green function line can be $\overline{G}^+$ or $\overline{G}^-$ but they must be chosen so that the diffuson connects $\overline{G}^+\overline{G}^-$. Diffusons with different signs of the term $i\Delta E\tau_0$ result from different pairing, but this is irrelevant in the external vertices J_0 which only involve $\overline{G}$ and are short-ranged, so that the small energy or field difference may be neglected. Hence all the external vertices are equal and can be approximated as δ-functions connecting r, R and r', R'. This observation leads to the exhilarating realisation that each of these vertices is equivalent to the vertex $J(\mathrm{r}, \mathrm{r}') = J_0\,\delta(\mathrm{r} - \mathrm{r}')$ considered above, where because of the definition of the diffuson one need not appeal to time-reversal symmetry to express it in the trace form of Equation (76). Then by inspection

$$F_1 = \left(\frac{e^2\hbar}{4\pi L_z^2}\right)^2 J_0^2 \int d\mathrm{r}\, d\mathrm{r}' \left\{2\,|d(\mathrm{r}, \mathrm{r}', \Lambda + \delta\Lambda)|^2 + 2\,\mathrm{Re}\left[d(\mathrm{r}, \mathrm{r}', \Lambda + \delta\Lambda)\, d(\mathrm{r}', \mathrm{r}, \Lambda + \delta\Lambda)\right]\right\} \tag{83}$$

Recalling that $J_0 = 2l_0^2/\hbar^2$ for arbitrary field, and using the spectral representation of

$d(\mathrm{r}', \mathrm{r}, \Lambda + \delta\Lambda)$ [Equation (49)] to perform the integrations over r, r', yields

$$F_1 = 2\left(\frac{e^2}{h}\right)^2 \frac{l_0^4}{L_x^4} \sum_j \left[\frac{1}{\xi_j^4} + \mathrm{Re}\left(\frac{1}{\xi_j^4}\right)\right], \tag{84}$$

where we have used the orthonormality of the $\chi_j(\mathrm{r})$. The contribution of the other type of diagram to F is exactly equal to the second term in the square brackets, hence

$$F(\Delta E, \Delta B) = 2\left(\frac{e^2}{h}\right)^2 \frac{l_0^4}{L_x^4} \sum_j \left[\frac{1}{\xi_j^4} + 2\,\mathrm{Re}\left(\frac{1}{\xi_j^4}\right)\right], \tag{85}$$

where the dependence on ΔE, ΔB comes through the dependence of the eigenvalues ξ_j^2 of the diffusion equation (64) on these quantities. Equation (85) gives the full correlation function for $T = 0$. It is valid at arbitrary magnetic field (except for fields near $B = 0$ where it is straightforward to include the cooperon contribution) as long as the SCBA is reasonable. This expression is identical to the conventional theory which neglects Landau quantisation effects. Thus we have obtained the striking new result (Xiong and Stone 1991) that the conventional theory goes over unchanged to the high-field limit except for the dependence of the constants l_0^2 and τ_0 on the magnetic field, which we now show does not appear in $\mathrm{var}(g)$ just as at low fields.

8.3 Variance of conductance

The variance of g is obtained from Equation (85) simply by setting $\Delta E = \Delta B = 0$ in the evaluation of ξ_j^2, which we have already found above gives $\xi_j^2 = l_0^2(k_m^2 + q_n^2) = l_0^2(m\pi/L_x^2 + n\pi/L_y^2)$. Inserting this into Equation (85) we immediately see that the factor l_0^4/L_x^4 cancels to give

$$\mathrm{var}(g) = \frac{6}{\pi^4}\left(\frac{e^2}{h}\right)^2 \sum_{m=1,n=0}^{\infty} \frac{1}{\left[m^2 + n^2(L_x/L_y)^2 + (L_x/\pi l_{\mathrm{in}})^2\right]^2}, \tag{86}$$

where the generalised inelastic diffusion length is defined as

$$l_{\mathrm{in}}^2 = \frac{l_0^2}{\tau_0}\tau_{\mathrm{in}} \equiv D_0(B)\tau_{\mathrm{in}}. \tag{87}$$

