- Lecture I
  - motivate/discuss celestial objects where rotation influences buoyancy driven flows
  - discuss energetics, waves, balance
- Lecture II
  - stability theory for rotating convection - what can we glean from it
  - motivate non-hydrostatic quasi-geostrophy
  - derive and investigate semi-analytic solutions (skeleton for strongly NL flows)

Lecture III

- investigate fully NL rotating convection from QG perspective
- assess how the theory holds up c.f. experiments and DNS
- comments on broader view and future outlook

NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3}\vartheta.$

Vert. Vorticity

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

Vert. Velocity

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\,\theta$$

Temp. Fluct.

$$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

Mean. Temp.

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Two control parameters

$$\widetilde{Ra} = Ra E^{4/3}, \; Pr$$

Reduced BCs.

$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

- Isotropic velocity magnitudes (non-hydrostatic dynamics)
- No vertical inertial advection (hallmark of QG theory)
- Horizontal dissipation only (no vertical dissipation - filtered mtm. Ekman bl's)
- Vertical diffusion for mean temperature only (TBL can develop)
- Slow inertial waves only. No fast (unbalanced) waves $\omega_{wave}^2 = \frac{k_Z^2}{k_\perp^2}$

NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3}\vartheta.$

Vert. Vorticity

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

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Temp. Fluct.

$$\partial_t \theta + J[\psi, \theta] + w\partial_Z \overline{T} = \frac{1}{Pr}\nabla_\perp^2 \theta$$

Mean. Temp.

$$\partial_Z(\overline{w\theta}) = \frac{1}{Pr}\partial_{ZZ}\overline{T}$$

Two control parameters

$$\widetilde{Ra} = RaE^{4/3}, \; Pr$$

Reduced BCs.

$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

Conserved quantities: Energy (volume averaged) and Potential Vorticity (pointwise)

$$E = \frac{1}{2}\left\langle |\nabla^\perp \psi|^2 + w^2 \right\rangle_V, \qquad Q_{PV} = \left(\zeta + \frac{\widetilde{Ra}}{Pr}\partial_Z\left(\frac{\theta}{\partial_Z \overline{T}}\right)\right) + J[w, \theta]$$

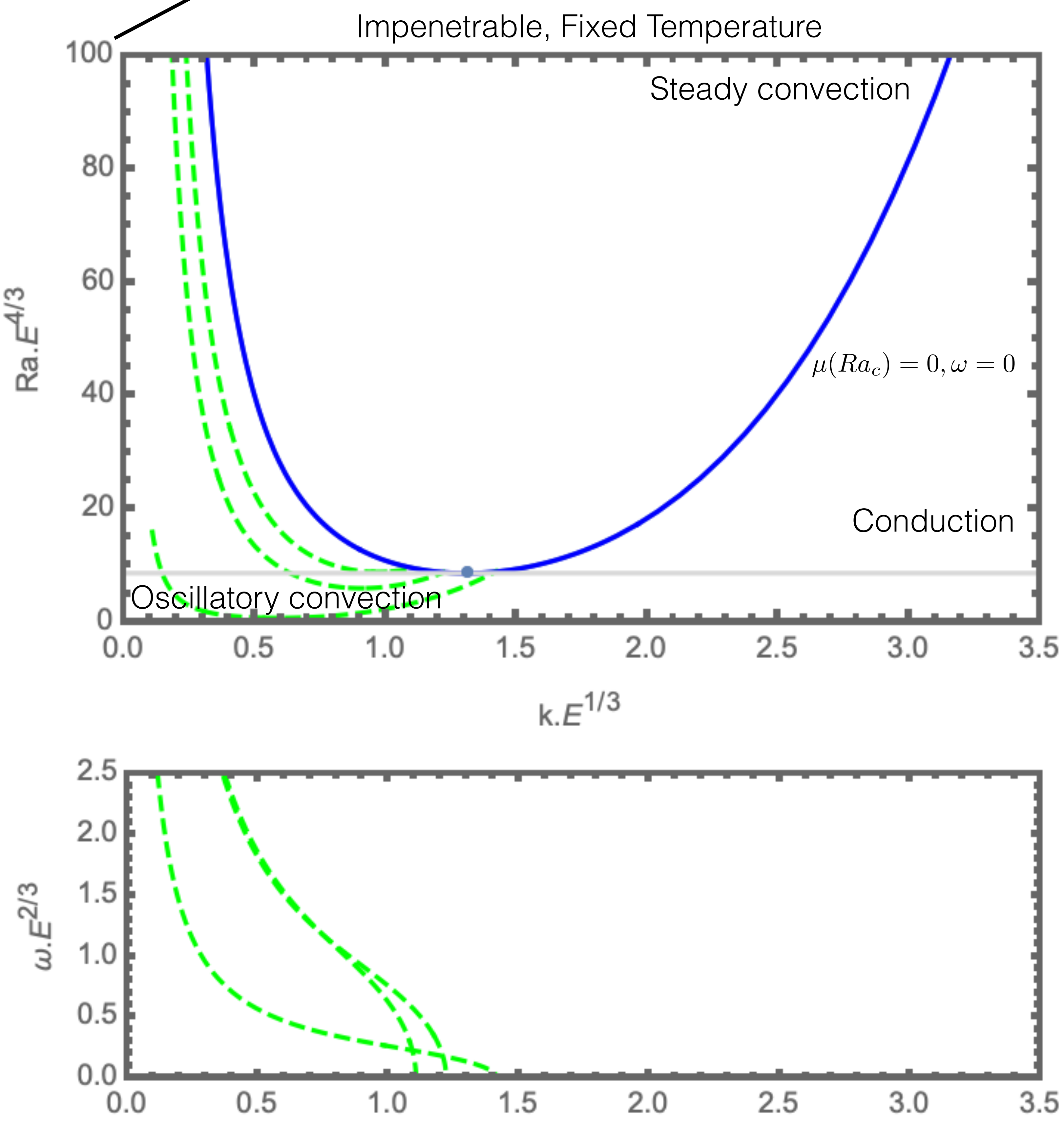
Enstrophy not conserved. Forward cascade?

Linear Stability Theory: Captured

Asymptotic validity of NH-QGE

$$\widetilde{Ra} < \mathcal{O}(E^{-1/3})$$

$@\ E = 10^{-15} \implies Ra = 10^{22}$



Single-Mode Solutions

$$p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + \epsilon \vartheta.$$

$$\partial_t \zeta + \cancel{J[\psi, \zeta]} - \partial_Z w = \nabla_\perp^2 \zeta$$

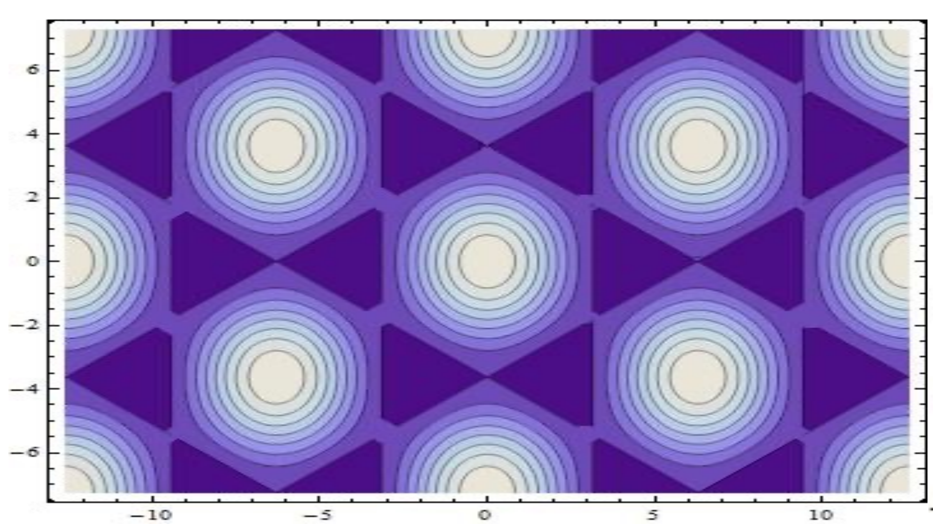
$$\partial_t w + \cancel{J[\psi, w]} + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr} \theta$$

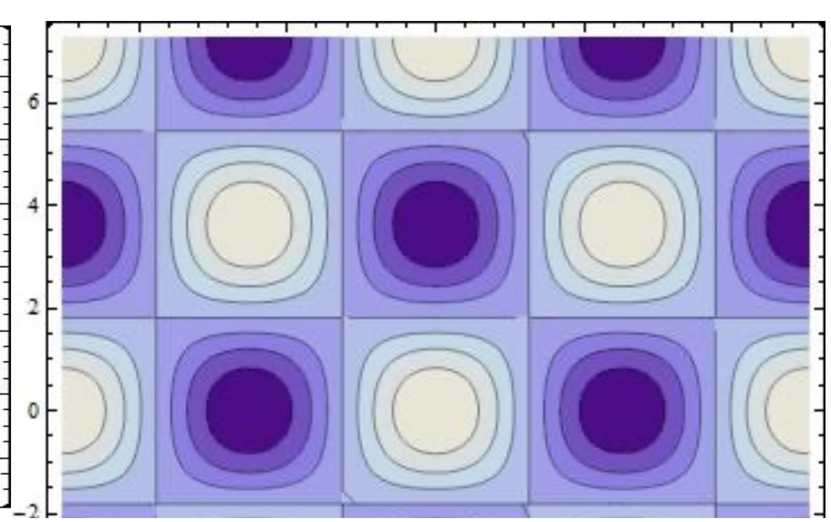
$$\partial_t \theta + \cancel{J[\psi, \theta]} + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Pose: $\quad w = \hat{W}(Z,t) h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0 \qquad J[h,h] = 0$

$h(x,y)$ top view





Single-Mode Solutions

$$p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + \epsilon\vartheta.$$

$$\cancel{k_\perp^2 \partial_t \hat{\Psi}} + \partial_Z \hat{W} = -k_\perp^4 \hat{\Psi}$$

$$\cancel{\partial_t \hat{W}} + \partial_Z \hat{\Psi} = -k_\perp^2 \hat{W} + \frac{\widetilde{Ra}}{Pr} \hat{\Theta}$$

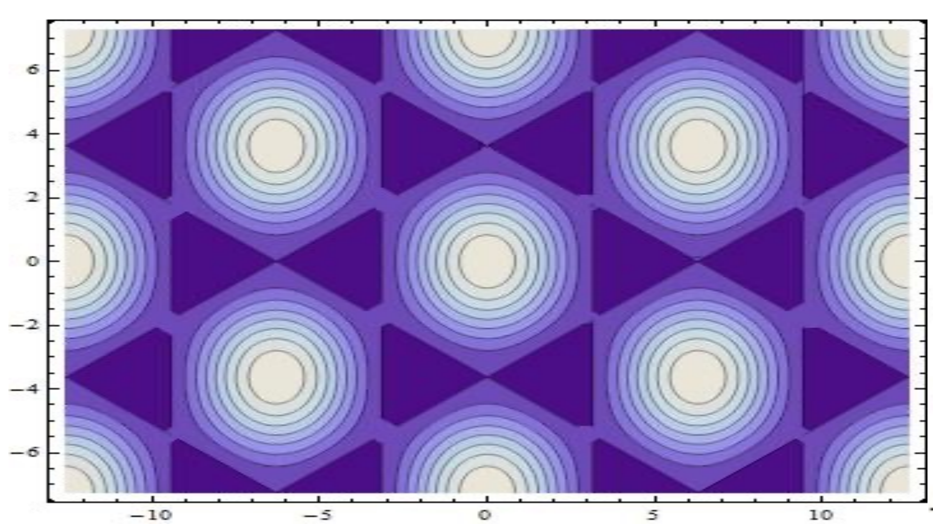
$$\cancel{\partial_t \hat{\Theta}} + w\partial_Z \overline{T} = -\frac{1}{Pr} k_\perp^2 \hat{\Theta}$$

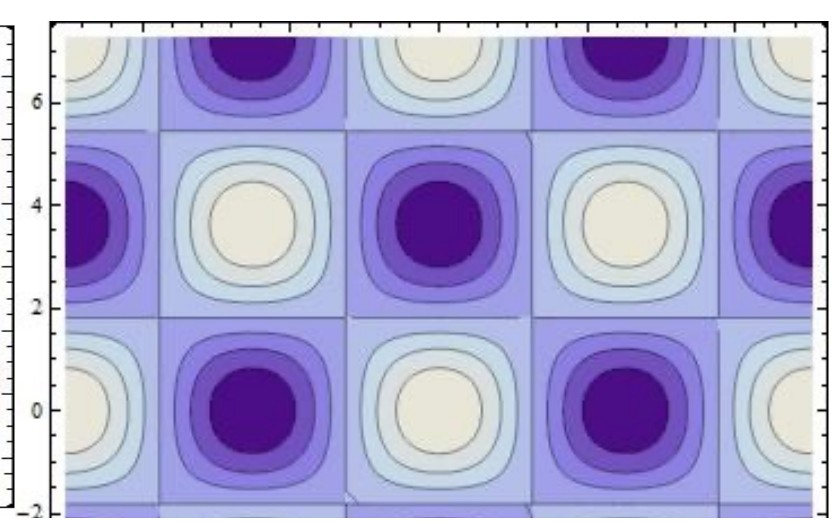
$$\partial_Z (\hat{W}\hat{\Theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Steady convection

Pose: $w = \hat{W}(Z,t)h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0 \qquad J[h,h] = 0$

$h(x,y)$ top view





Single-Mode Solutions

Pose:

$$w = \hat{W}(Z)h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$$

Find:

$$\partial_{ZZ}\hat{W} - k_\perp^2 \left[\widetilde{Ra}\partial_Z \overline{T} + k_\perp^4\right]\hat{W} = 0,$$

$$\partial_Z \overline{T} = -\left(\frac{k_\perp^2}{k_\perp^2 + Pr^2 \hat{W}^2}\right) Nu \qquad Nu = \left[\int_0^1 \left(\frac{k_\perp^2}{k_\perp^2 + Pr^2 \hat{W}^2}\right) dZ\right]^{-1}$$

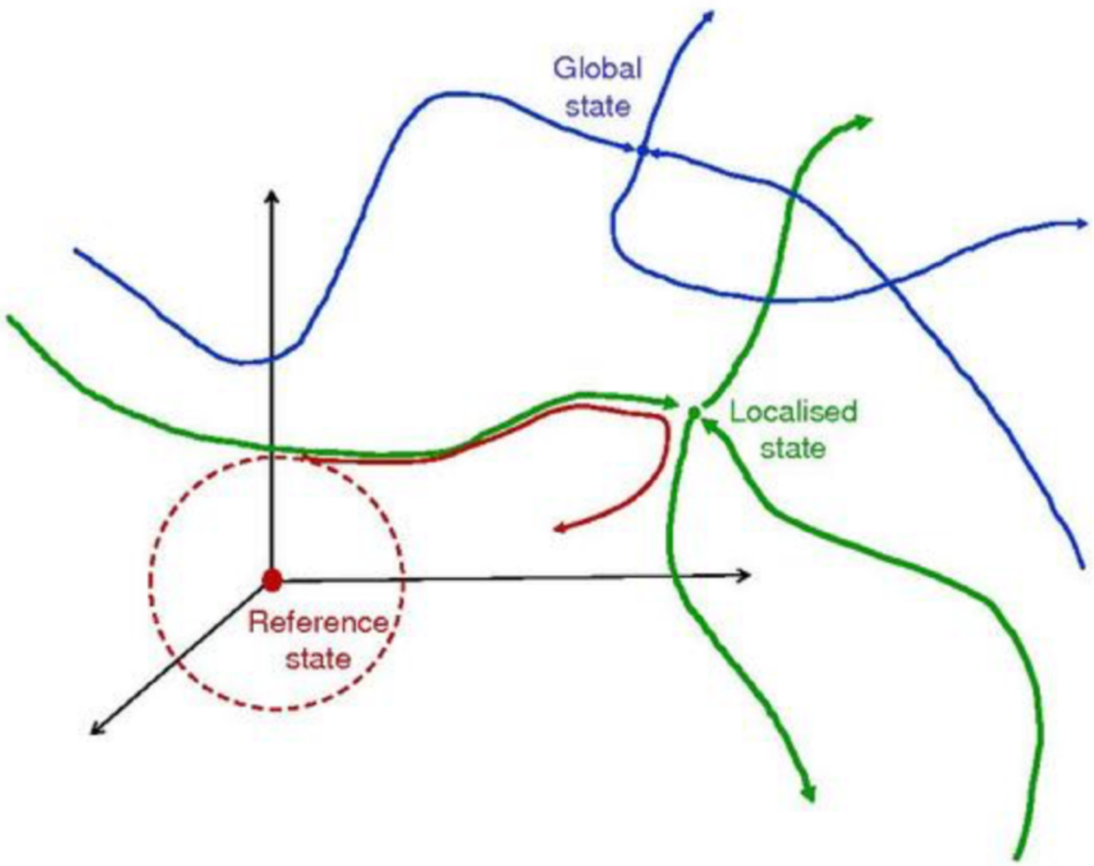
Bassom & Zhang GAFD 1994 Julien & Knobloch PoF 1996, JFM 1998

Single-Mode Solutions

Pose: $$w = \hat{W}(Z)h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$$

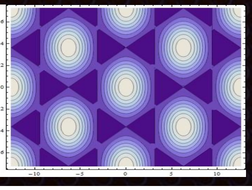
Find: $$\partial_{ZZ}\hat{W} - k_\perp^2\left[\widetilde{Ra}\partial_Z\overline{T} + k_\perp^4\right]\hat{W} = 0,$$

$$\partial_Z\overline{T} = -\left(\frac{k_\perp^2}{k_\perp^2 + Pr^2\hat{W}^2}\right)Nu \qquad Nu = \left[\int_0^1\left(\frac{k_\perp^2}{k_\perp^2 + Pr^2\hat{W}^2}\right)dZ\right]^{-1}$$