In the limit $l_{\mathrm{in}} \to \infty$ ($T \to 0$) this equation implies that $\mathrm{var}(g) \approx (e^2/h)^2$, independent of the size of the sample, degree of disorder and magnetic field (insofar as the ergodic hypothesis is satisfied), hence the term 'universal conductance fluctuations'. At $B = 0$ a factor of two is needed to include the cooperon contribution (which again is equal to the diffuson by time-reversal symmetry), and a factor of 4 is needed to account for spin degeneracy in the absence of spin-orbit interactions or Zeeman splitting larger than $k_B T$. Spin-orbit interactions reduce both diffuson and cooperon contributions by a factor of 4 due to suppression of the triplet channels (Altshuler and Shklovskii 1986); we are able to show that this suppression also occurs at high fields by an extension of the analysis presented above. The specific value of $\mathrm{var}(g)$ at $T = 0$ can easily be evaluated from Equation (86) but in typical experiments, which are done in multi-probe geometries, this value depends on the probe configuration and it is not possible to make

precise comparisons with Equation (86). When $l_{\rm in} \ll L$ the variance of g depends on their ratio (we assume $L_x = L_y$ here), and since many terms contribute to the sum in Equation (86) we may accurately convert it to an integral giving in 2D

$$\mathrm{var}(g) = \frac{24}{\pi}\left(\frac{l_{\rm in}}{L}\right)^2\left(\frac{e^2}{h}\right)^2. \tag{88}$$

Note that inelastic scattering does reduce the fluctuations as a power of the system size, making the system self-averaging as the size goes to ∞.

8.4 Correlation ranges E_c and B_c

The functions $F(\Delta E, 0)$ and $F(0, \Delta B)$ can be obtained quantitatively from Equations (64) and (85) and plotted to find their precise half-widths in terms of the system parameters. Detailed evaluations in various limits have been given elsewhere (Lee and Stone 1985, Lee *et al.* 1987). However the parametric scale of E_c can be determined for the energy correlation function quite easily because ΔE appears as only a constant shift of the eigenvalue ξ_j^2. With no inelastic scattering the sum in Equation (85) is very rapidly convergent and E_c can be determined simply by looking at its largest term, which is the smallest eigenvalue $|\xi_0^2|^2 = \pi^4 l_0^4/L_x^4 + (\Delta E \tau_0/\hbar)^2$. The sum will decay substantially when the second term is comparable to the first, *i.e.* $\Delta E \approx \hbar l_0^2/\tau_0 L_x^2$. Thus

$$E_c(B) \approx \frac{\hbar D_0(B)}{L_x^2}, \tag{89}$$

which is the generalisation to arbitrary field of the well-known low-field result which was discussed in Section 3.4. By inserting the high and low-field limits of l_0 and τ_0 one finds

$$E_c = \begin{cases} \hbar l^2/(2\tau L_x^2), & l \ll r_c \\ (r_c^2 \nu \sin\theta(E))/L_x^2, & l \gg r_c. \end{cases} \tag{90}$$

It is easily shown from Equation (85) that $l_{\rm in}$ simply replaces L_x in these relations when $l_{\rm in} \ll L_x$.

The determination of the magnetic field correlation length from Equation (85) is slightly more involved as ΔB does not enter only as an eigenvalue shift, and one must solve the differential equation (64) for $\xi_j^2(\Delta B)$. As noted above in the discussion of WL effects in 2D, the solutions will give the analogue of Landau levels when $l_{\rm in} \ll L$, in this case with the field replaced by ΔB and mass $m = \hbar^2/2l_0^2$, but with the conventional charge e. The correlation length can again be obtained by looking at the lowest eigenvalue, $\xi_0^2(\Delta B) = (\Delta B l_0^2)/(\hbar c/e) + \tau_0/\tau_{\rm in}$, and finding the scale of ΔB at which the two terms become comparable. Thus

$$B_c(B) \approx \frac{(\hbar c/e)}{D_0 \tau_{\rm in}}, \tag{91}$$

which again generalises the low field result discussed in Section 3.4 to arbitrary field. In generalising the theory we obtain a field-dependent B_c as recently observed in low-mobility GaAs by Geim *et al.* (1991); see Figure 3 and the chapter by Main. If $\tau_{\rm in}$ is

only weakly field-dependent on a scale B_c, then our theory predicts that when $r_c \ll l$

$$B_c \propto \frac{1}{D_0(B)} \propto \frac{1}{r_c^2 \nu} \propto B^{3/2}, \tag{92}$$

where it must be noted that the WN limit for the random potential has been assumed, and this may have a limited region of applicability to GaAs.

9 Summary and conclusions

I have discussed in detail the two types of experimentally observed quantum interference effects in disordered conductors, weak localisation and universal conductance fluctuations. A third type of interference effect for non-interacting electrons is associated with thermodynamic properties of mesoscopic systems such as persistent currents and can be treated by the same techniques, but has yet to be unambiguously observed. I have shown that both weak localisation and universal conductance fluctuations can be treated in a real-space formulation based on effective diffusion equations for the diffuson and cooperon contributions, and for the case of the diffuson (which does not rely on time-reversal symmetry) the approach can be generalised within the self-consistent Born approximation to the limit where the Landau level spacing is much greater than the disorder broadening. Since this generalisation is new a few comments about its expected range of validity are in order.

It is well known from the study of the quantum Hall effect that localisation effects become important rapidly as a function of magnetic field even in high (zero-field) mobility two-dimensional electron gases. From the perturbative point of view taken above this may be seen as due to the fact that no matter how large is $k_F l$, it is the parameter $1/N$ which measures the localisation effects in high field, and this parameter approaches unity in the quantum Hall regime. Even when $N \approx 10$ the beginning of Hall plateaus are observed and the SCBA breaks down in these intervals of field and the theory presented here does not apply. Conversely, at the centres of the LLs one expects extended states to exist even in the infinite system; however they are believed to be described by a strong-coupling fixed point whose properties are different in general from those of the perturbative unstable fixed point described by the SCBA. Nonetheless, by the universality hypothesis one expects var(g) to be independent of size and disorder at this fixed point as we find in the SCBA, so at most the precise constant and correlation scales could be different. Moreover, the mesoscopic systems of interest are likely to be far from the infinite volume behaviour (at least in the conducting region) and may reasonably be described by perturbation theory. Thus, in summary, we expect the theory presented here to give a reasonable description of mesoscopic transport fluctuations in a Landau quantised 2DEG even in the quantised Hall regime in the transition region between the plateaus, with the major *caveat* that the disordered potential is short-ranged compared to the cyclotron radius.

The theory presented here highlights further the universality of the transport fluctuation phenomena. The details of the bare quantum states, as long as they are extended, are unimportant. This is because at long wavelengths the only coherent scattering (represented by the diffuson) will satisfy the same diffusion equation; only the diffusion constant reflects the nature of the underlying states. Since the diffusion constant cancels

in the variance of the conductance (and in other physical properties such as the persistent current (Altshuler *et al.* 1991) one finds remarkably general behaviour. Recently a theoretical study of ballistic conductors by Jalabert *et al.* (1990) has also found fluctuation effects identical to those of disordered conductors with correlation ranges given by expressions similar to those obtained above. Here the multiple scattering comes from the geometry of the device which generates classically chaotic scattering for the cases considered. I have conjectured, but not proven, that a diffuson-type approximation to the modulus of the semi-classical propagator can be used to describe this limit also. Thus it seems that complex quantum scattering of almost any type leads to a single type of coherent fluctuation phenomena in normal electron transport.

Acknowledgments

I would particularly like to acknowledge the major contributions of my student S Xiong to the generalisation of the UCF theory to high magnetic field and for helpful comments on the manuscript. This work was supported by NSF grant DMR-8658135, by the AT&T Foundation, and by the Alfred P Sloan Foundation.

References

Abrahams E, Anderson P W, Licciardello D C, and Ramakrishnan T V, 1979. *Phys. Rev. Lett.*, 42, 673.

Abrikosov A A, Gorkov L P, Dyzaloshinski I E, 1965. *Methods of Quantum Field Theory in Statistical Physics*, New York, Pergamon.

Aharonov Y, and Casher A, 1984. *Phys. Rev. Lett.*, 53, 19.

Al'tshuler B L, Khmelnitskii D E, Larkin A I, and Lee P A, 1980. *Phys. Rev. B*, 22, 5142.

Al'tshuler, B L, Aronov A G, and Spivak B Z, 1981. *JETP Lett.*, 33, 94.

Al'tshuler, B L, 1985. *JETP Lett.*, 41, 648

Al'tshuler B L, and Shklovskii B I, 1986. *Sov. Phys. JETP*, 64, 127.

Altshuler B L, Gefen Y, and Imry Y, 1991. *Phys. Rev. Lett.*, 66, 88.

Anderson, P W, 1958. *Phys. Rev.*, 102, 1008.

Ando T, Matsumoto Y, and Umera Y, 1974. *J. Phys. Soc. Jpn.*, 36, 959.

Ando T, Matsumoto Y, and Uemura Y, 1975. *J Phys. Soc. Jpn.*, 39, 279.

Aronov A G, and Sharvin Yu V, 1987. *Rev. Mod. Phys.*, 59, 755.

Baranger H U, and Stone A D, 1989. *Phys. Rev. B*, 40, 8169.

Beenakker C W J, and van Houten H, 1989. *Phys. Rev. Lett.*, 63, 1857.

Benoit A D, Washburn S, Umbach C P, Laibowitz R B, and Webb R A, 1986. *Phys. Rev. Lett.*, 57, 1765.

Bergmann G, 1984. *Phys. Rep.*, 107, 11.

Birge N O, Golding B, and Haemmerle W H, 1989. *Phys. Rev. Lett.*, 62, 195.

Büttiker M, 1986. *Phys. Rev. Lett.*, 57, 1761.

Büttiker, M, 1988. *Phys. Rev. B*, 38, 9375.

Carra P, Chalker J T, and Benedict K A, 1989. *Ann. Phys. (NY)*, 194, 1.

Chakravarty S, and Schmid A, 1986. *Phys. Rep.*, 140, 193.