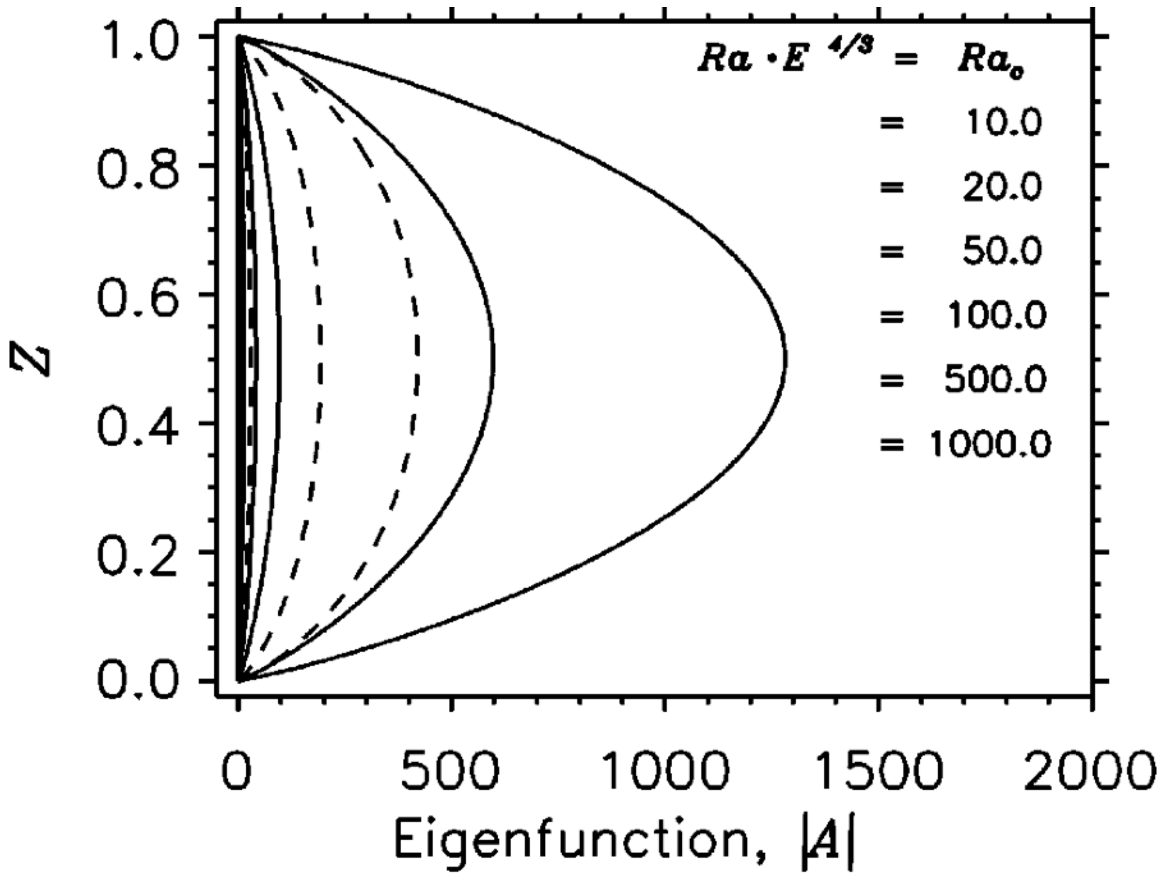


Unstable solutions form a skeleton that impact the terrain of phase space

Bassom & Zhang GAFD 1994 Julien & Knobloch PoF 1996, JFM 1998

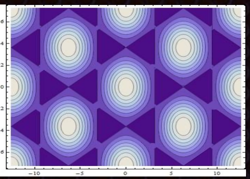


Single-Mode Solutions

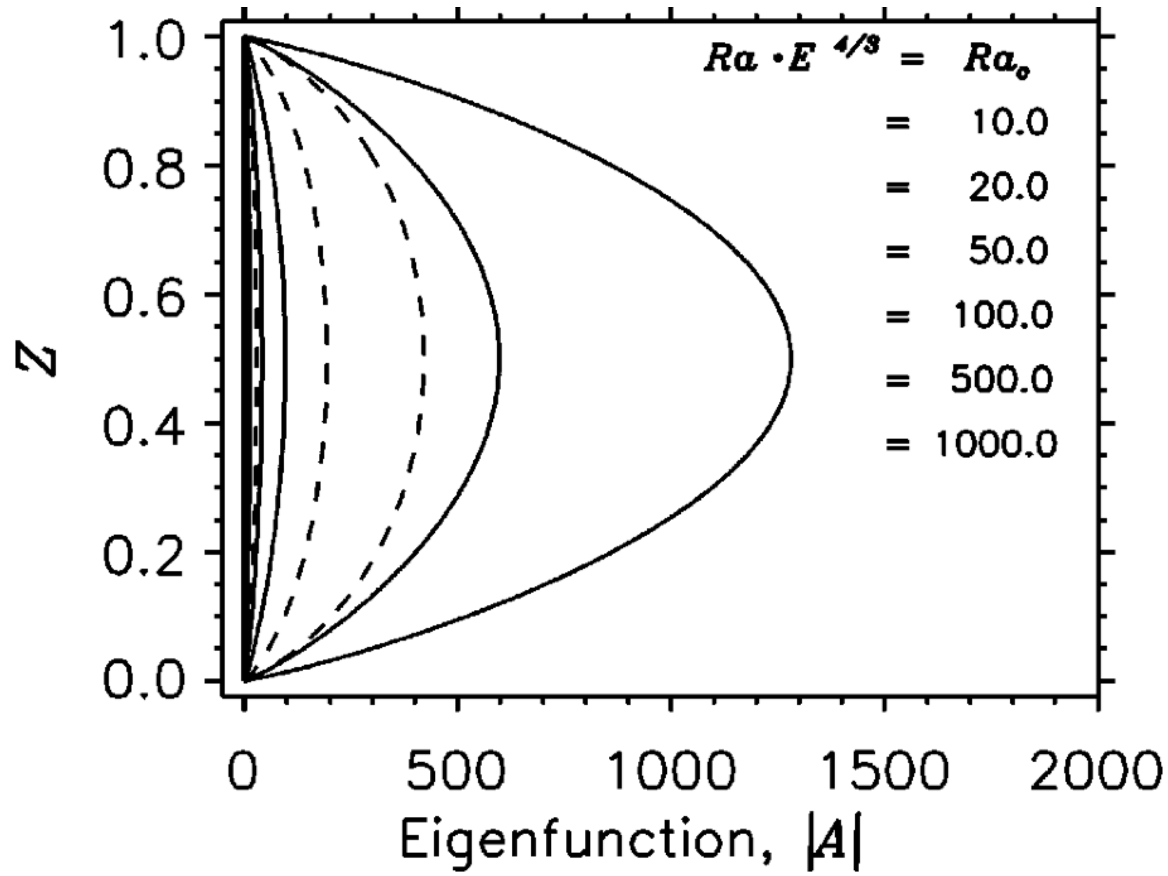


Sinusoidal vertical structure

For both steady and oscillatory convection

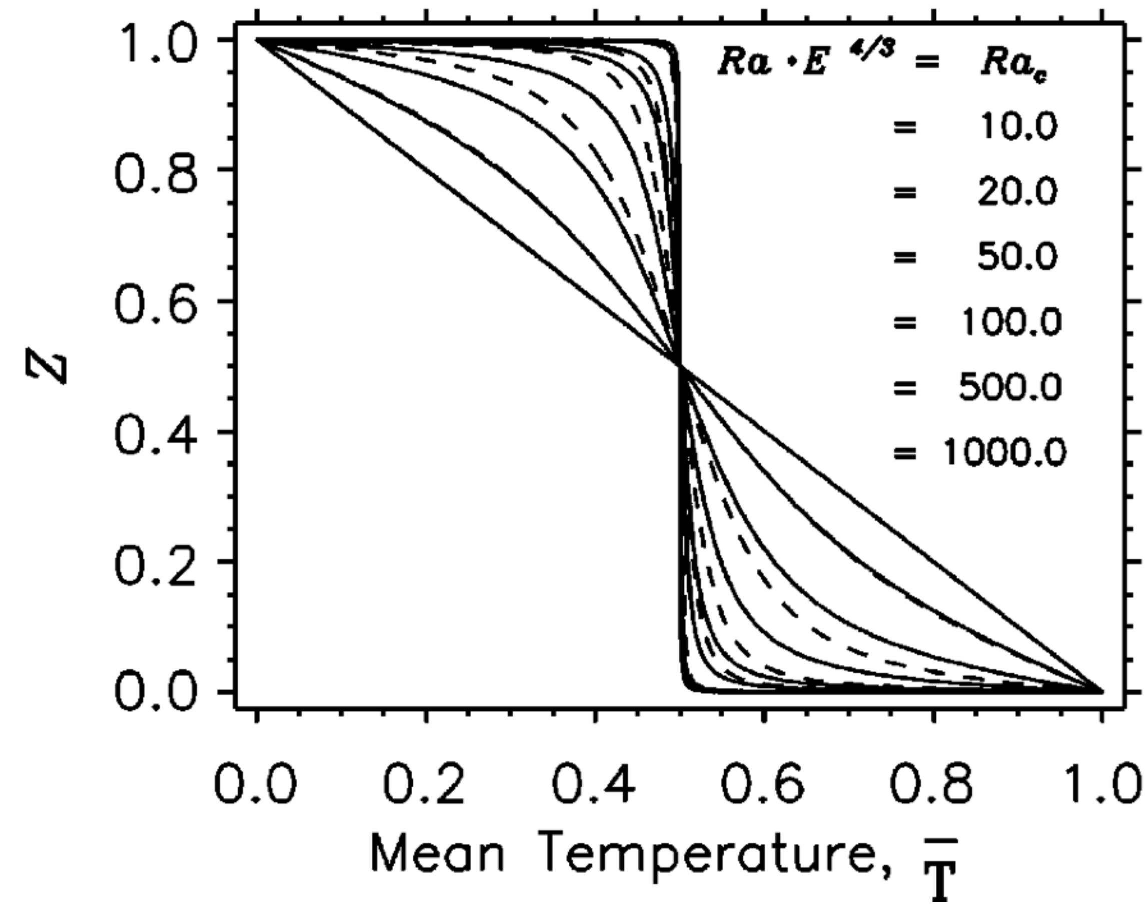


Single-Mode Solutions



Sinusoidal vertical structure

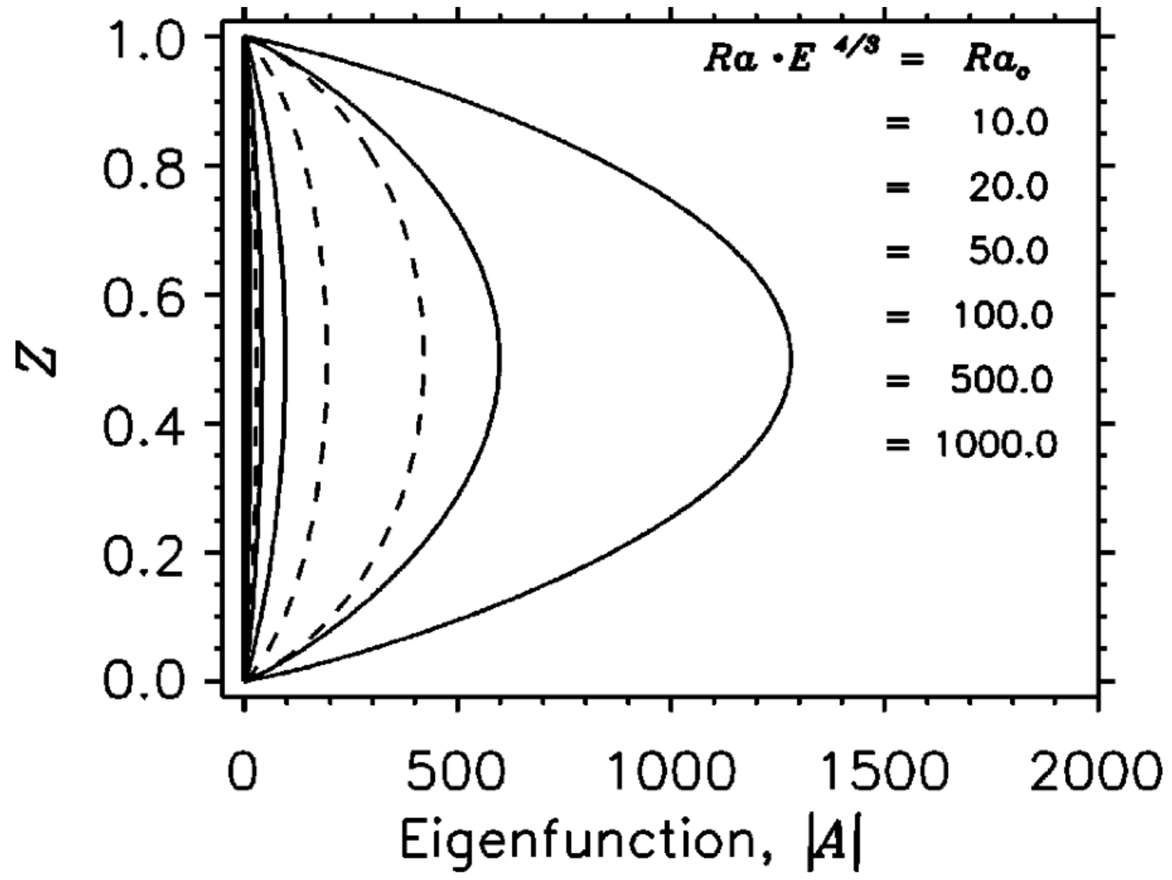
For both steady and oscillatory convection

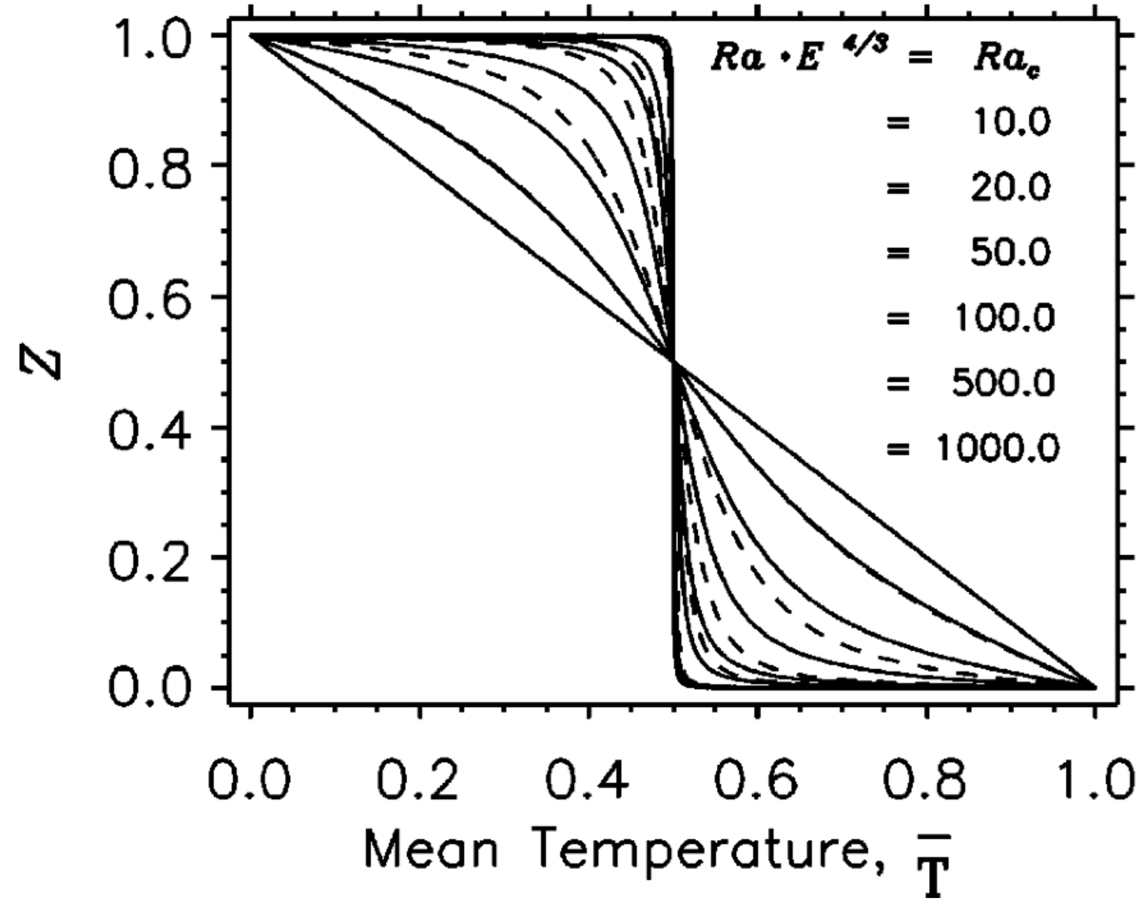


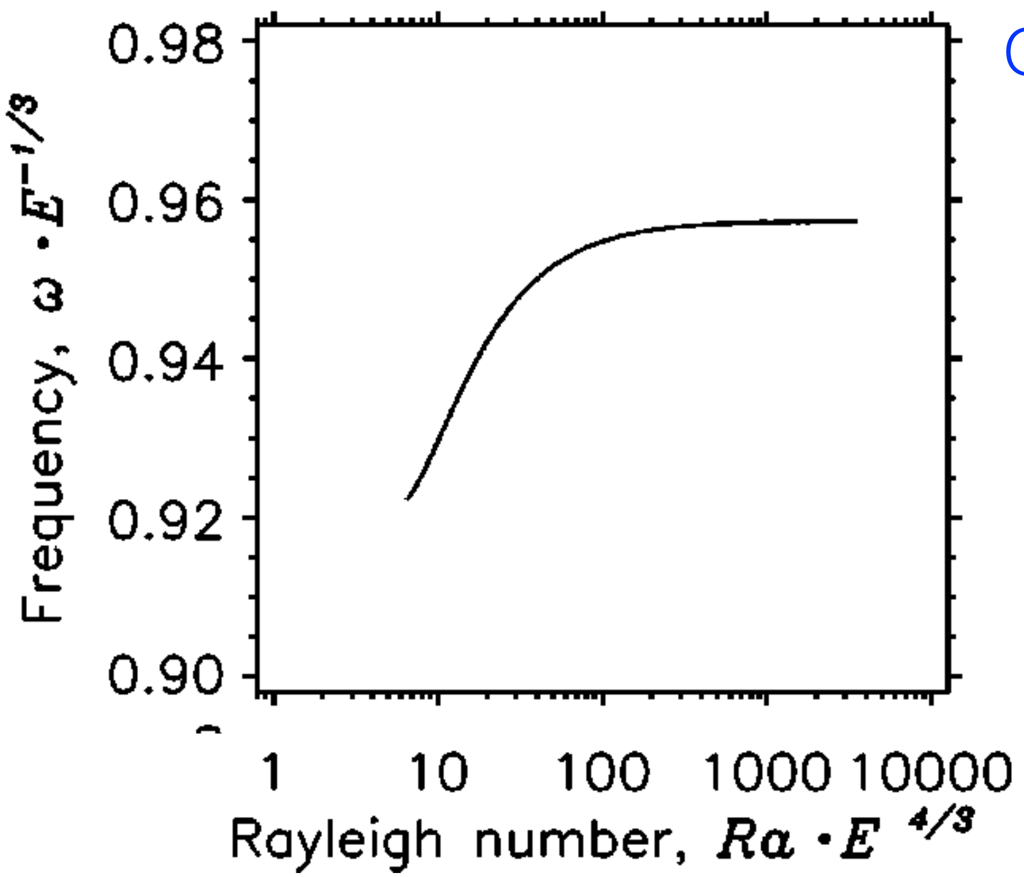
Development of TBL + Isothermal Interior

$$\partial_Z \overline{T} \propto \widetilde{Ra}^{-1}$$

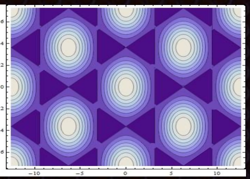
Nonlinear Solutions - Exact Coherent Structures (ECS)