Economou E N, and Soukoulis C M, 1981. *Phys. Rev. Lett.*, 46, 618.

Edwards, S F, 1958. *Phil. Mag.*, 3, 1020.

Feng S, Lee P A, and Stone A D, 1986. *Phys. Rev. Lett.*, 56, 1960.

Fisher D S, and Lee P A, 1981. *Phys. Rev. B*, 23, 6851.
Geim A K, Main P C, Beton P H, Streda P, Eaves L, Wilkinson C D W, and Beaumont S P, 1991. *Phys. Rev. Lett.*, 67, 3014.
Gefen Y, Imry Y, and Azbel M Ya, 1984. *Phys. Rev. Lett.*, 52, 129.
Glazman L I, Lesovick G B, Kmelnitskii D E, and Shekhter R I, 1988. *JETP Lett.*, 48, 218.
Greenwood, D A, 1958. *Proc. Phys. Soc. (London)*, 71, 585.
Gutzwiller M C, 1990. *Chaos in Classical and Quantum Mechanics*, Berlin, Springer-Verlag, pp. 184–190, 283–287.
Imry Y, 1986. In *Directions in Condensed Matter Physics*, Eds. Grinstein G, and Mazenko G, Singapore, World Scientific, p. 101.
Jain J K, and Kivelson S A, 1988a. *Phys. Rev. Lett.*, 60, 1542.
Jain J K, and Kivelson S A, 1988b. *Phys. Rev. B*, 37, 4276.
Jalabert, R A, Baranger H U, and Stone A D, 1990. *Phys. Rev. Lett.*, 65, 2442.
Kubo, R, 1956, *Canad. J Phys.*, 34, 1274.
Kubo R, Miyake S I, and Hashitsume N, 1900. In *Solid State Physics*, Eds. Seitz F, and Turnbull D, New York, Academic Press, 17, 288.
Landauer R, 1957. *IBM J. Res. Dev.*, 1, 233.
Landauer R, 1970. *Phil. Mag.*, 21, 863.
Langer J S, and Neal T, 1966. *Phys. Rev. Lett.*, 16, 984.
Lee P A, and Ramakrishnan T V, 1985a. *Rev. Mod. Phys.*, 57, 287.
Lee P A, and Stone A D, 1985b. *Phys. Rev. Lett.*, 55, 1622.
Lee P A, Stone A D, and Fukuyama H, 1987. *Phys. Rev. B*, 35, 1039.
Levy L P, Dolan G, Dunsmuir J, and Bouchiat H, 1990. *Phys. Rev. Lett.*, 64, 2094.
Luo J K, Ohno H, Matsuzaki K, Umeda T, Nakahara J, and Hasegawa H, 1989. *Phys. Rev. B*, 40, 3461.
Mathur H, and Stone A D, 1991. In preparation.
Prange R E, and Girvin S M (Eds.), 1987. *The Quantum Hall Effect*, New York, Springer-Verlag.
Sanquer M, Mailly D, Pichard J-L, and Pari P, 1989. *Europhys. Lett.*, 8, 471.
Skocpol W J, Mankiewich P M, Howard R E, Jackel L D, Tennant D M, and Stone A D, 1987. *Phys. Rev. Lett.*, 58, 2347.
Sharvin D Yu, and Sharvin Yu V, 1981. *JETP Lett.*, 34, 272.
Stone A D, 1985. *Phys. Rev. Lett.*, 54, 2692.
Stone A D, and Imry Y, 1986. *Phys. Rev. Lett.*, 56, 189.
Stone A D, and Szafer A, 1988. *IBM J. Res. Dev.*, 32, 384.
Stone A D, 1989. *Phys. Rev. B*, 39, 10736.
Stone A D, Szafer A, McEuen P L, and Jain J K, 1990. *Ann. N Y Acad. Sci.*, 581, 21.
Streda P, Kucera J, and MacDonald A H, 1987. *Phys. Rev. Lett.*, 59, 1973.
Szafer A, and Stone A D, 1989. *Phys. Rev. Lett.*, 62, 300.
Timp G, Behringer R, Sampere S, Cunningham J E, and Howard R E, 1989. In *Nanostructure Physics and Fabrication*, Eds. Reed M A, and Kirk W P, Boston, Academic Press, p. 000.
van Houten H, Beenakker C W J, and van Wees B J, 1990. In *Semiconductors and Semiconductors*, Ed. Reed M A, New York, Academic Press.
van Wees B J, Van Houten N; Beenakker C W J, Williamson J G W, Kouwenhoven L, Van Der Marel D, and Foxon C T, 1988. *Phys. Rev. Lett.*, 60, 848.
Washburn S, and Webb R A, 1986. *Adv. Phys.*, 35, 375.
Xiong S, and Stone A D, 1991. In preparation.