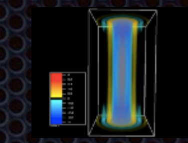


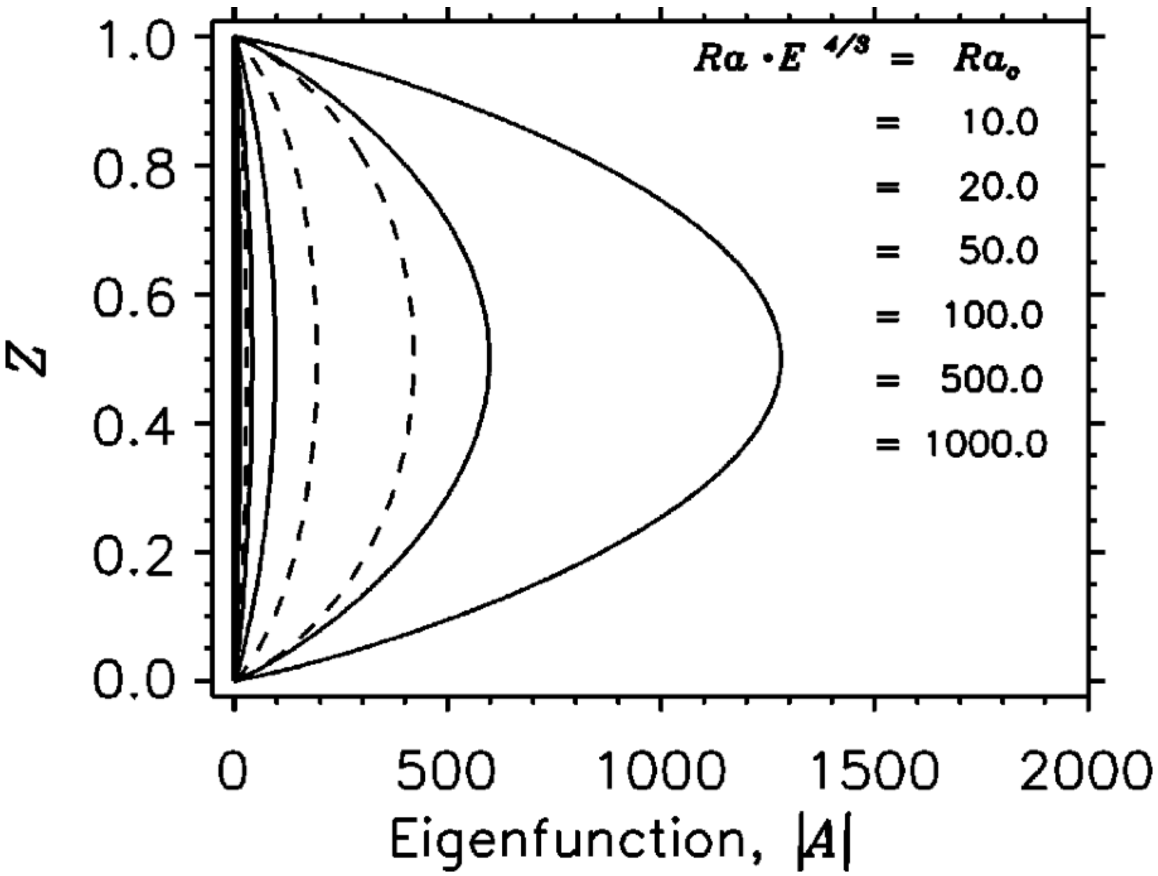


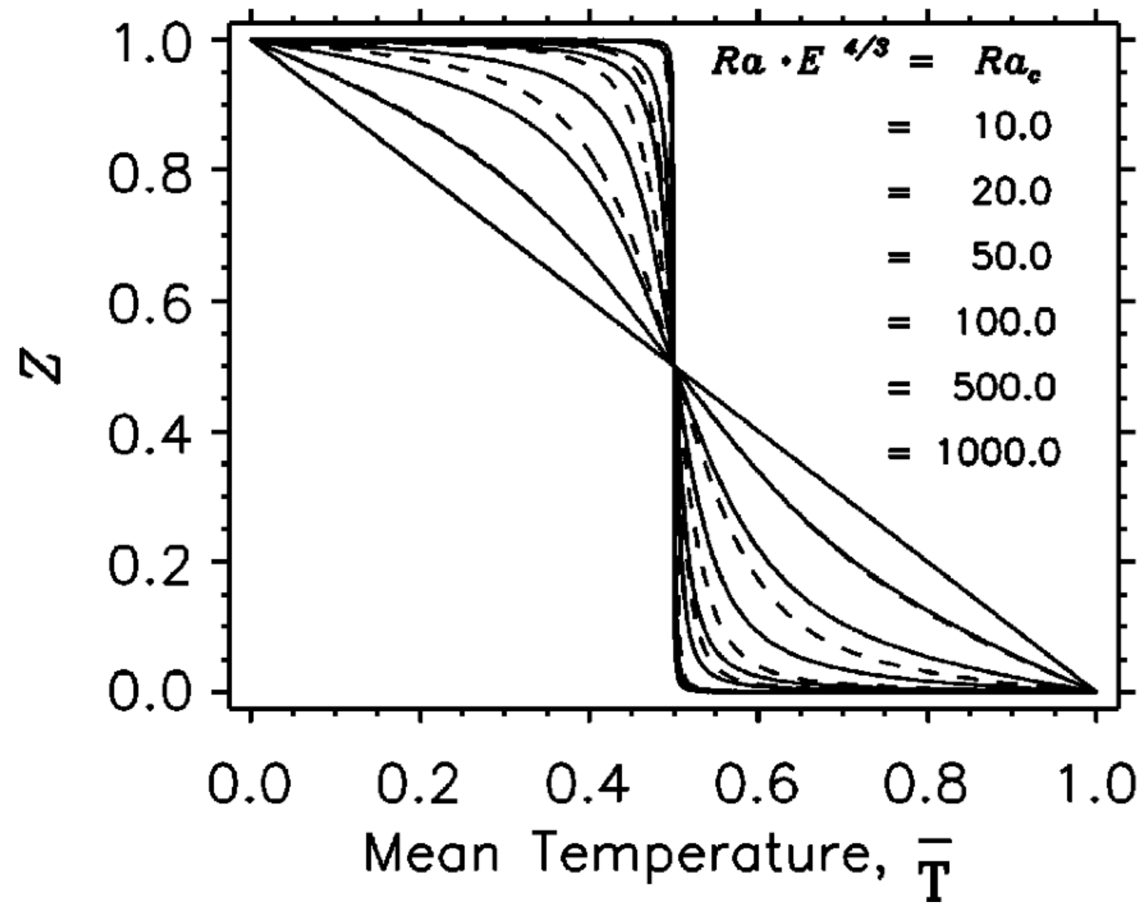
Oscillations saturate

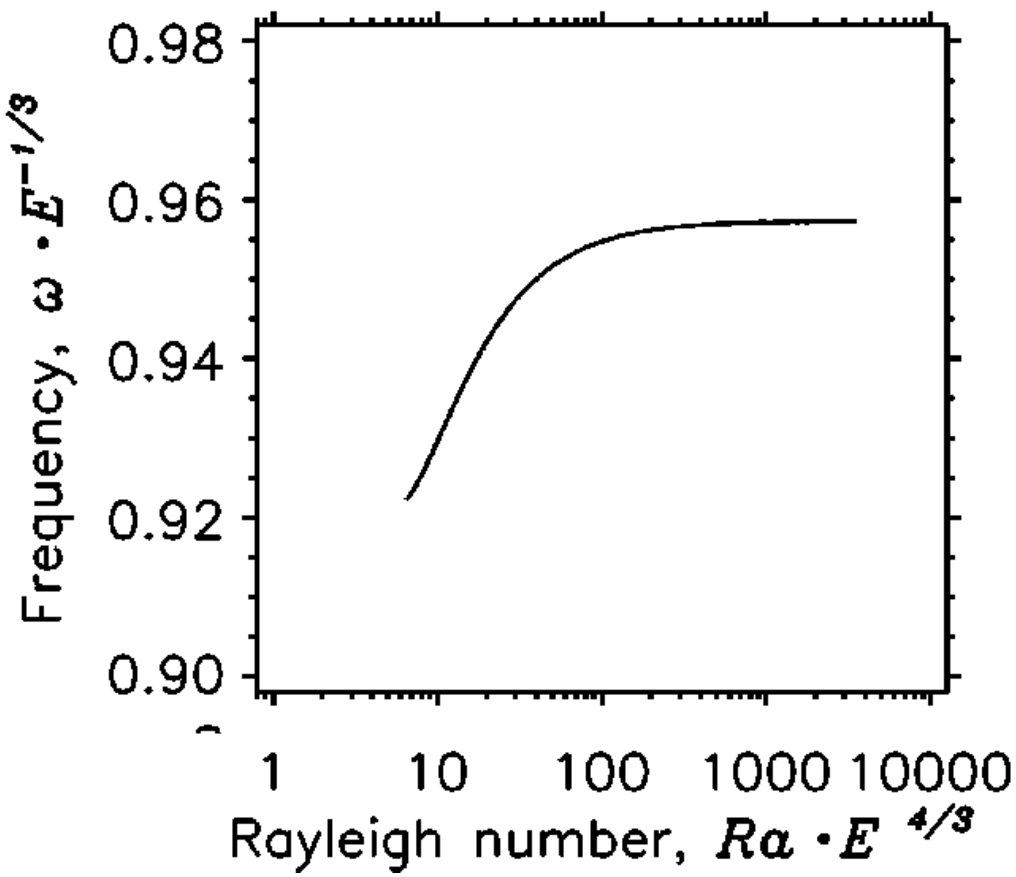


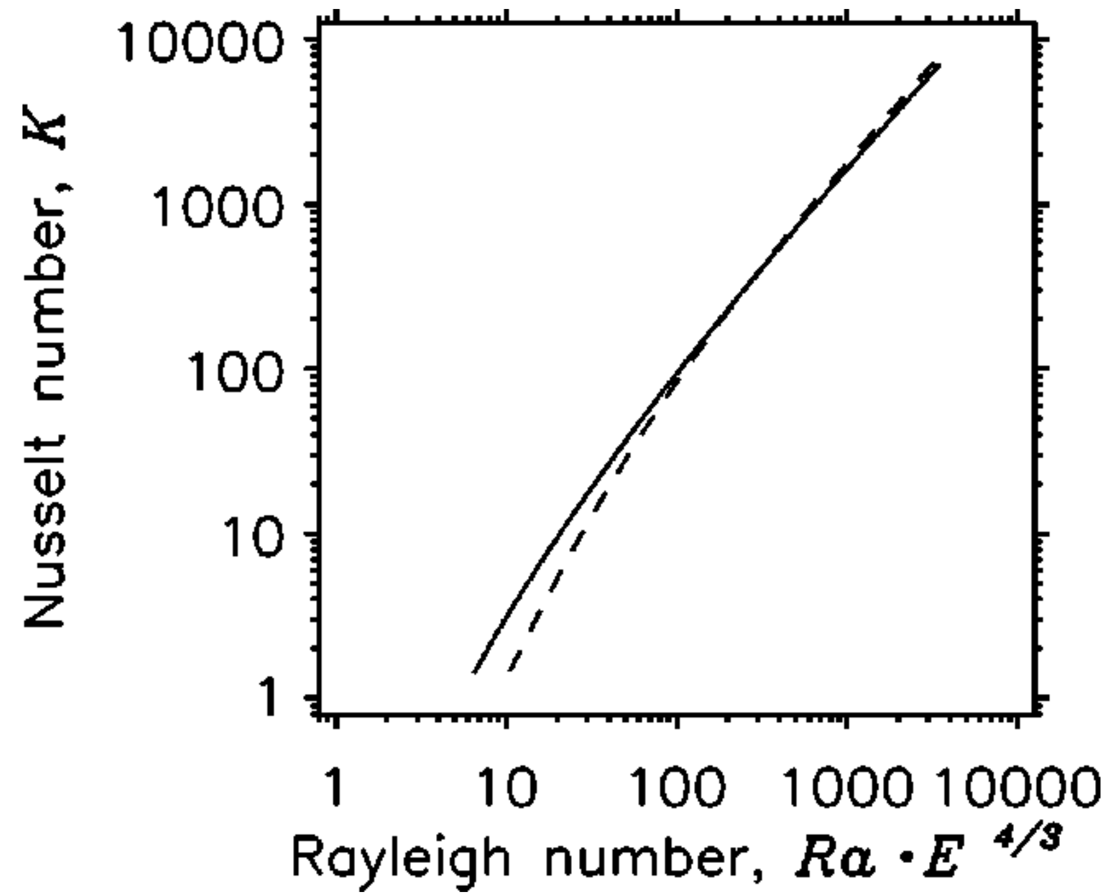
Nonlinear Solutions - Exact Coherent Structures (ECS)



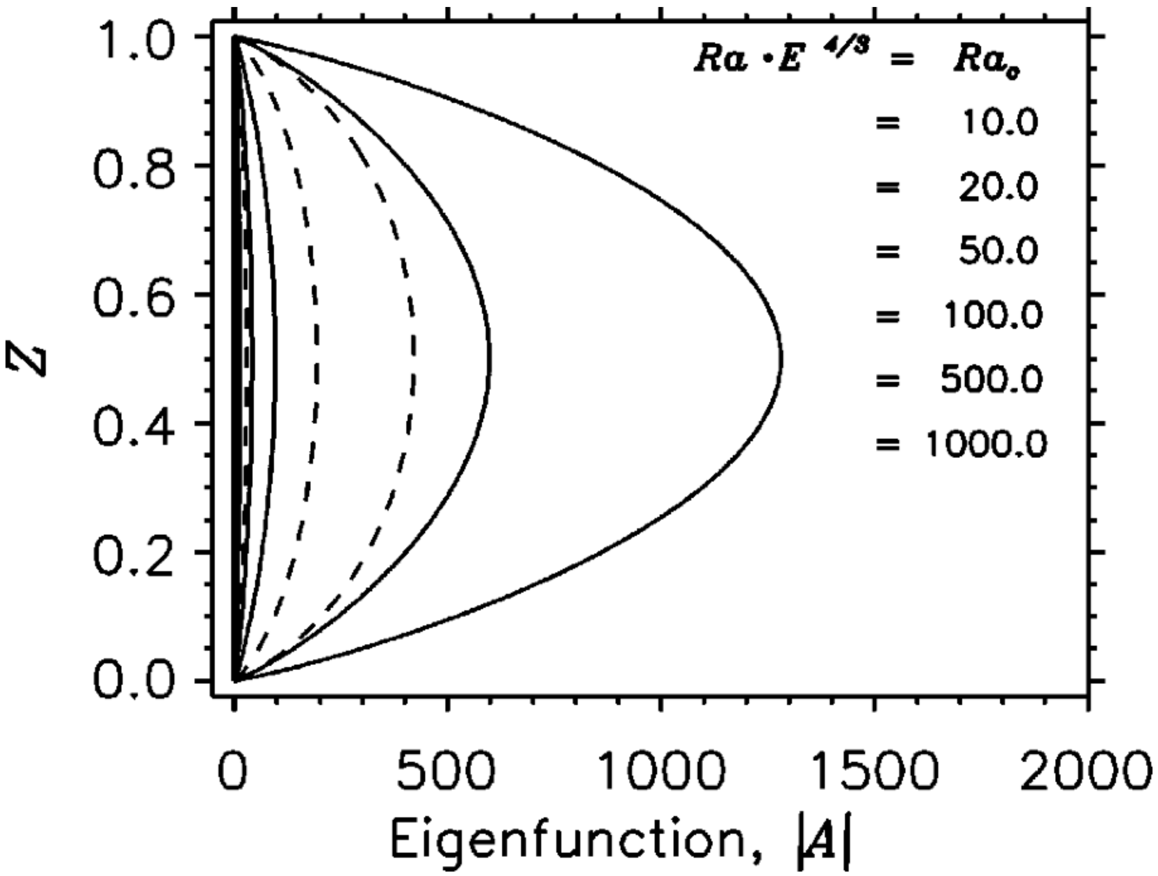


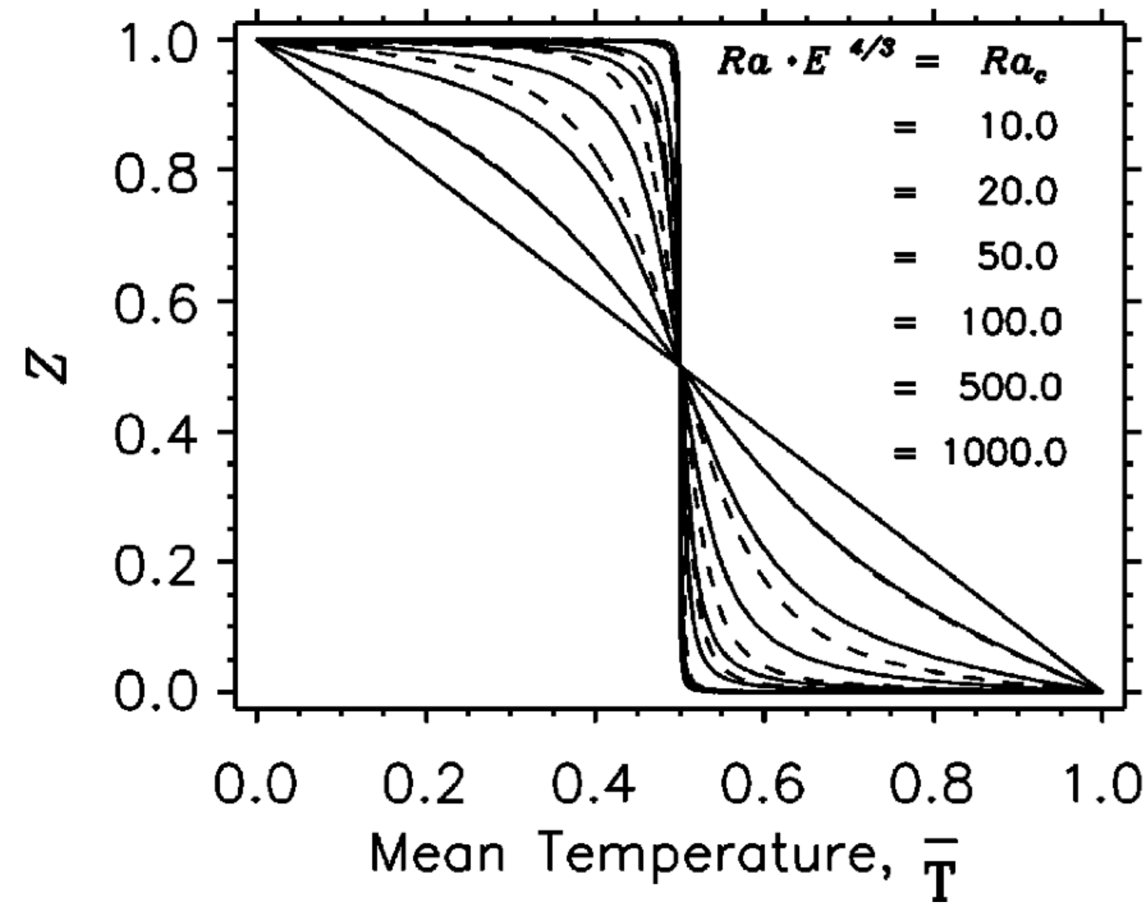




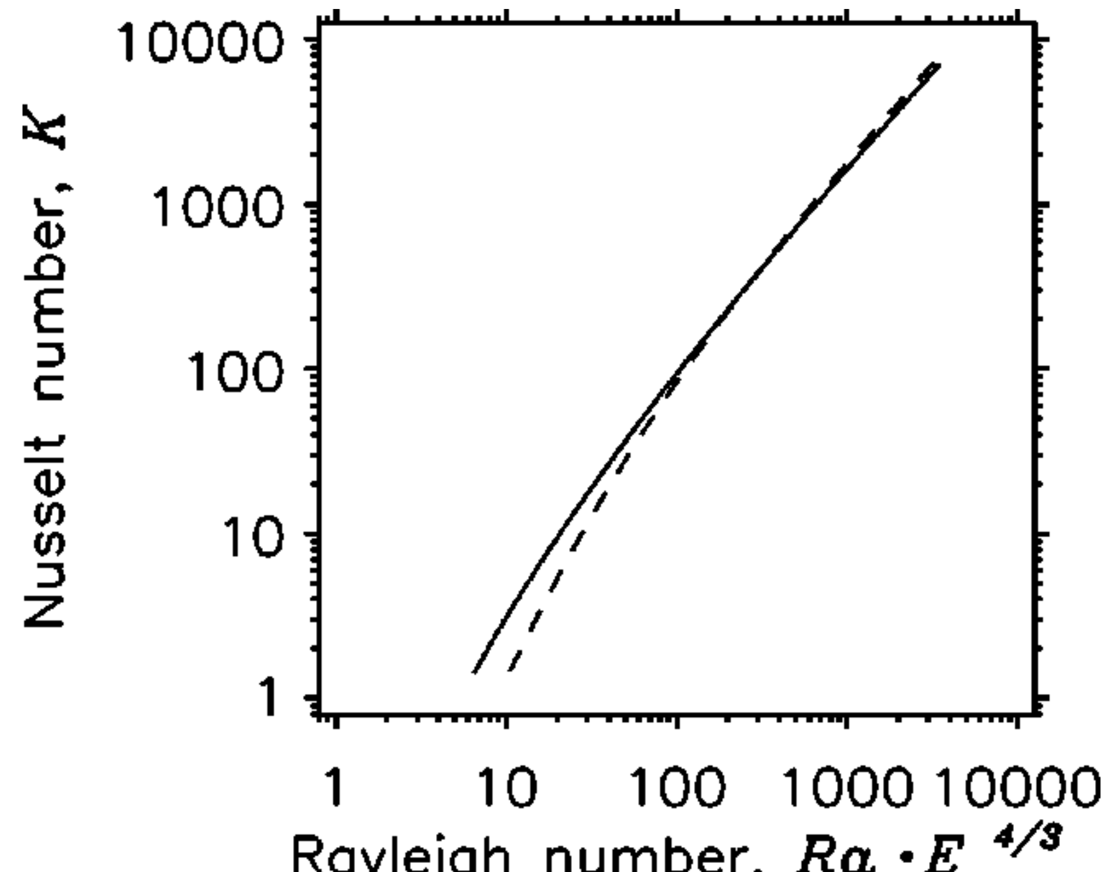


Nonlinear Solutions - Exact Coherent Structures (ECS)





HX: $Nu \propto \widetilde{Ra} \ln \widetilde{Ra}$



NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3}\vartheta.$

Vert. Vorticity

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

Vert. Velocity

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\,\theta$$

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$$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

Mean. Temp.

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Two control parameters

$$\widetilde{Ra} = Ra E^{4/3}, \ Pr$$

Reduced BCs.

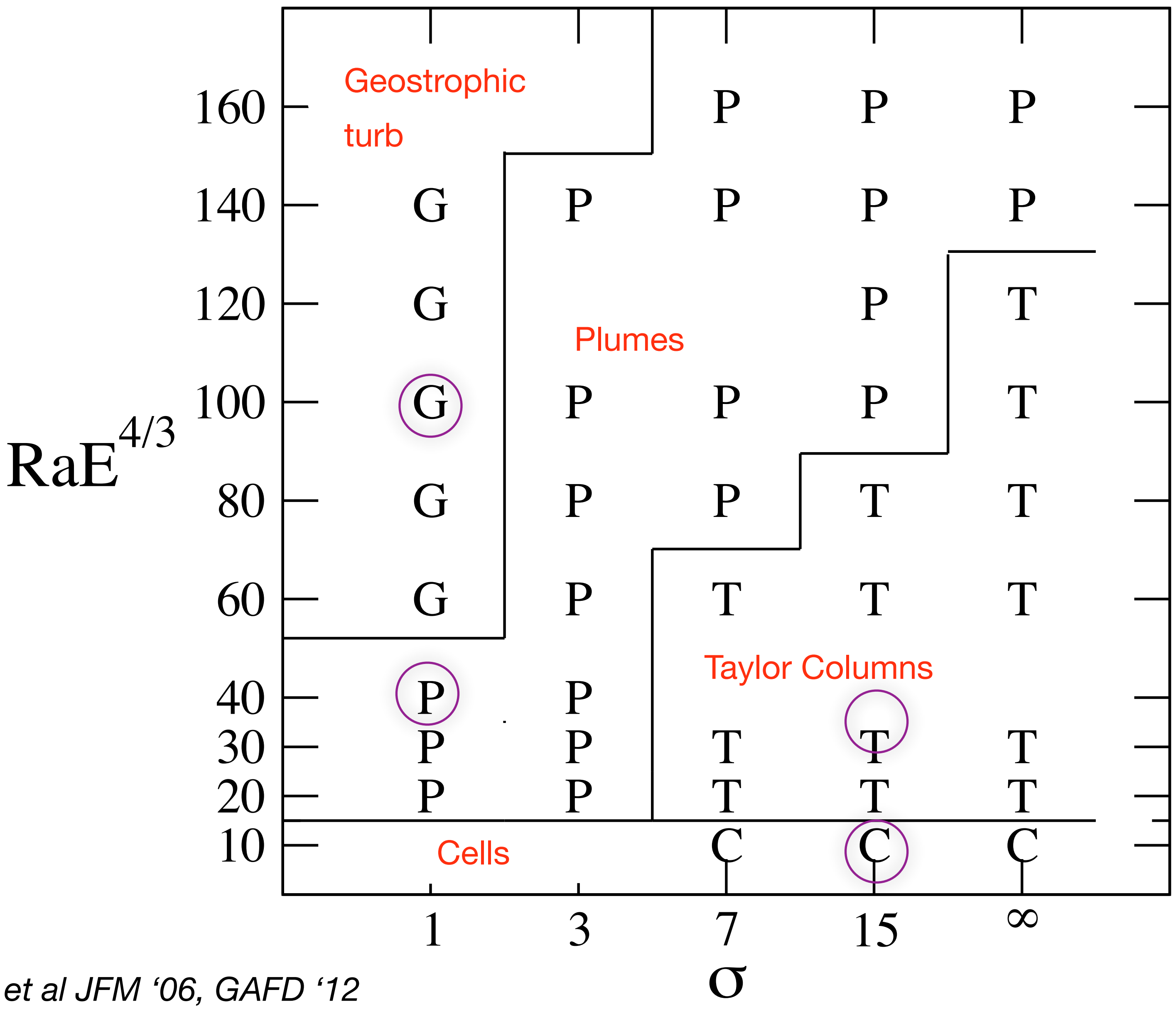
$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

Simulations for plane-layer performed on HPC platforms

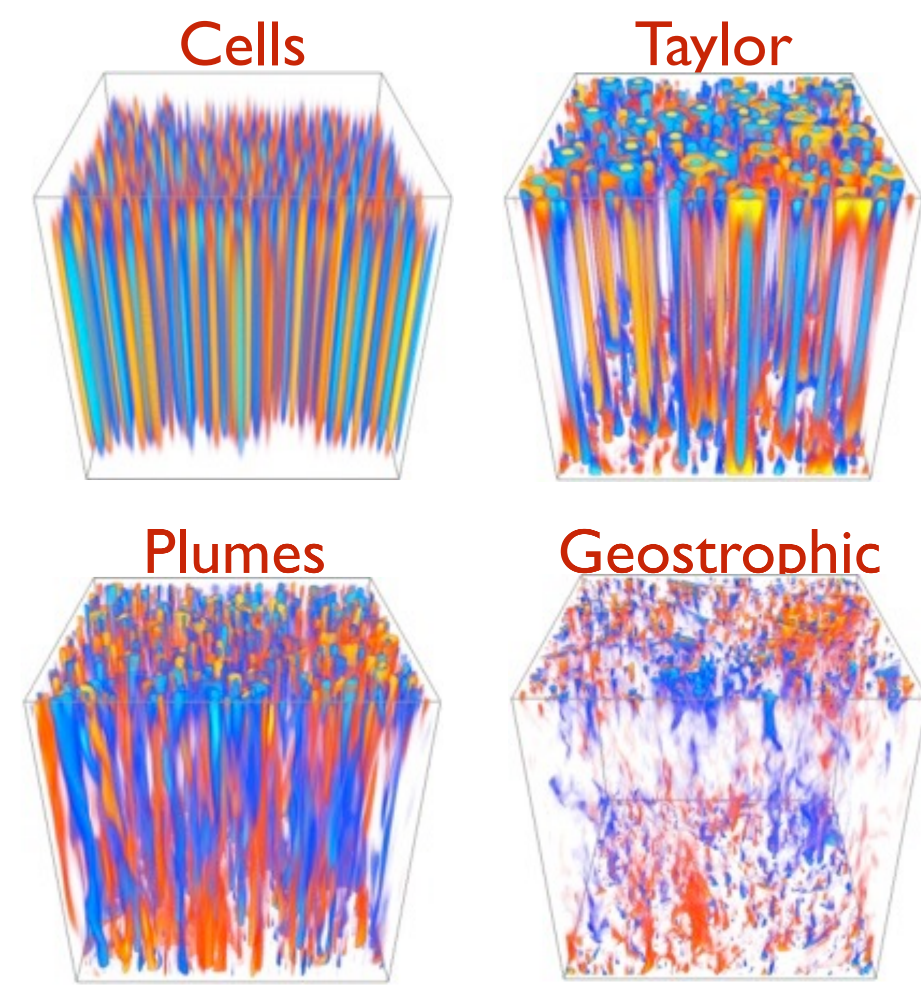
- Fast parallel algorithm
- Sparse Fourier-Chebyshev Spectral Method
- 3rd order Implicit/Explicit timestepping

KJ & Watson JCP 2008, Marti et al G[3] 2016, Burns et al PRR 2020

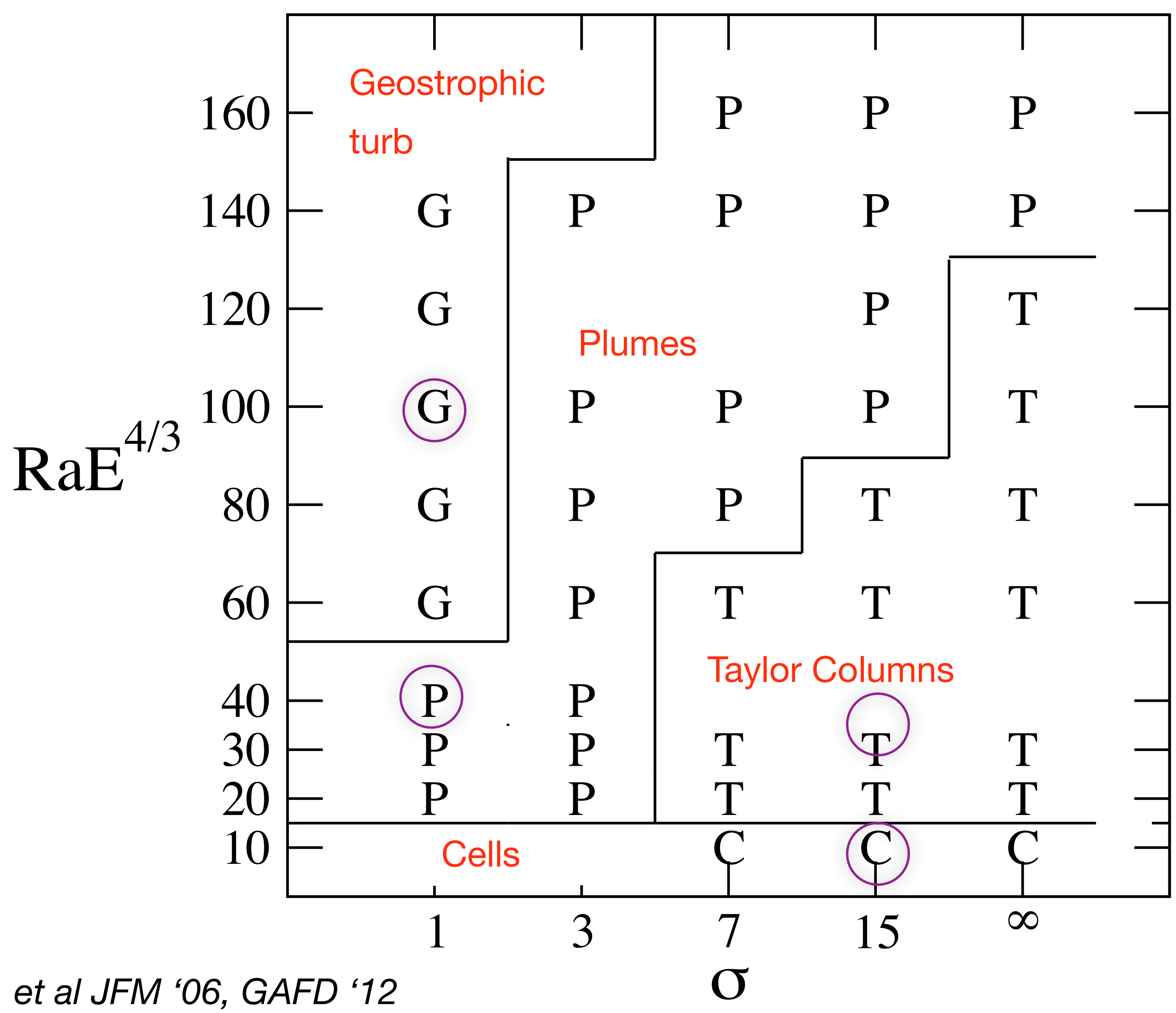
Quasi-Geostrophic RBC Flow Regimes
Mapping Parameter Space



J. et al JFM '06, GAFD '12

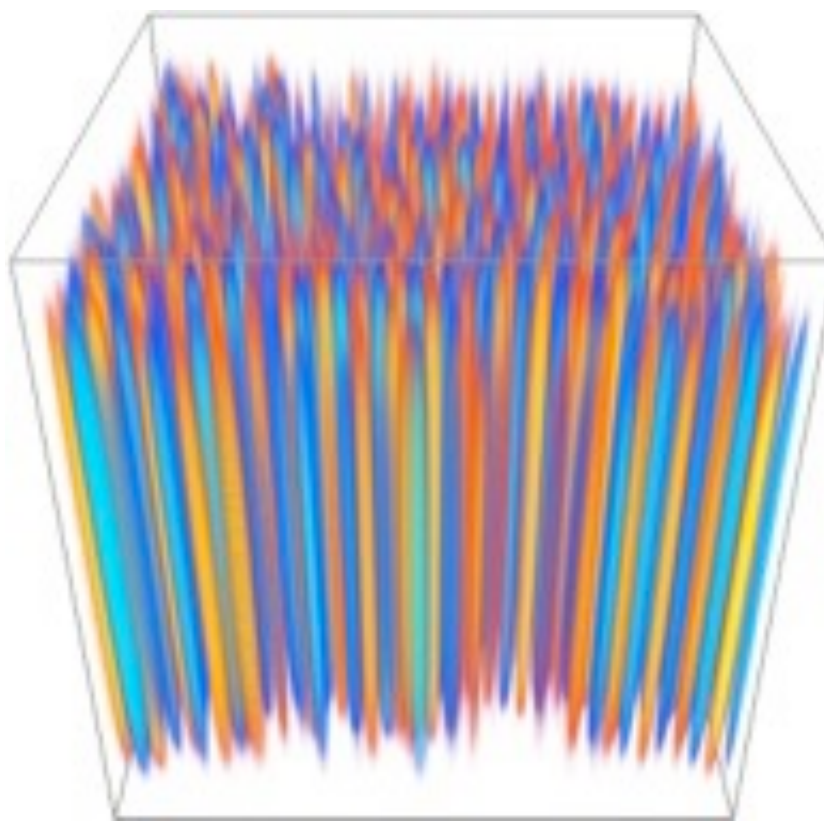


Quasi-Geostrophic RBC Flow Regimes
Cellular Regime



J. et al JFM '06, GAFD '12

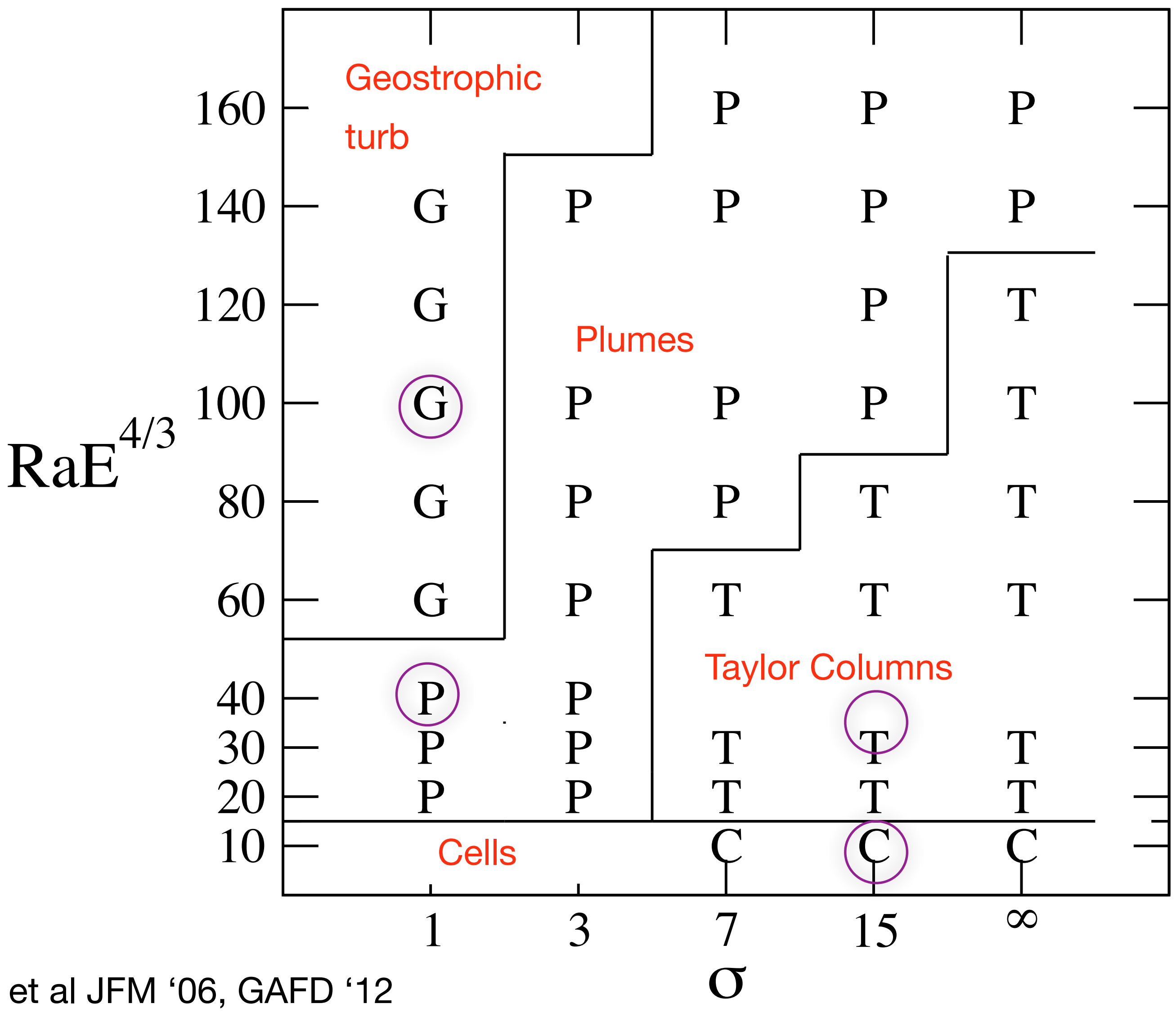
Thermal anomaly
$\theta \propto T - \overline{T}(Z)$



$\widetilde{Ra} = 10, \ Pr = 15$

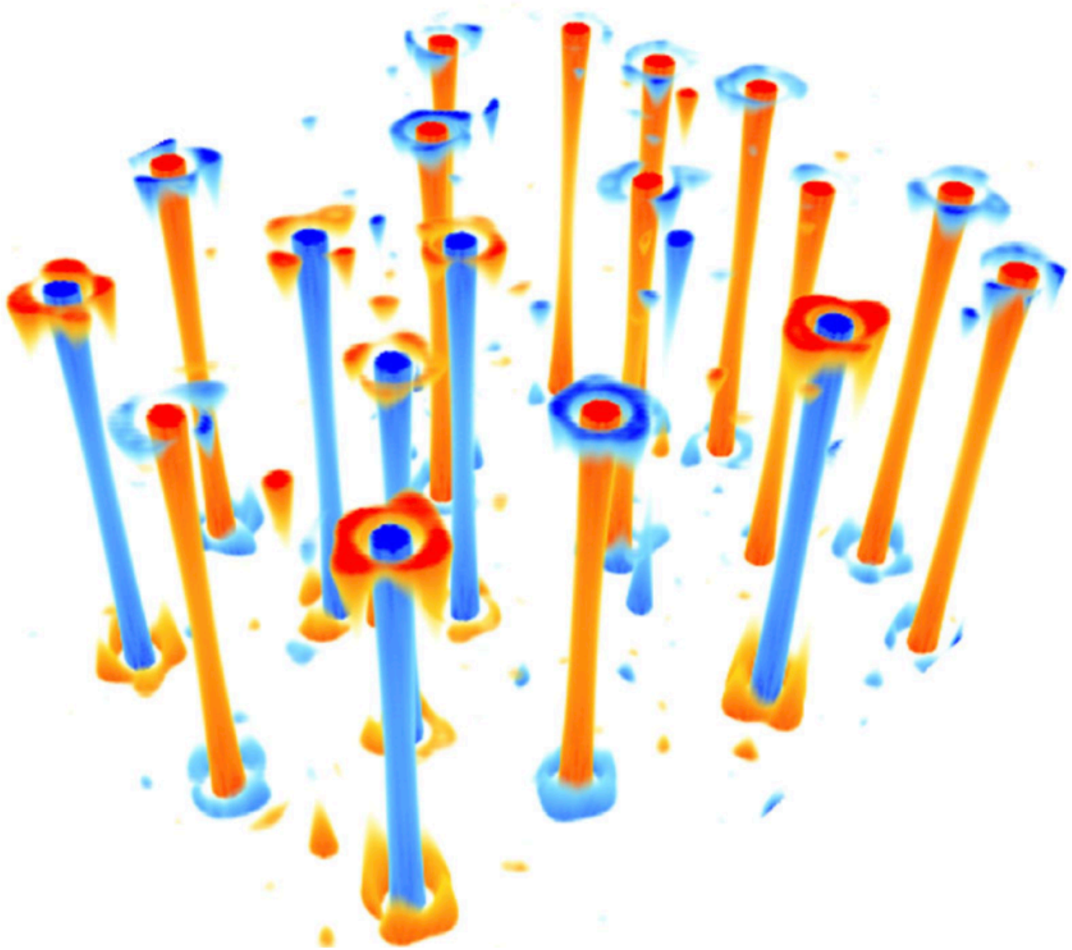
KJ. & Knobloch PoF 99, JMP '07

Quasi-Geostrophic RBC Flow Regimes
Convective Taylor Columns



J. et al JFM '06, GAFD '12

Thermal anomaly
$\theta \propto T - \overline{T}(Z)$



$\widetilde{Ra} = 40, \ Pr = 7$

Grooms, KJ, Knobloch, Weiss, PRL 2010

Quasi-Geostrophic RBC Flow Regimes
Convective Taylor Columns

Instability of TBL

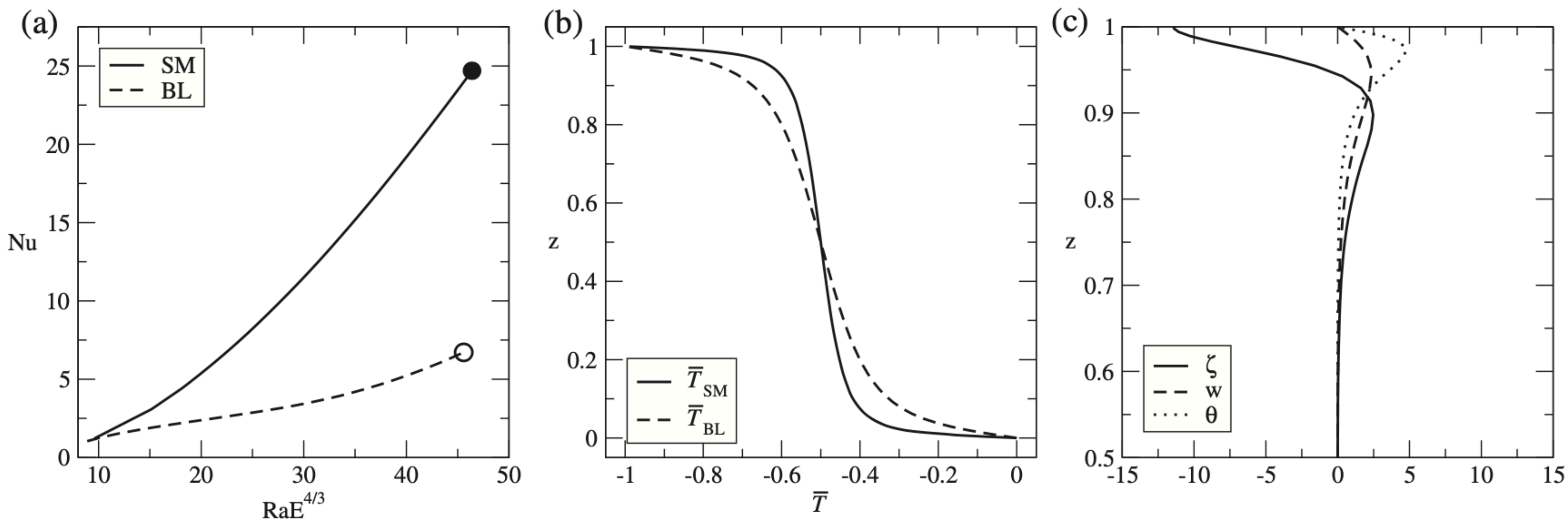
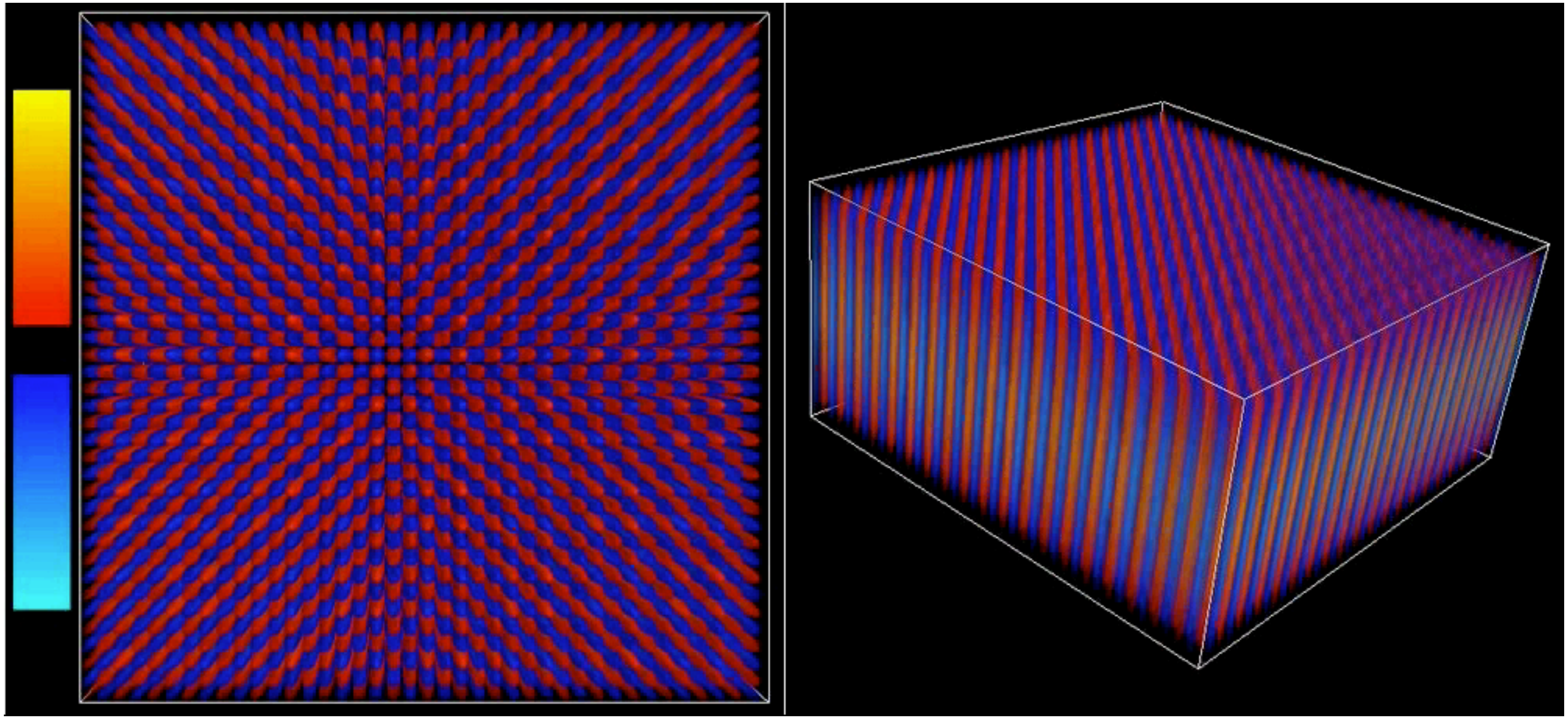


Figure 17. Comparison of single mode theory with the boundary layer stability analysis. (a) Nusselt number as a function of $\widetilde{Ra}$ for both single mode ($k_\perp = 1.3084$) and boundary layer instability. (b) Mean temperature profiles for single mode and boundary layer instability at $\widetilde{Ra} = 45.6$, shown as open and filled circles in (a). (c) Eigenfunctions of the boundary layer instability at $\widetilde{Ra} = 45.6$ with instability wavenumber $k_\perp = 4.8154$.

KJ, et al GAFD '10

Quasi-Geostrophic RBC Flow Regimes
Convective Taylor Columns

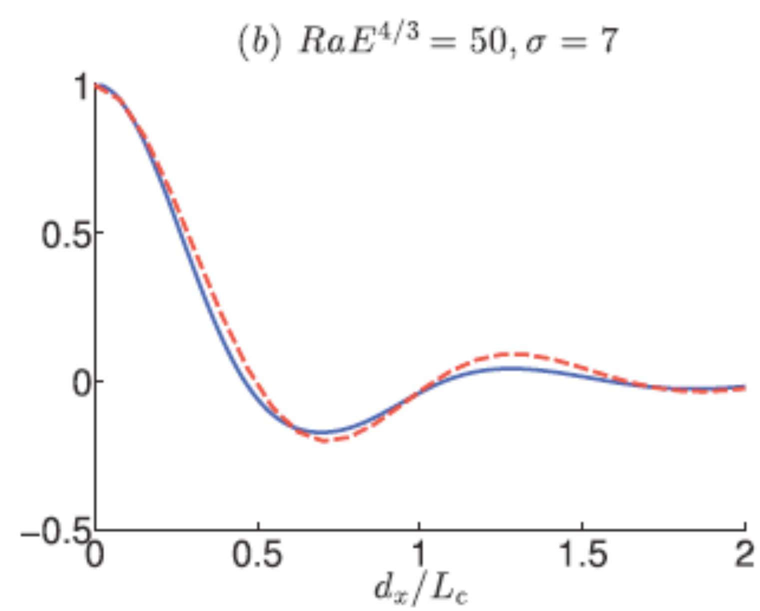


$RaE^{4/3} = 40, \sigma = 7$

Sprague,, KJ, et al JFM '06

Quasi-Geostrophic RBC Flow Regimes
Convective Taylor Columns

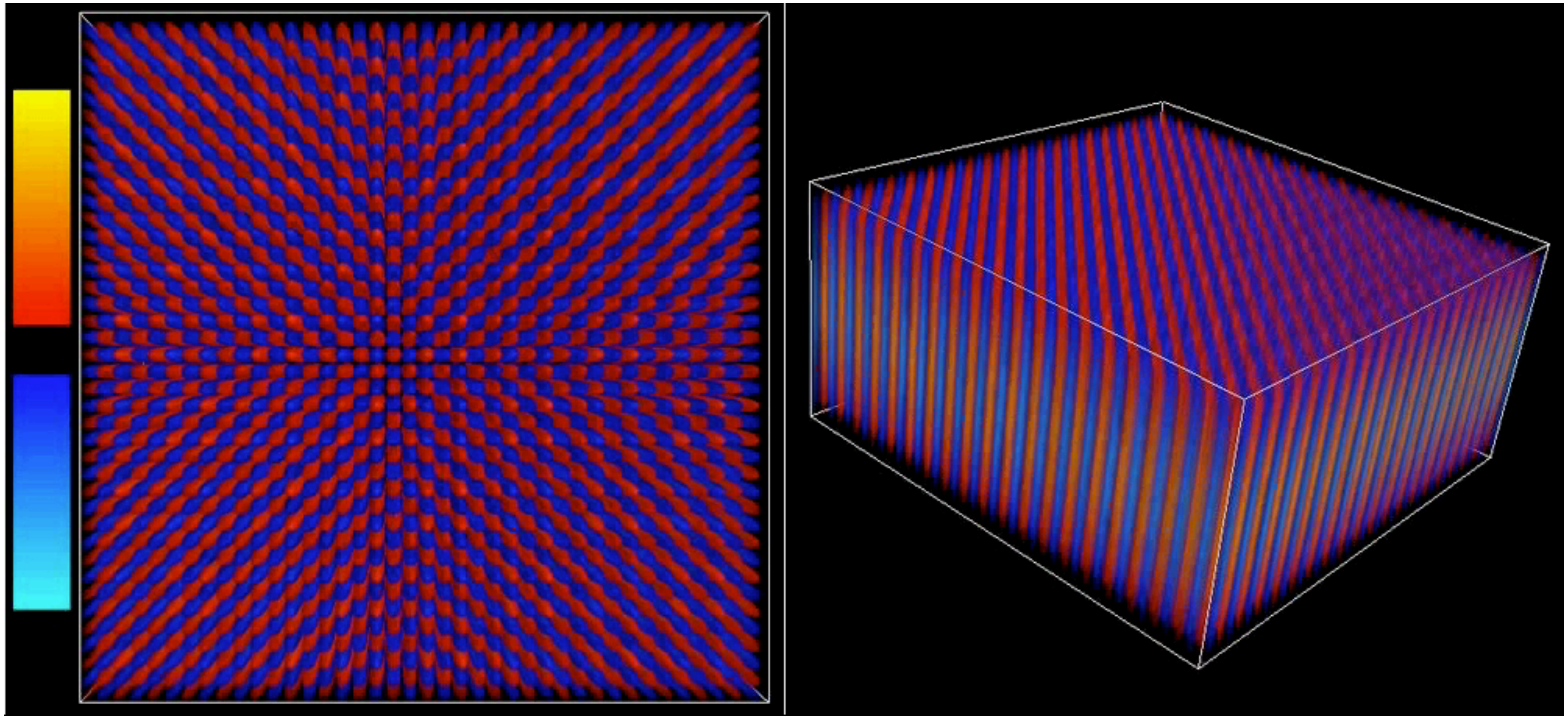
Circulation: $K \equiv \int \omega dA \approx 0$



Nieves et al PoF ’14

Grooms et al PRL ’10

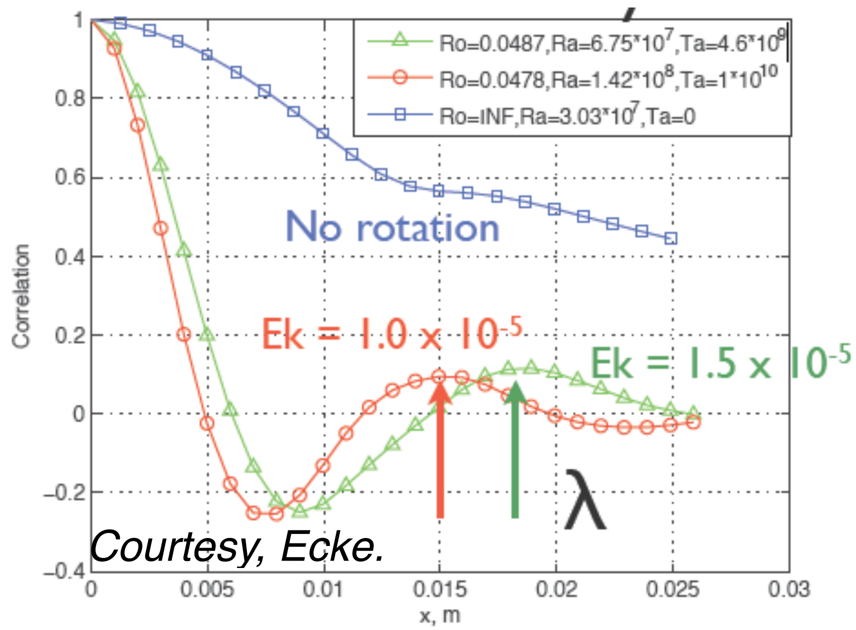
Auto-correlation of temperature fluctuations captures radial structure

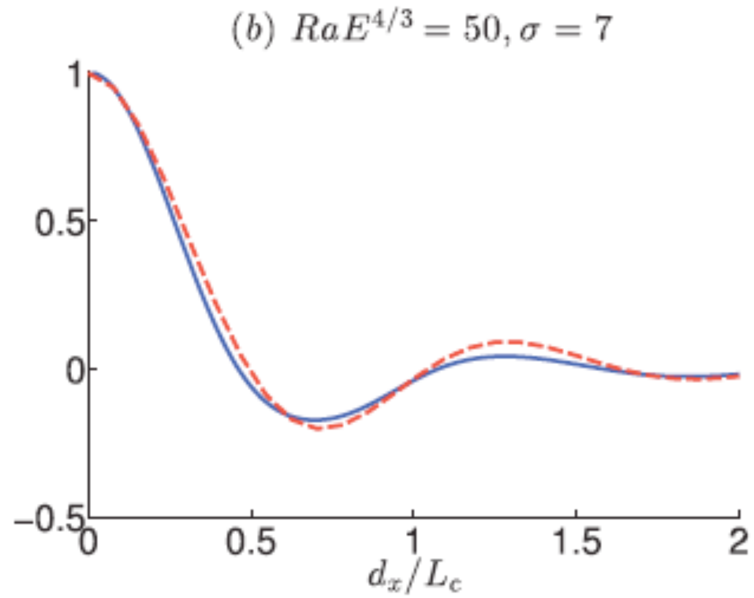


$RaE^{4/3} = 40, \sigma = 7$

Sprague,, KJ, et al JFM ‘06

Quasi-Geostrophic RBC Flow Regimes
Convective Taylor Columns

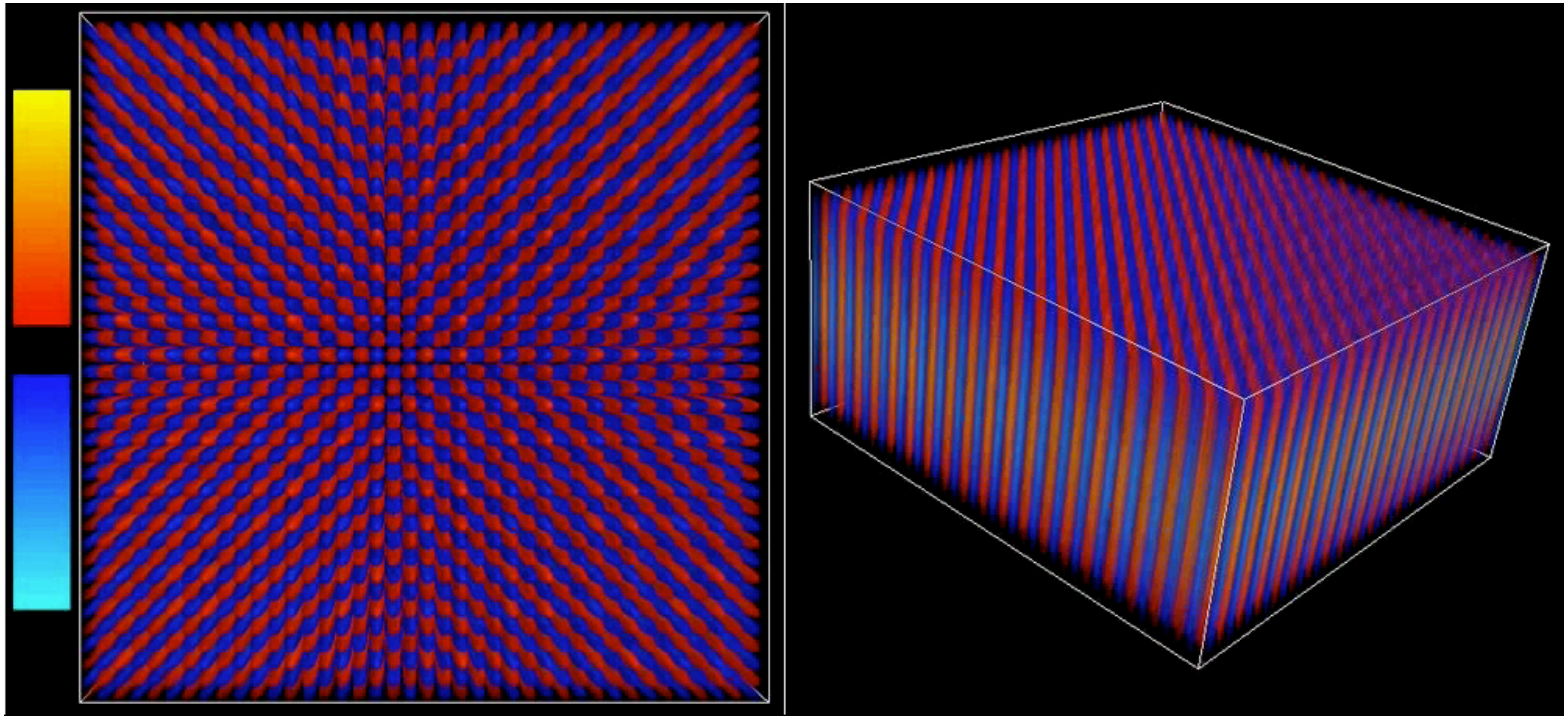




Nieves et al PoF '14

Grooms et al PRL '10

Auto-correlation of temperature fluctuations captures radial structure

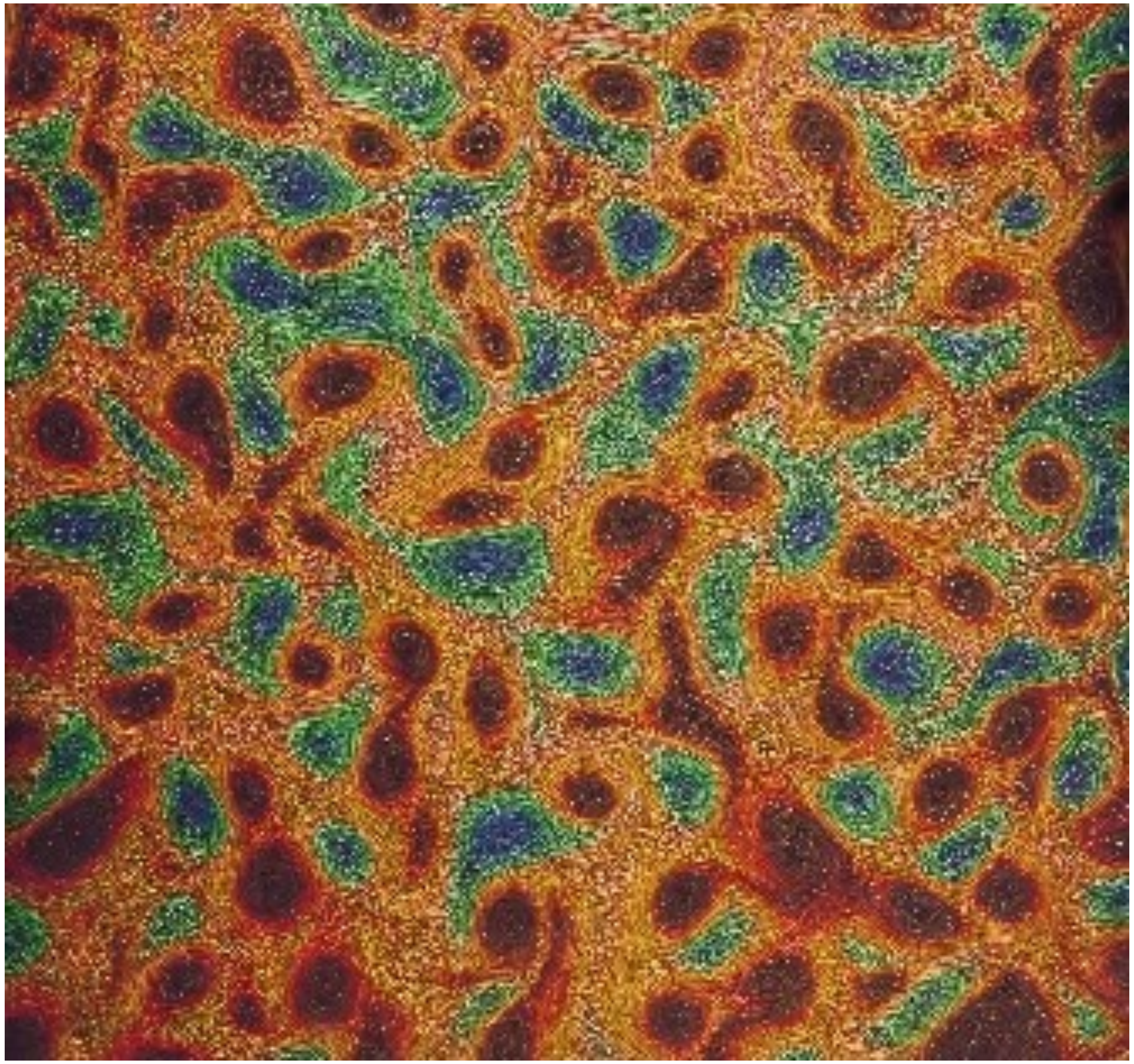


$RaE^{4/3} = 40, \sigma = 7$

Sprague,, KJ, et al JFM '06

Quasi-Geostrophic RBC Flow Regimes
Convective Taylor Columns

Top view - Temperature

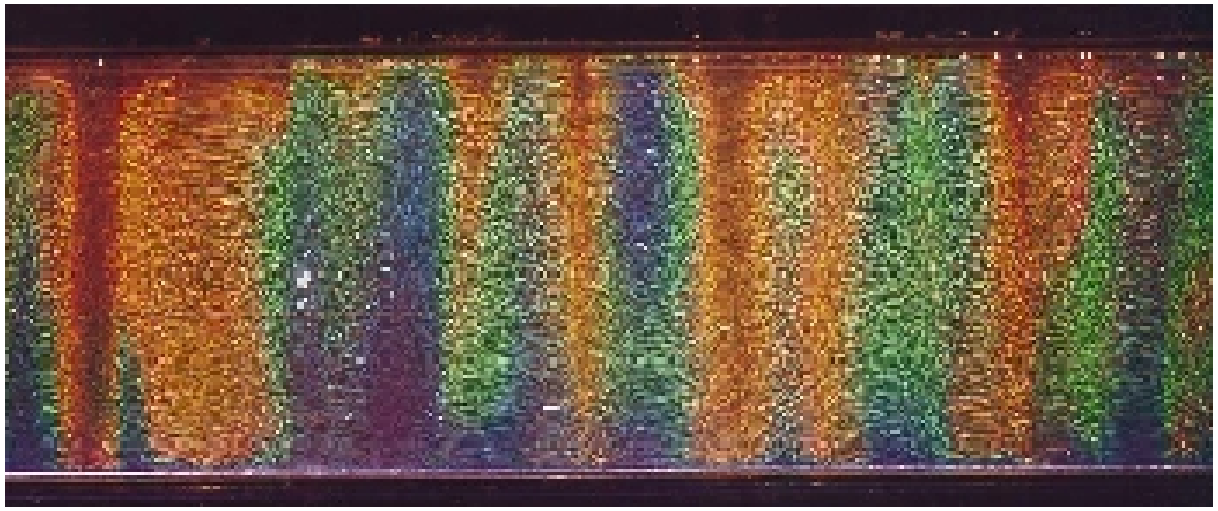


$$(Ra \approx 10^7,\ Ek \approx 10^{-4},\ Pr \approx 7)$$

$$\widetilde{Ra} \approx 37$$

Sakai, JFM 1997

Side view



$$Ra = \frac{g\alpha \Delta T H^3}{\nu\kappa} \sim E^{-4/3} \qquad Ro_c = \sqrt{\frac{Ra}{Pr}}E \sim 0.1 \sim E^{1/3}$$

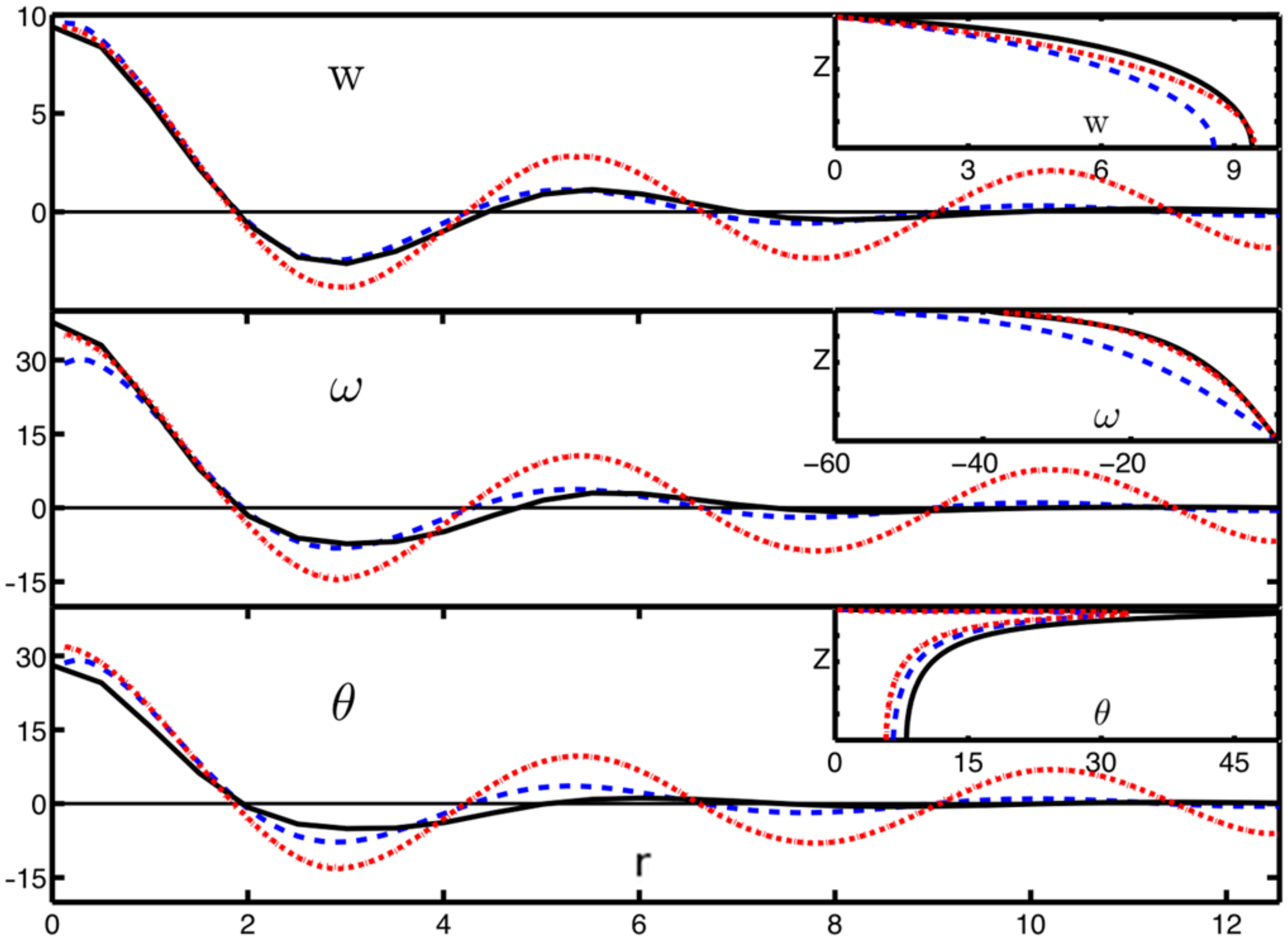
Nonlinear Single-mode Solutions

Pose: $\phi = \hat{\Phi}(r)h(r) + c.c., \quad \nabla_\perp^2 h + k_\perp^2 h = 0$

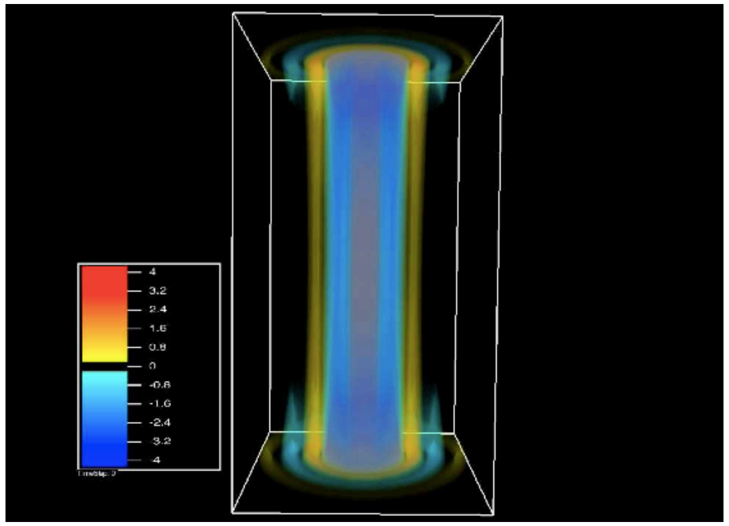
Find: $$\partial_z^2 \phi + \nabla_r^2(\widetilde{\mathrm{Ra}}\partial_z \bar{T} + \nabla_r^4)\phi = 0,$$

$$\partial_z \bar{T} = -\frac{\mathrm{Nu}}{1 + f_t + c_f \sigma^2 \langle (\partial_r \phi)^2 \rangle}.$$

turbulence CTC's



Spatially Localized (SL)



$$h(r) = J_0(k_\perp r) + iY_0(k_\perp r), \;\; k_\perp = |k_\perp| e^{i\alpha}$$

Grooms, KJ, Knobloch, Weiss, PRL 2010

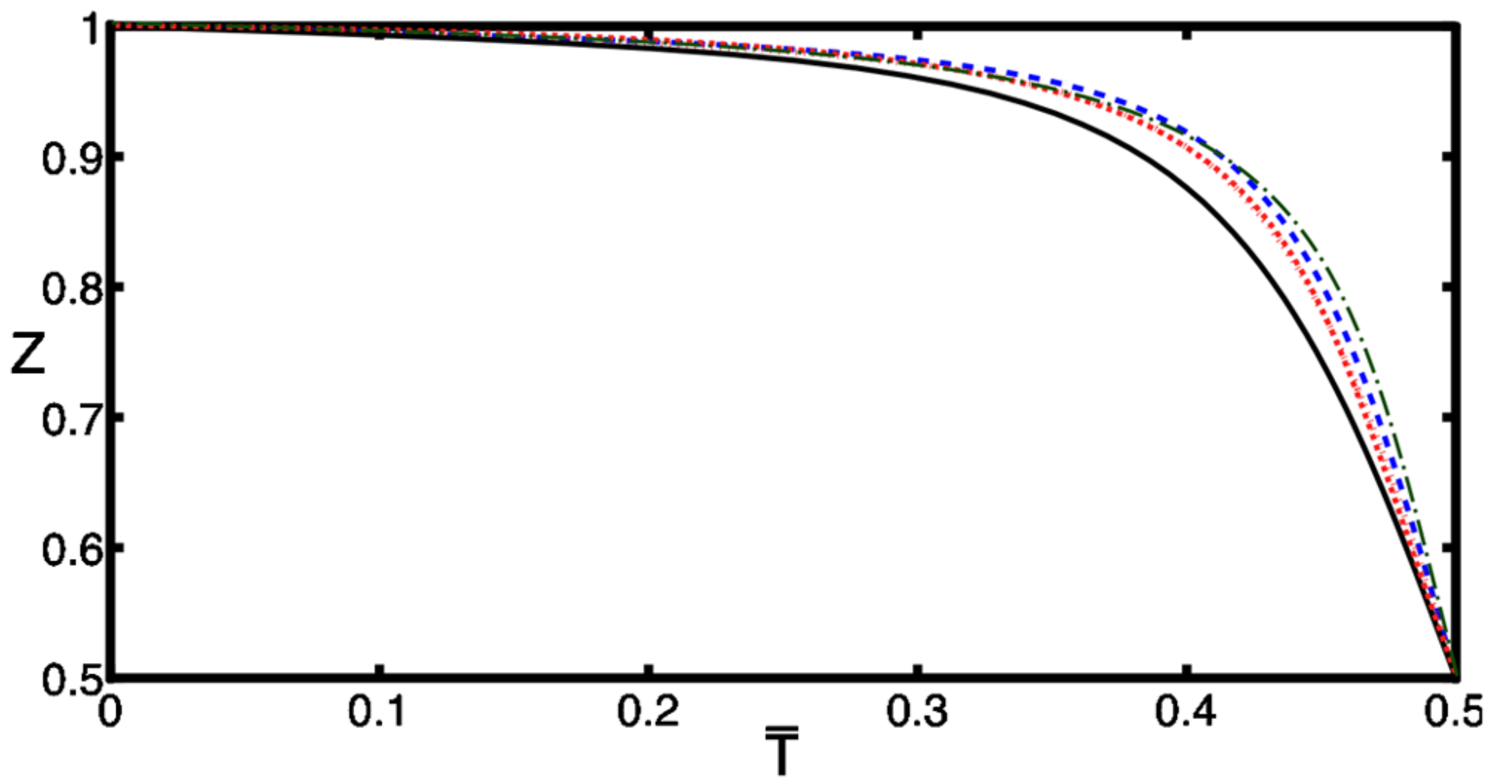
Nonlinear Single-mode Solutions

Pose: $\phi = \hat{\Phi}(r)h(r) + c.c., \quad \nabla_\perp^2 h + k_\perp^2 h = 0$

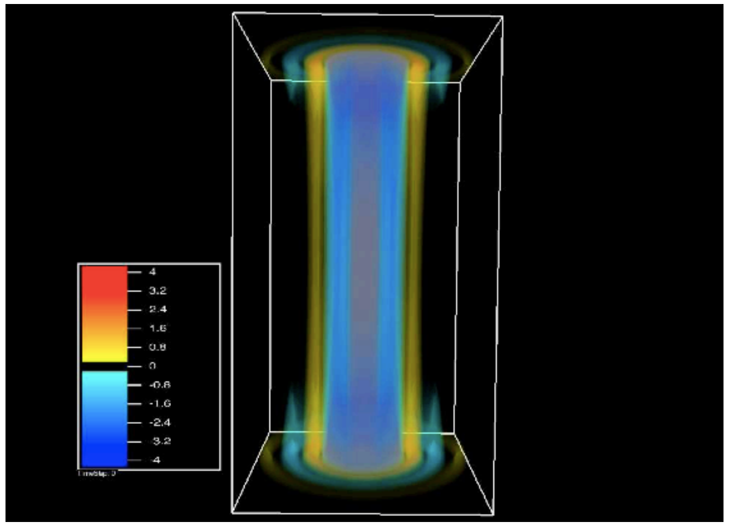
Find: $$\partial_z^2 \phi + \nabla_r^2(\widetilde{\mathrm{Ra}}\partial_z \bar{T} + \nabla_r^4)\phi = 0,$$

$$\partial_z \bar{T} = -\frac{\mathrm{Nu}}{1 + f_t + c_f \sigma^2 \langle (\partial_r \phi)^2 \rangle}.$$

turbulence CTC's



Spatially Localized (SL)

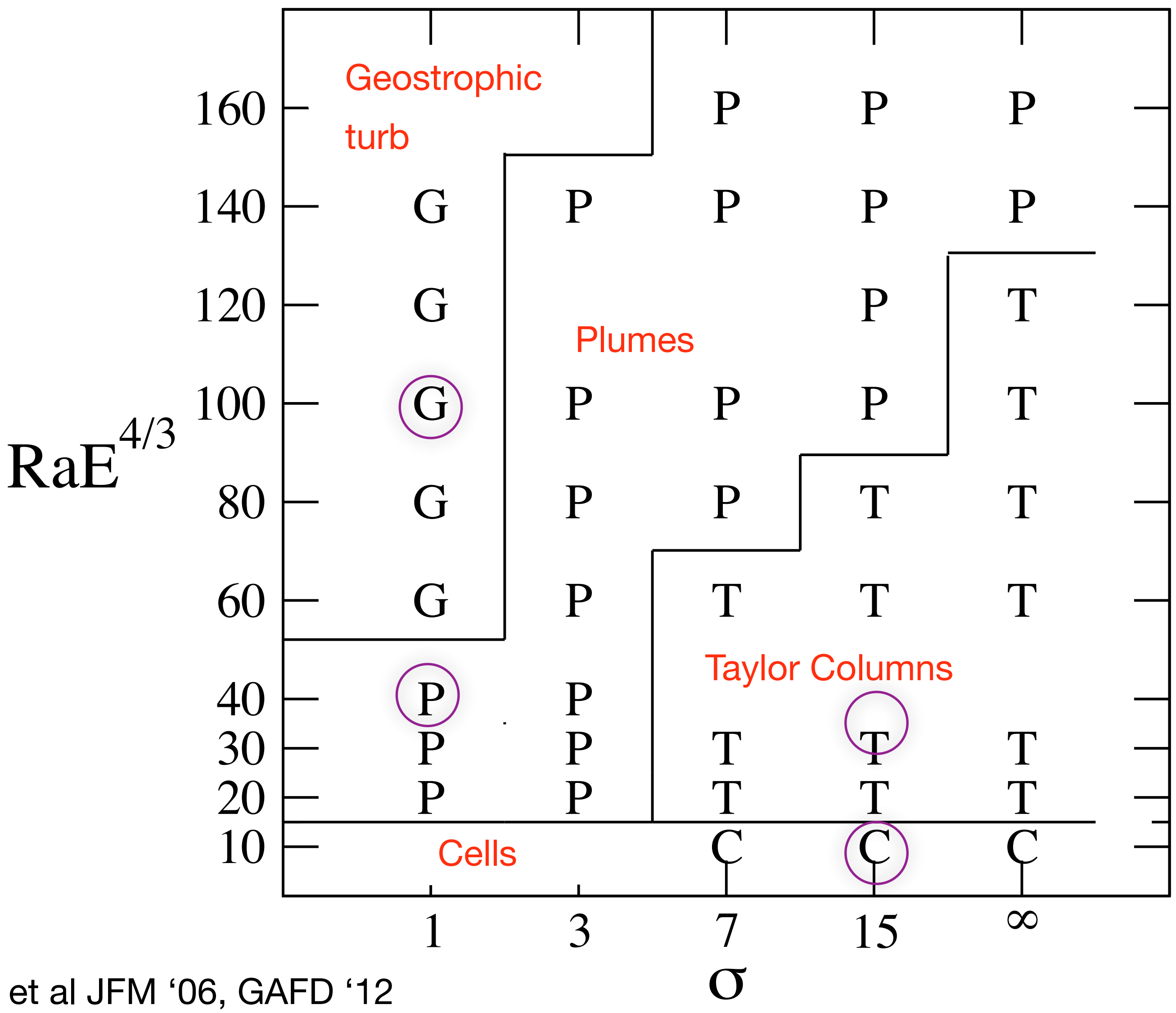


$$h(r) = J_0(k_\perp r) + iY_0(k_\perp r), \;\; k_\perp = |k_\perp| e^{i\alpha}$$

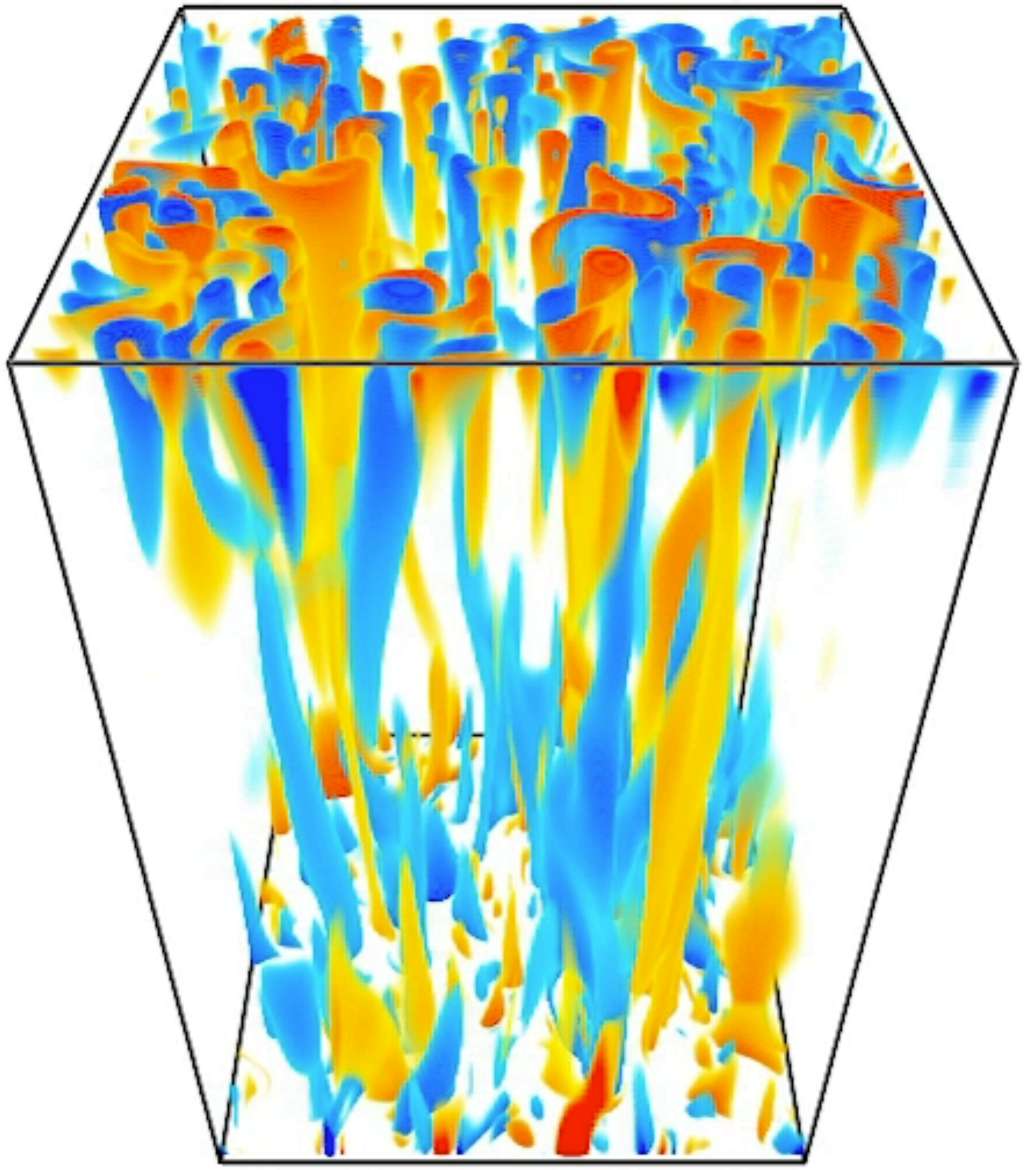
Grooms, KJ, Knobloch, Weiss, PRL 2010

Quasi-Geostrophic RBC Flow Regimes
Plume Regime

Courtesy M Calkins



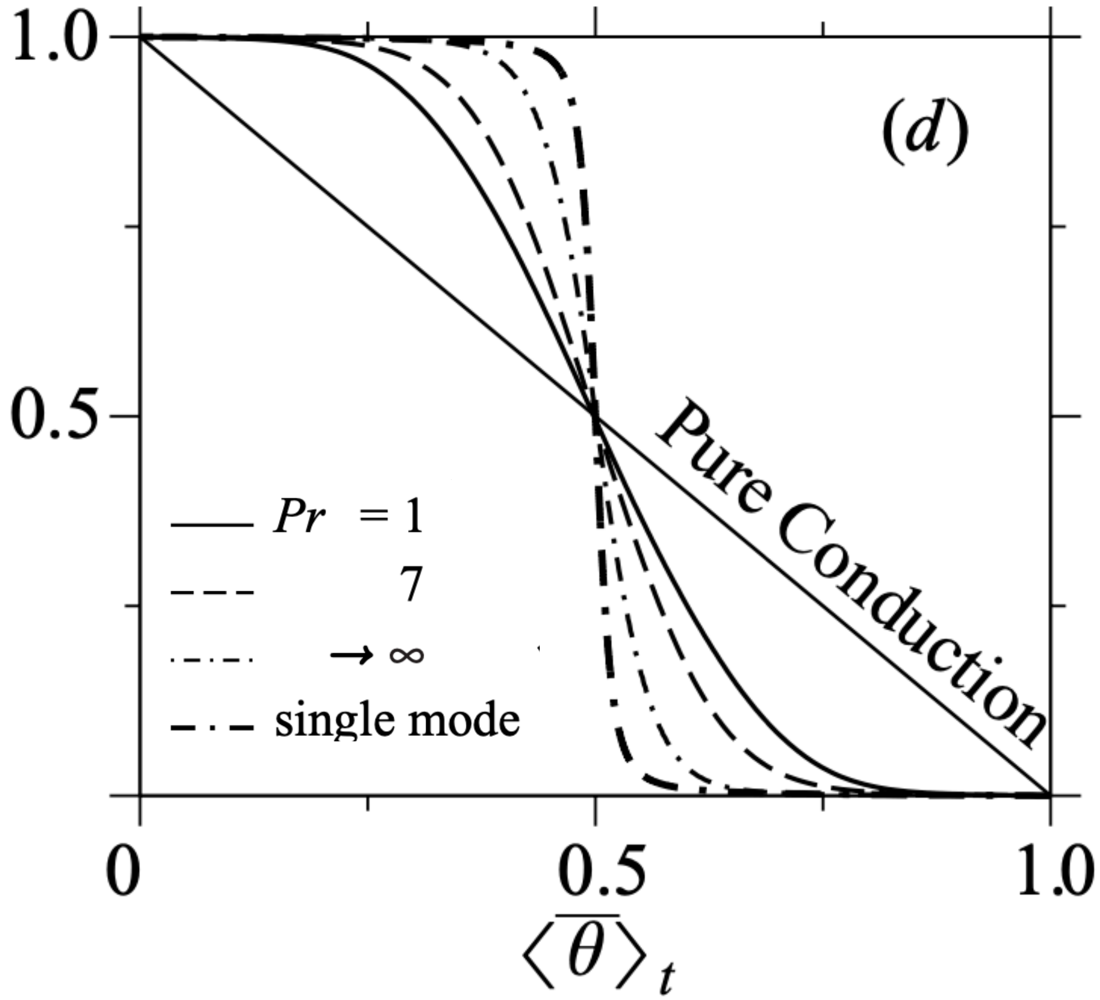
J. et al JFM '06, GAFD '12



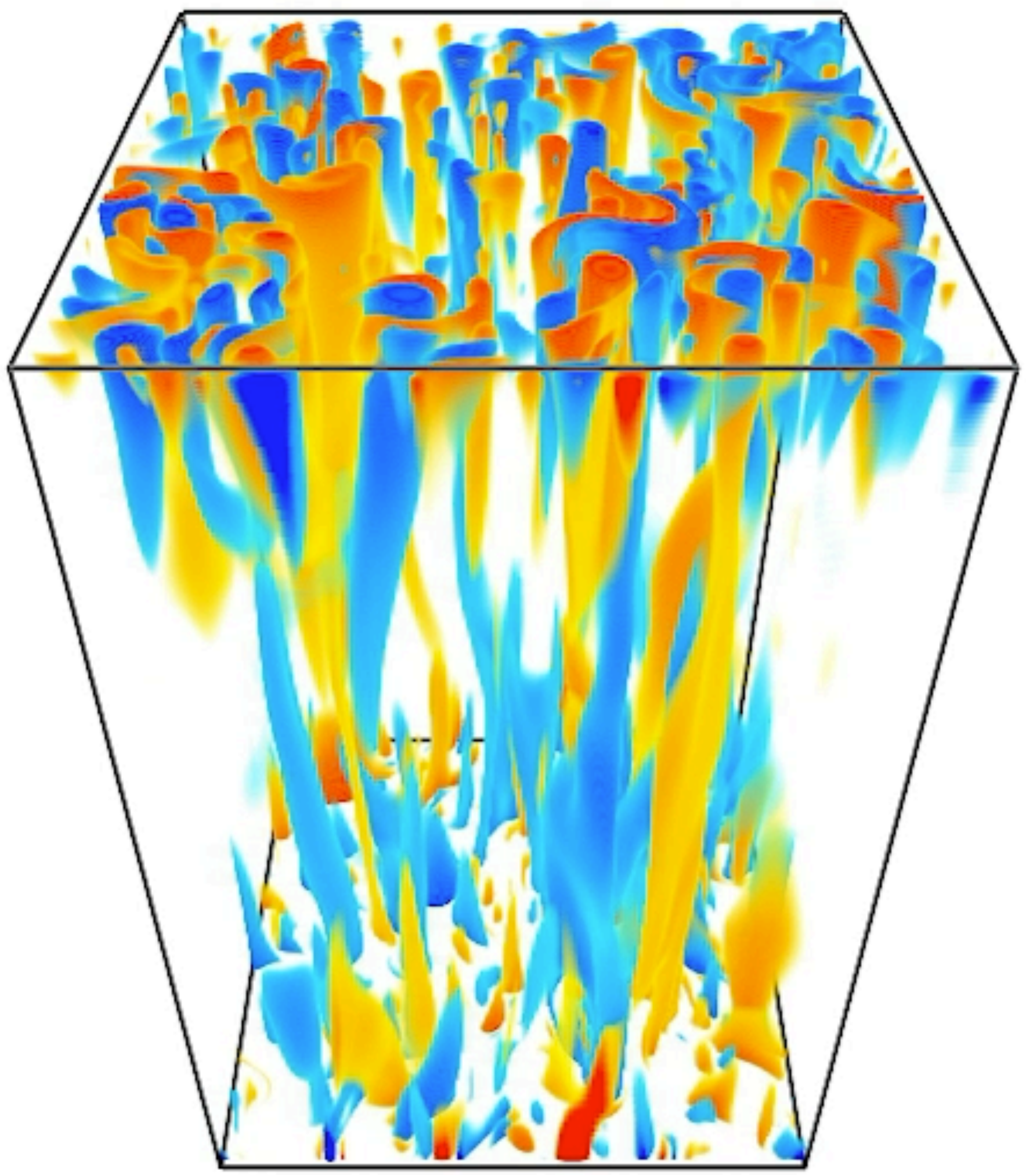
$$(RaE^{4/3} = 60, Pr = 2)$$

Quasi-Geostrophic RBC Flow Regimes
Plume Regime

Saturation of mean temperature gradient
lateral mixing



Courtesy M Calkins

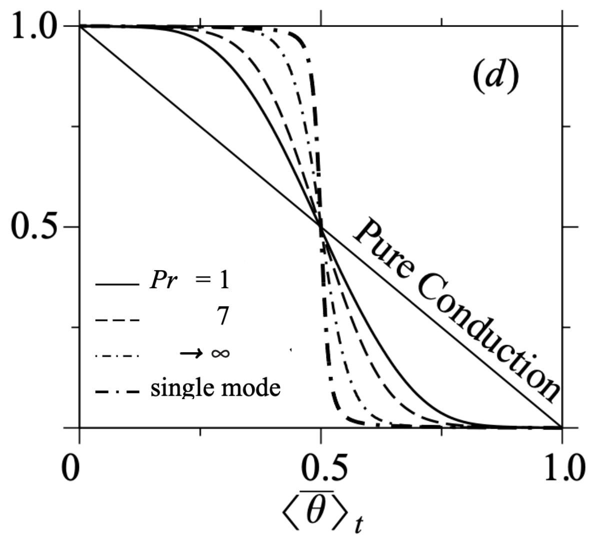


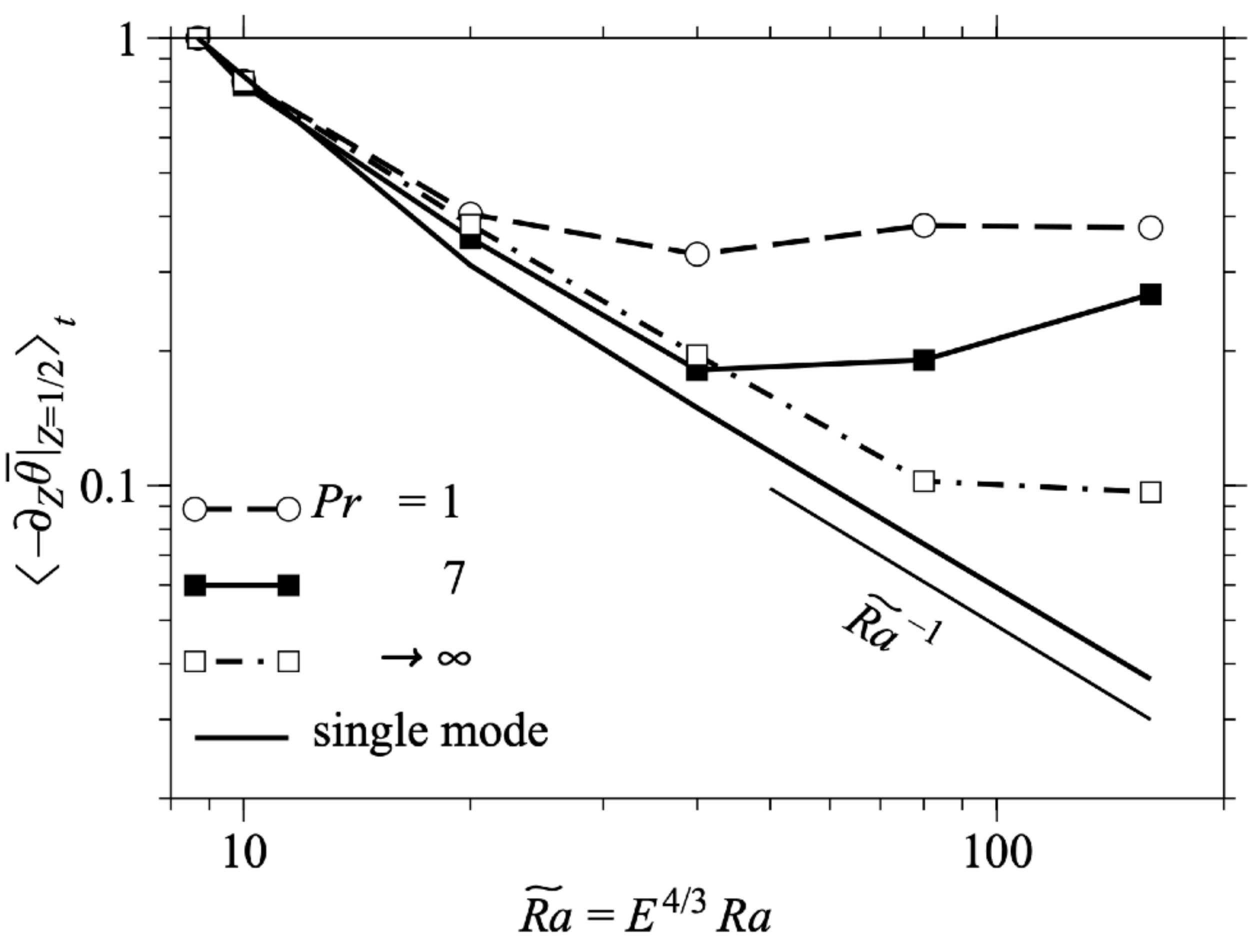
$$(RaE^{4/3} = 60, Pr = 2)$$

TBL's desynchronize

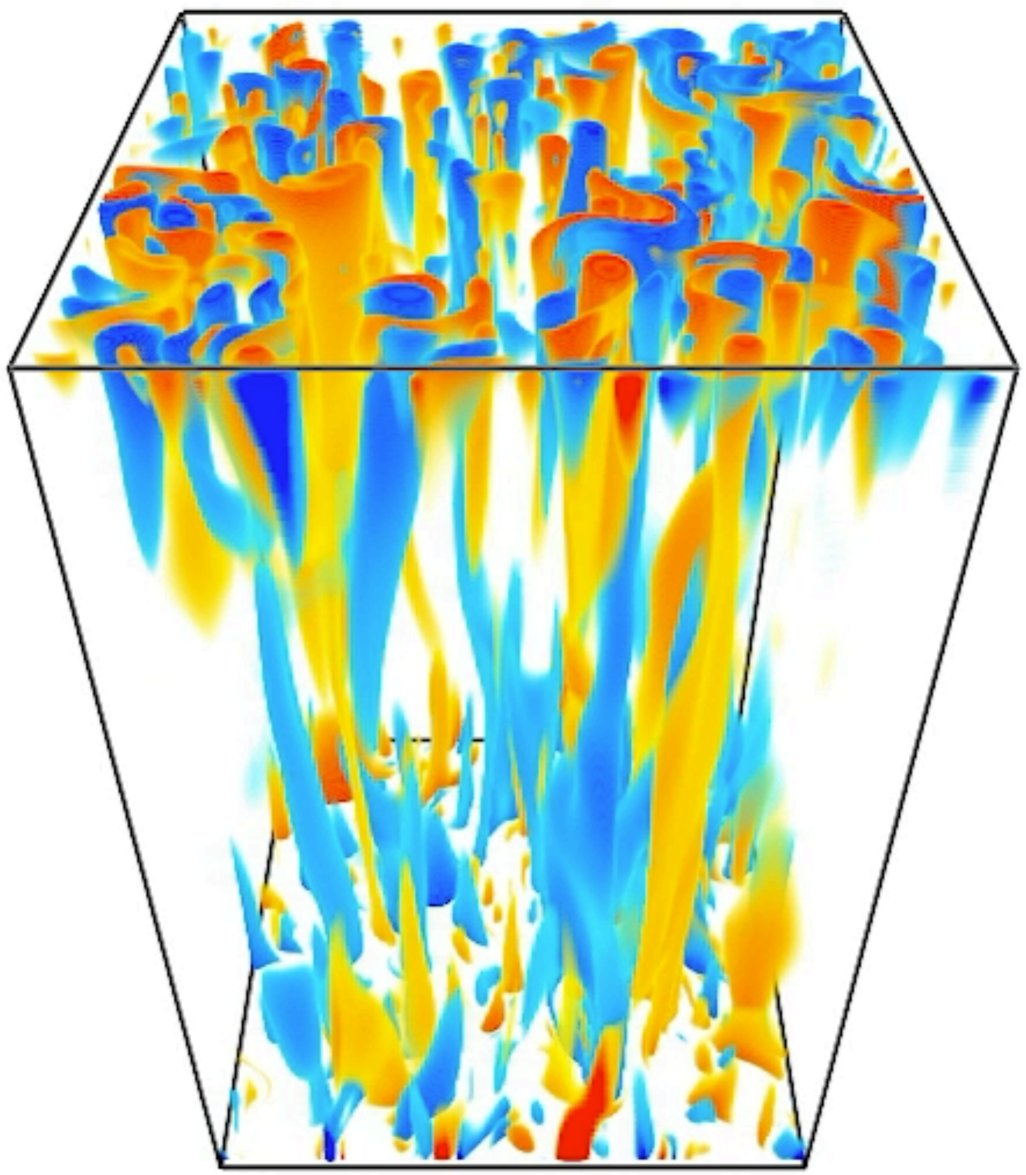
Quasi-Geostrophic RBC Flow Regimes
Plume Regime

Saturation of mean temperature gradient
lateral mixing





Courtesy M Calkins

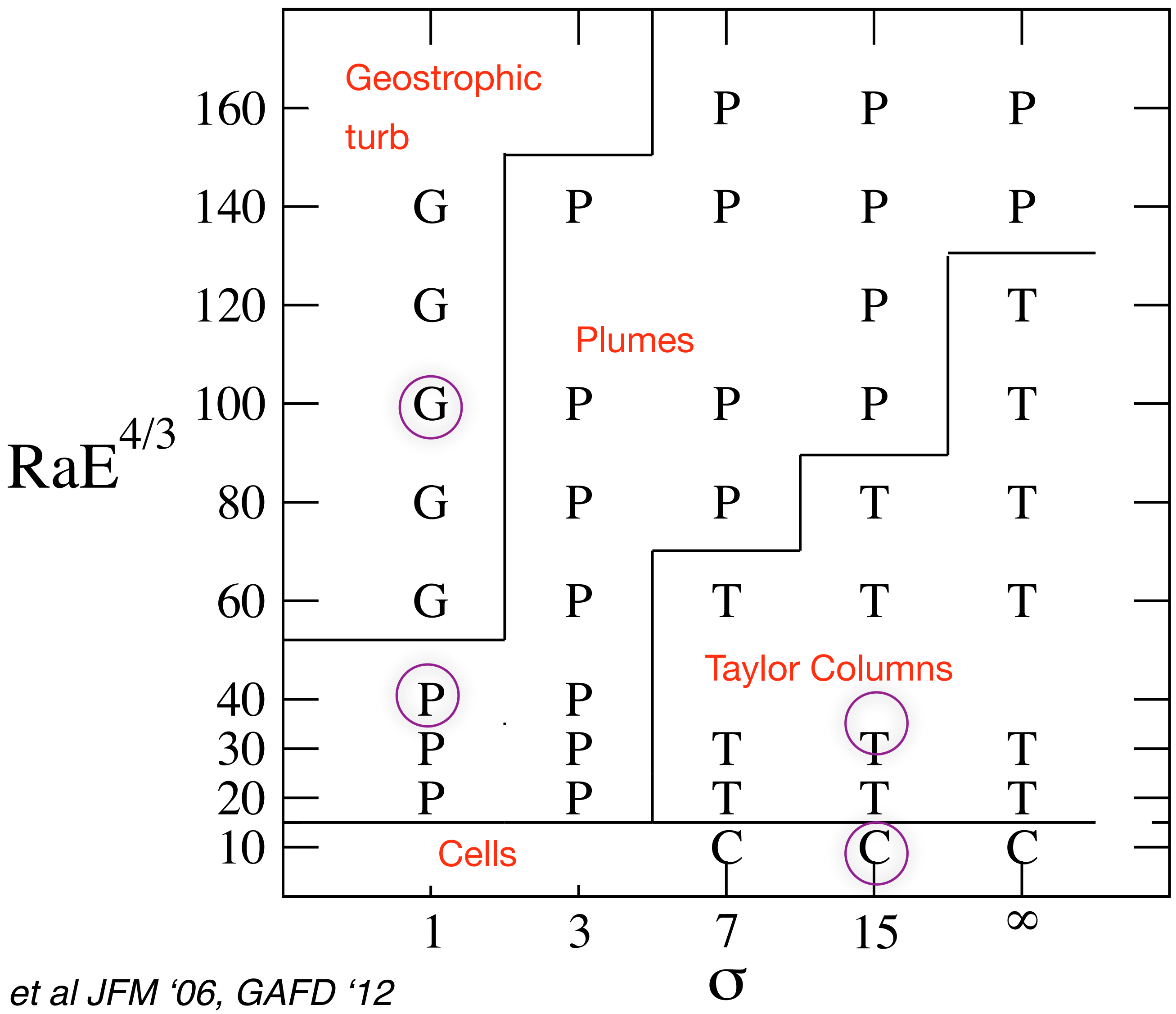


$$(RaE^{4/3} = 60, Pr = 2)$$

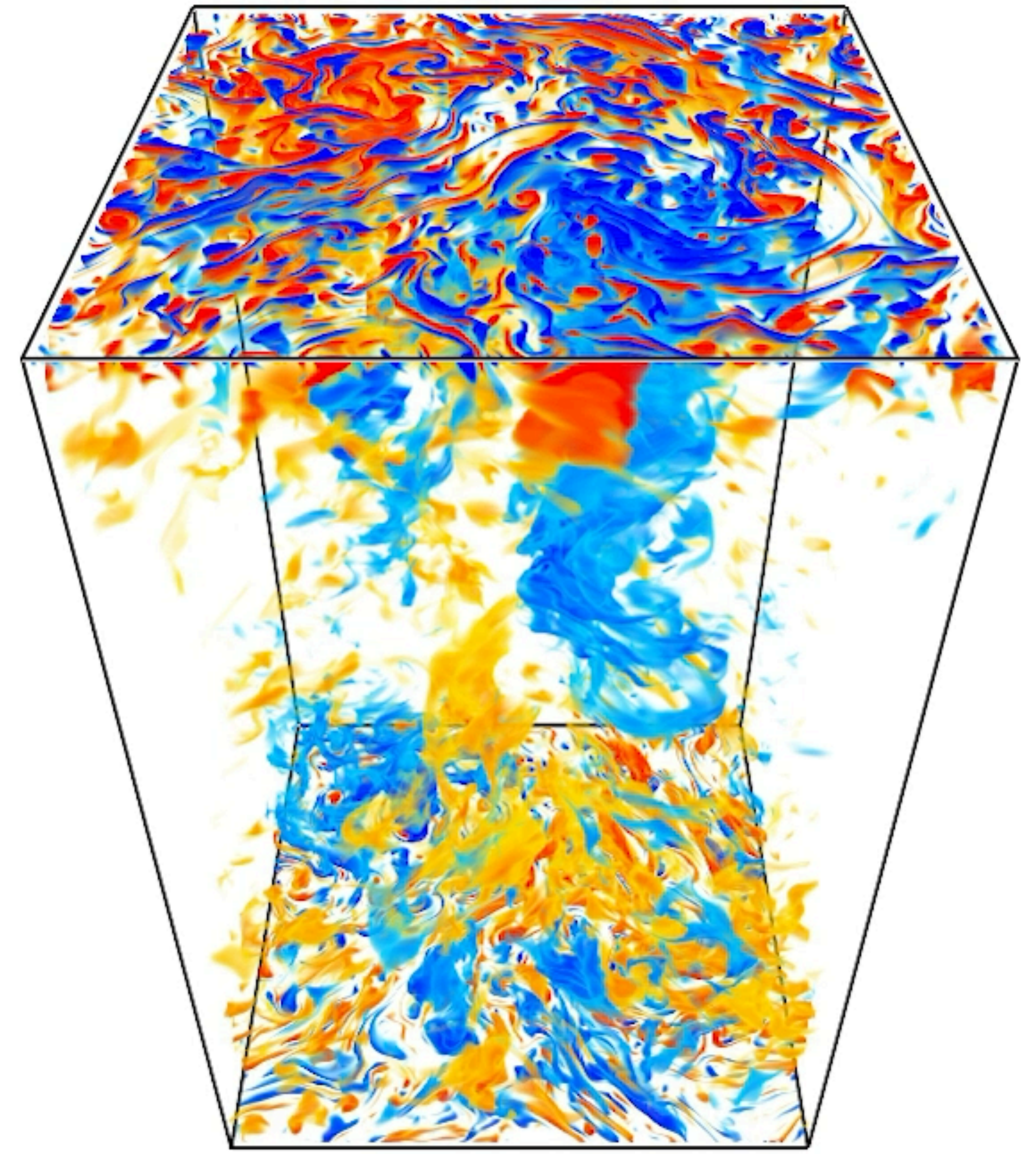
TBL's desynchronize

Quasi-Geostrophic RBC Flow Regimes
Geostrophic Turbulence

Courtesy M Calkins



J. et al JFM '06, GAFD '12

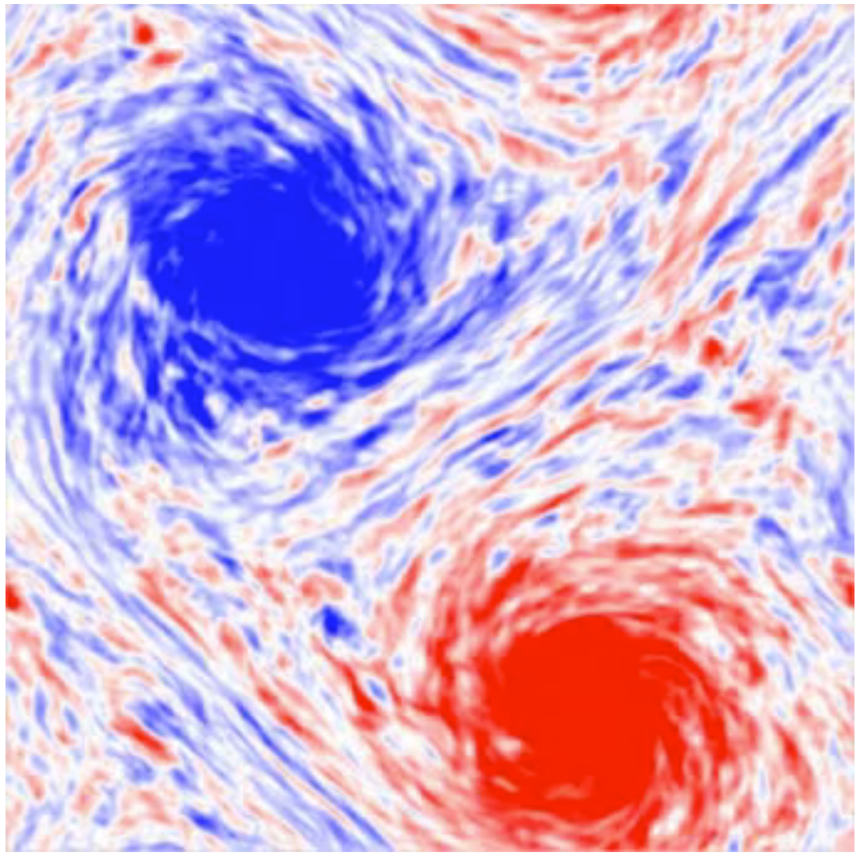


$$(RaE^{4/3} = 200, Pr = 1)$$

Maffei et al JFM 2021

Quasi-Geostrophic Inverse Cascade - Large Scale Vortices

Rubio et al PRL '14



Temperature anomaly: Ra $Ek^{-4/3}$=160, Pr=1

barotropic (depth averaged) - baroclinic decomposition

$$\psi = \langle \psi \rangle + \psi', \quad \mathbf{u}_{bt} = \nabla^{\perp} \langle \psi \rangle, \quad \mathbf{u}_{bc} = \nabla^{\perp} \psi'$$

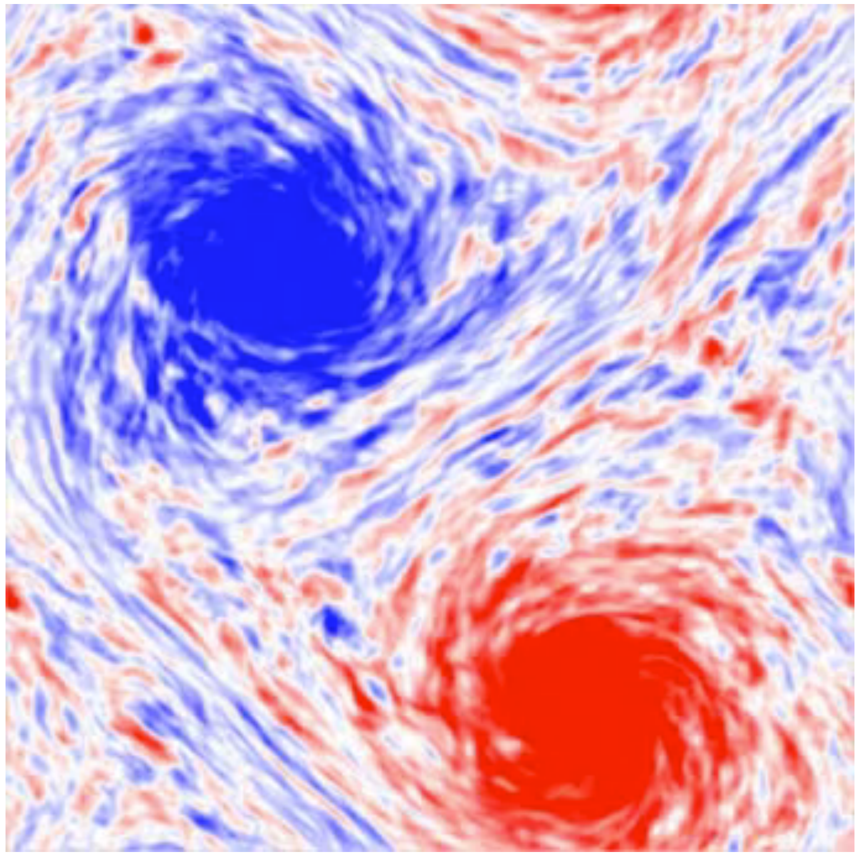
barotropic vorticity equation - baroclinically forced

$$\partial_t \langle \zeta \rangle = -J[\langle \psi \rangle, \langle \zeta \rangle] - \langle J[\psi', \zeta'] \rangle + \nabla_{\perp}^2 \langle \zeta \rangle$$

Dual cascade - inviscid conservation laws
area-averaged energy & enstrophy $\overline{|\nabla_{\perp} \langle \psi \rangle|^2}, \ \overline{(\nabla_{\perp}^2 \langle \psi \rangle)^2}$

Quasi-Geostrophic Inverse Cascade - Large Scale Vortices

Rubio et al PRL '14



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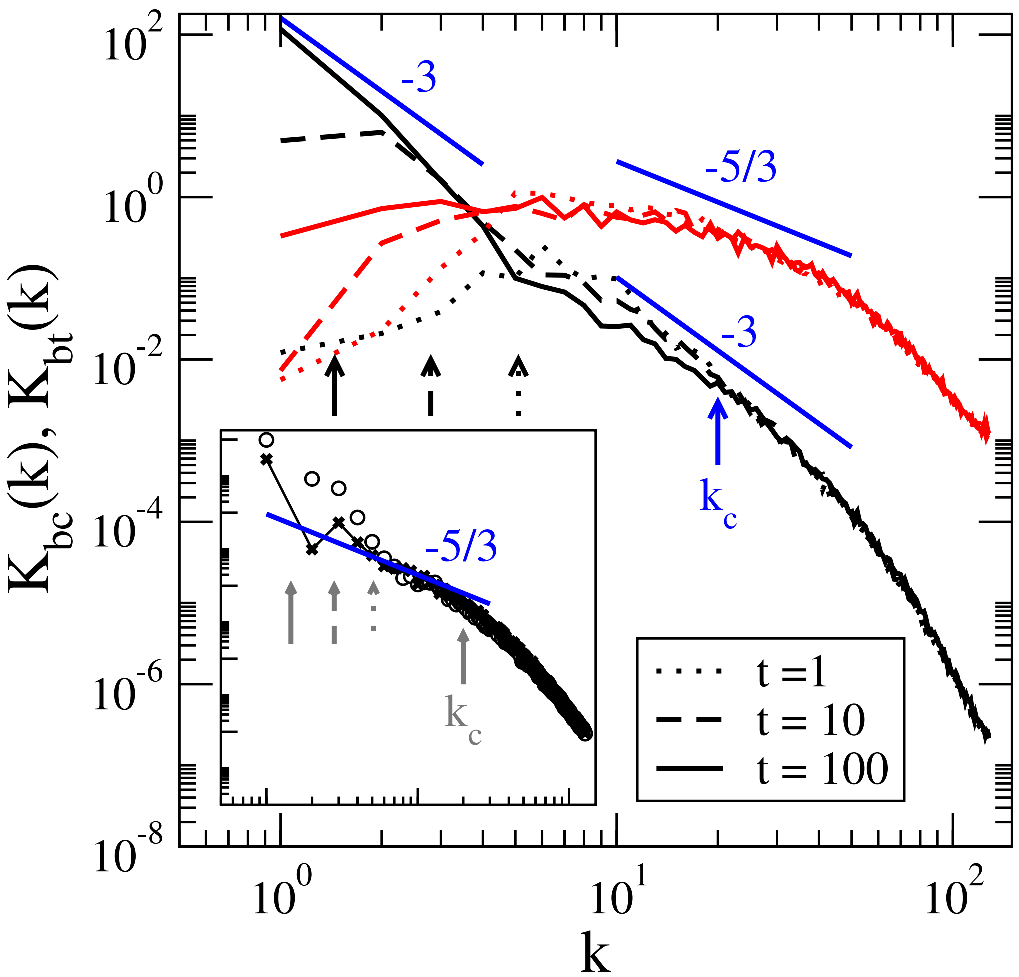
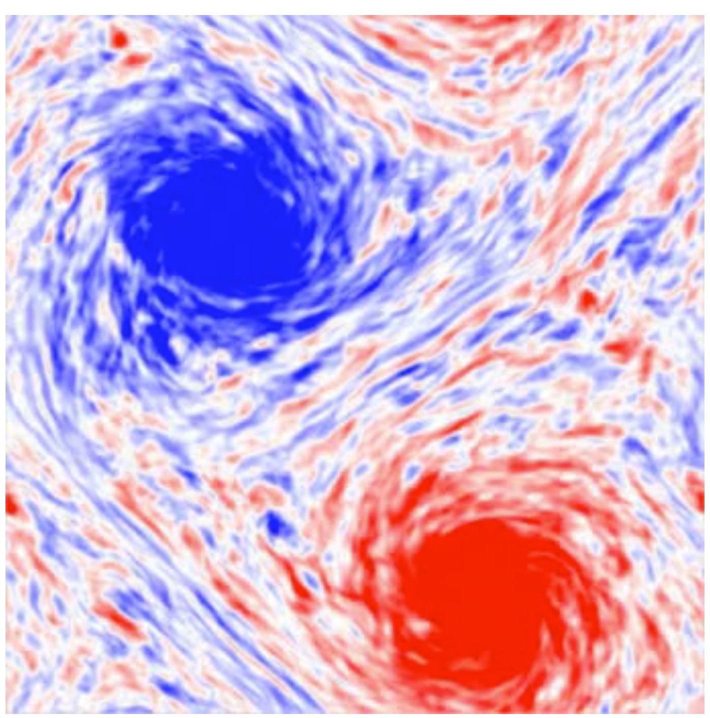
Quasi-Geostrophic Inverse Cascade - Large Scale Vortices

Rubio et al PRL '14





Temperature anomaly: Ra $Ek^{-4/3}$=160, Pr=1

barotropic (depth averaged) - baroclinic decomposition

$$\psi = \langle\psi\rangle + \psi', \quad \mathbf{u}_{bt} = \nabla^{\perp}\langle\psi\rangle, \quad \mathbf{u}_{bc} = \nabla^{\perp}\psi'$$

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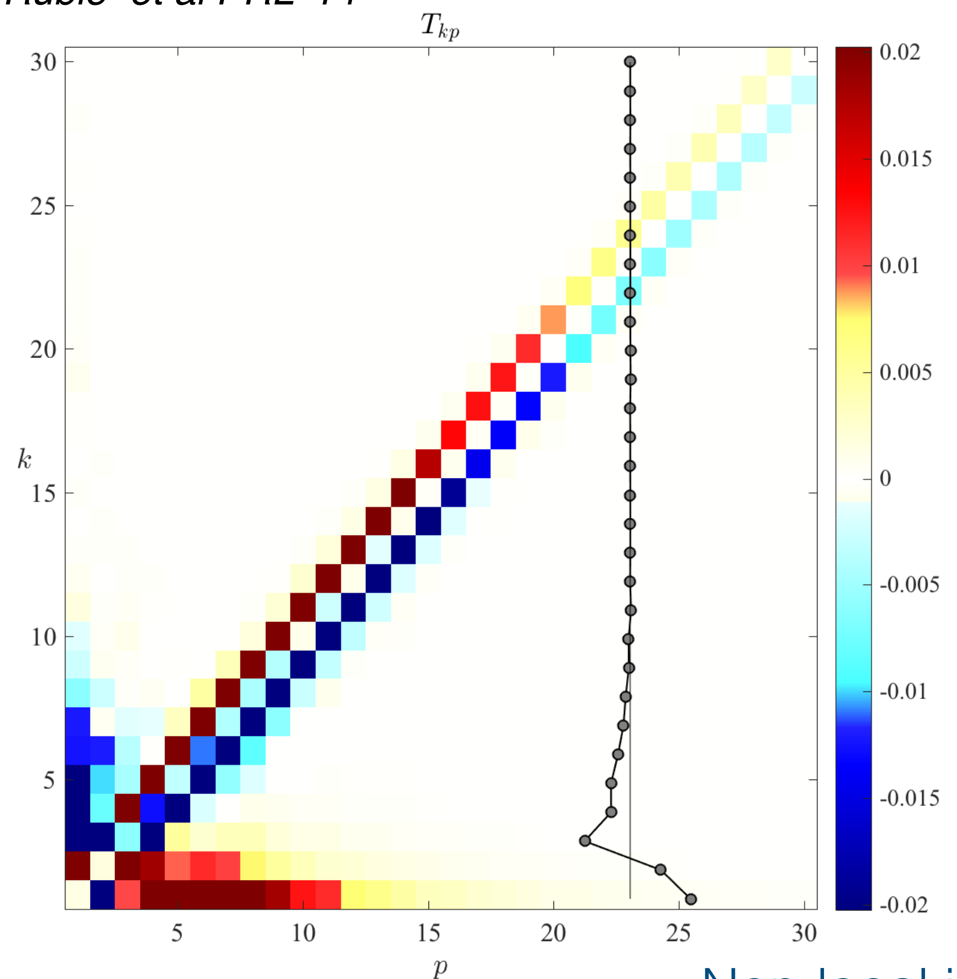
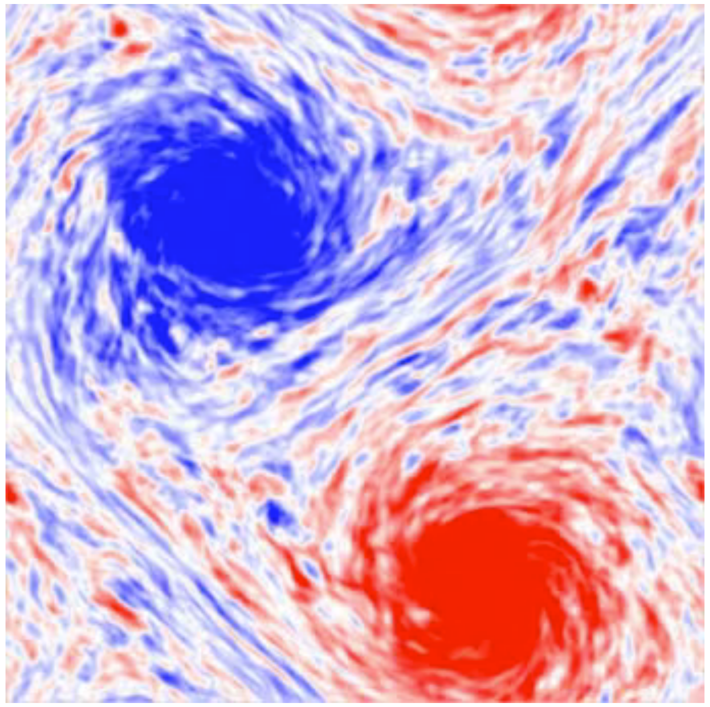
Quasi-Geostrophic Inverse Cascade - Large Scale Vortices

Rubio et al PRL '14



Non-local inverse cascade



Temperature anomaly: Ra $Ek^{-4/3}$=160, Pr=1

barotropic (depth averaged) - baroclinic decomposition

$$\psi = \langle\psi\rangle + \psi', \quad \mathbf{u}_{bt} = \nabla^{\perp}\langle\psi\rangle, \quad \mathbf{u}_{bc} = \nabla^{\perp}\psi'$$

barotropic vorticity equation - baroclinically forced

$$\partial_t\langle\zeta\rangle = -J[\langle\psi\rangle, \langle\zeta\rangle] - \langle J[\psi', \zeta']\rangle + \nabla_{\perp}^2\langle\zeta\rangle$$

Dual cascade - inviscid conservation laws

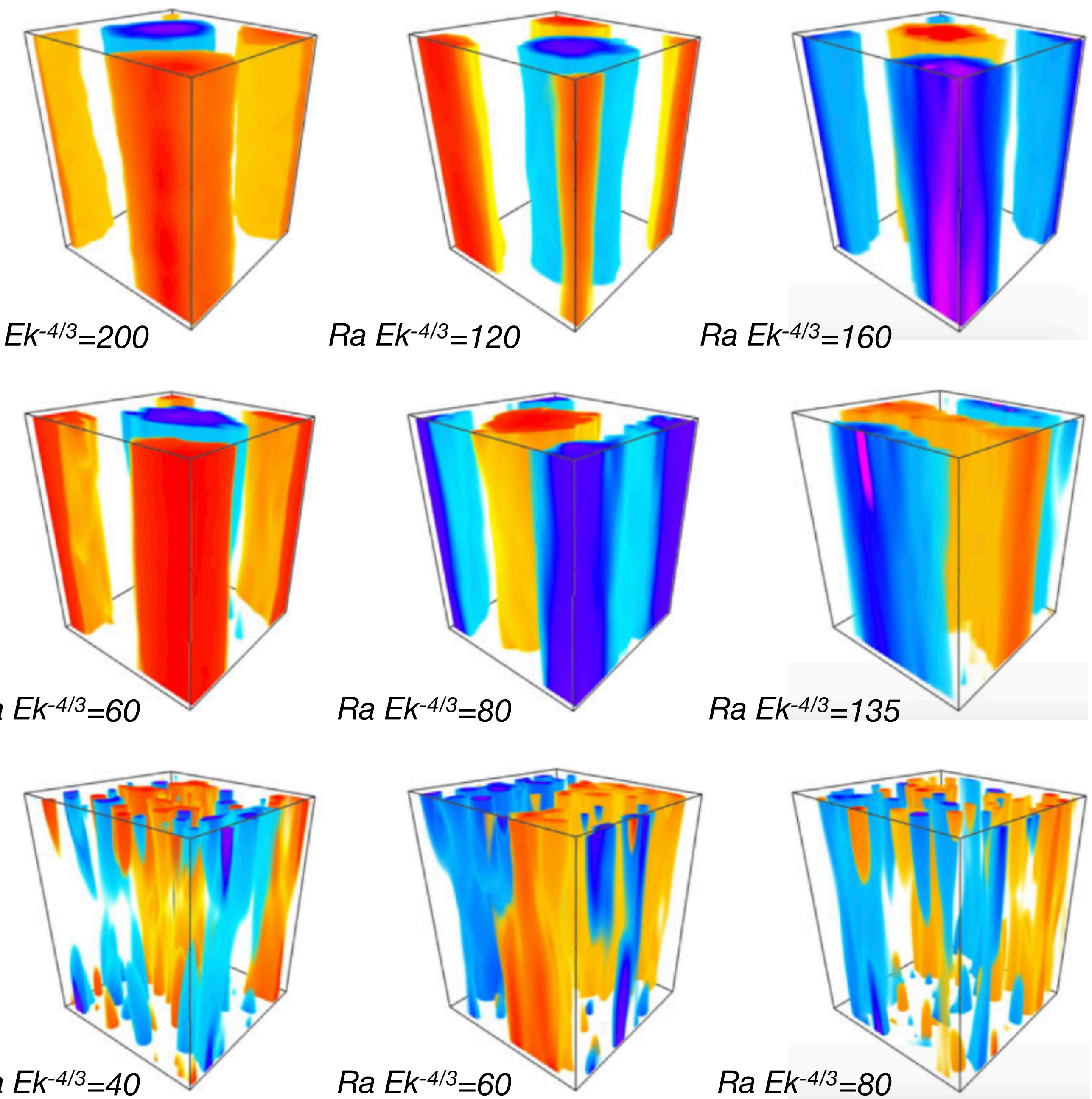
area-averaged energy & enstrophy $\overline{|\nabla_{\perp}\langle\psi\rangle|^2}, \ \overline{(\nabla_{\perp}^2\langle\psi\rangle)^2}$

$$\partial_t KE_{bt}(k_{\perp}) = T(k_{\perp}) + F(k_{\perp}) + D(k_{\perp})$$

Energy (upscale) & Enstrophy (downscale)

Quasi-Geostrophic Inverse Cascade - Large Scale Vortices

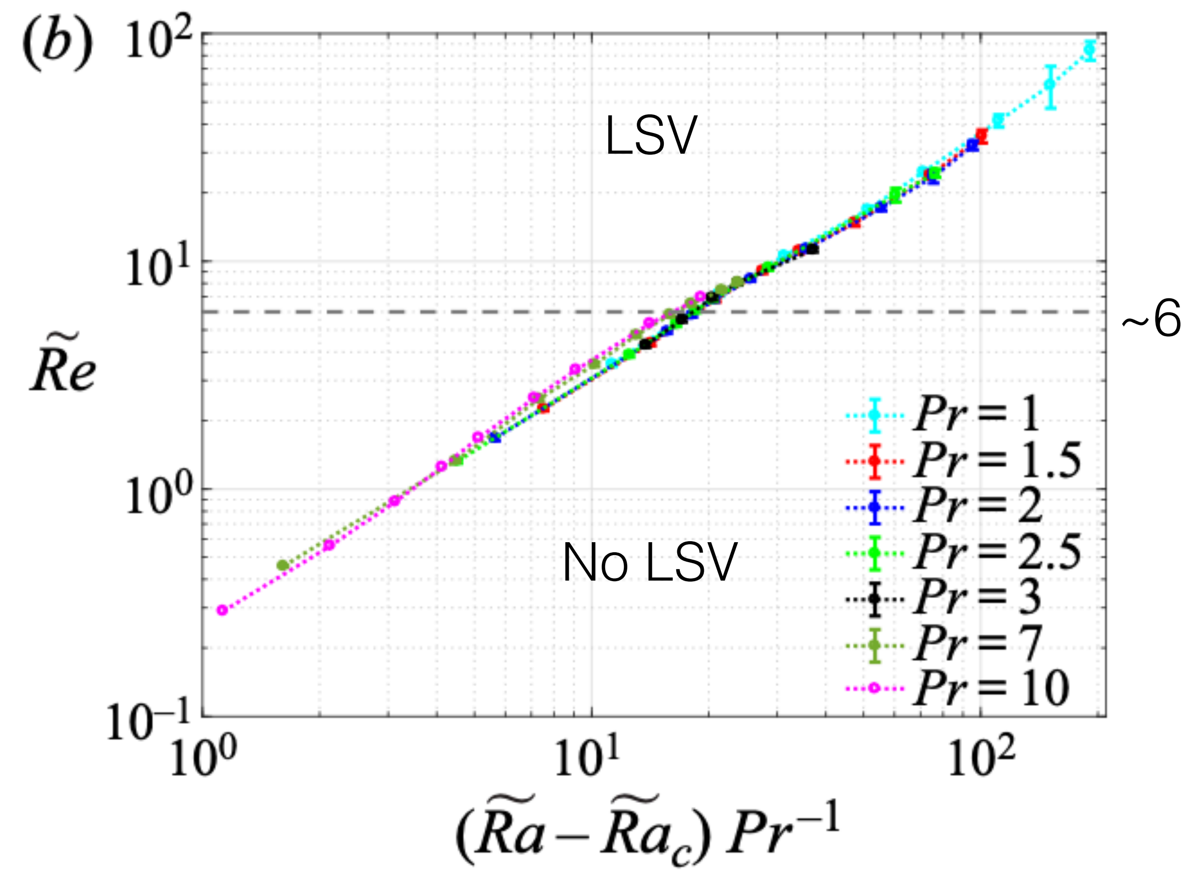
Maffei et al JFM 2020



Pr=2 *Pr*=3 *Pr*=7

ψ — Geostrophic Streamfunction

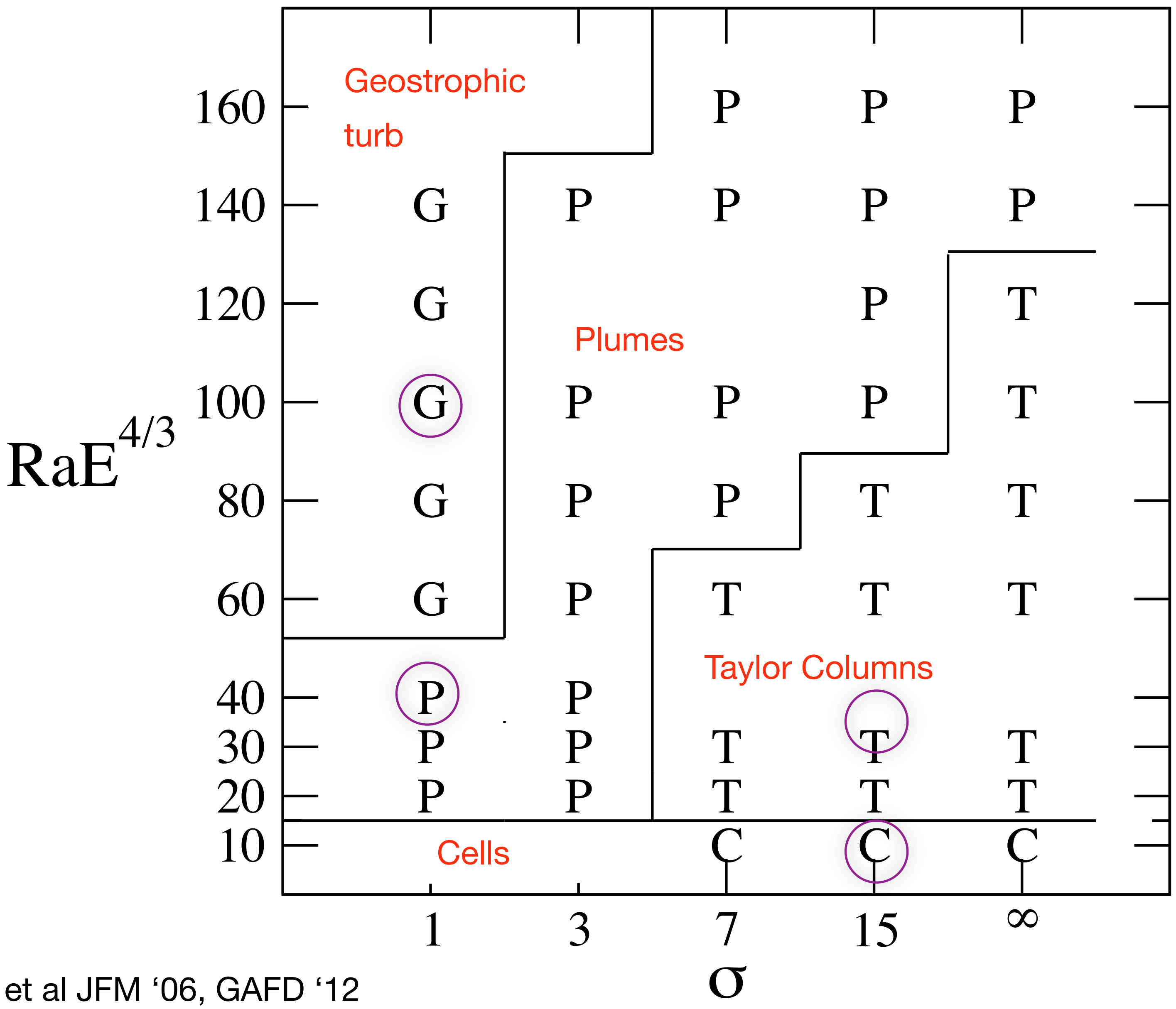
Maffei et al JFM 2020



LSV prevalent for all *Pr* when *Re* > 6

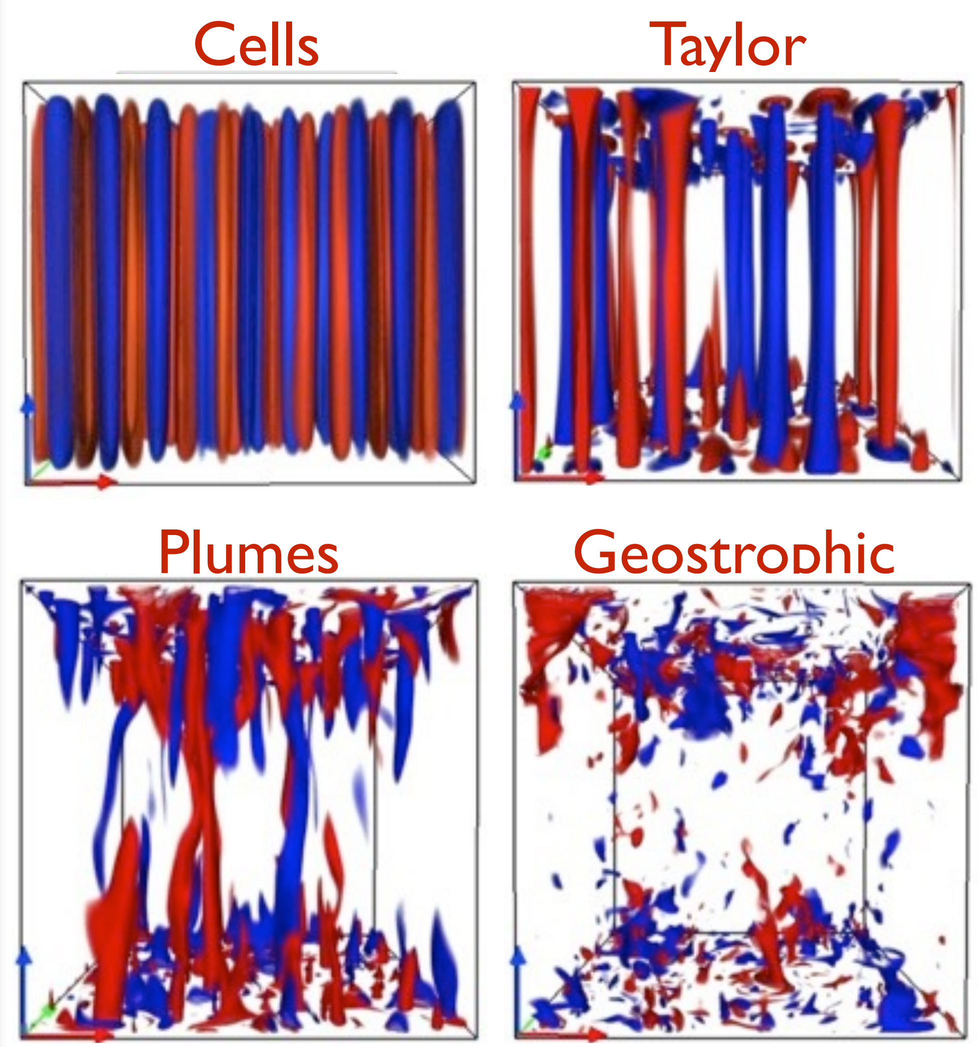
Quasi-Geostrophic RBC Flow Regimes by DNS

Mapping Parameter Space



J. et al JFM '06, GAFD '12

DNS @ $E = 10^{-7}$



Stellmach ..,KJ,.., Aurnou, PRL '14

Quasi-Geostrophic Inverse Cascade - Large Scale Vortices

Observations in the Lab?

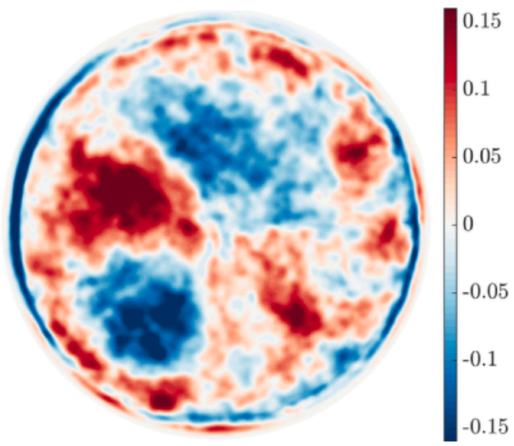
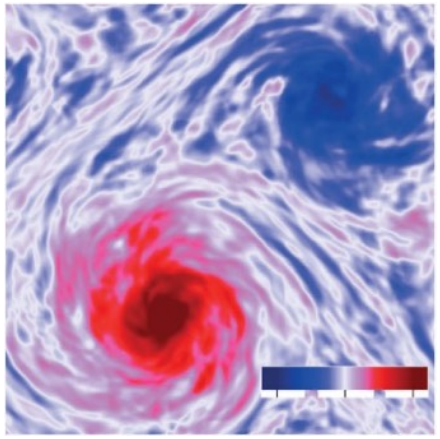


Fig. 4: Orientation-compensated mean vorticity field (in 1/s) at $Ra/Ra_C = 47$.

Madonia et al EPL 2021
APS DFD 2021 P06.00009 Flow measurements of turbulent rotating Rayleigh-Bénard convection in the geostrophic regime

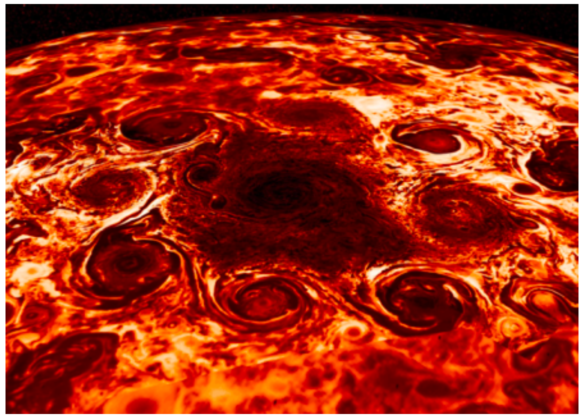
Direct Numerical Simulations?



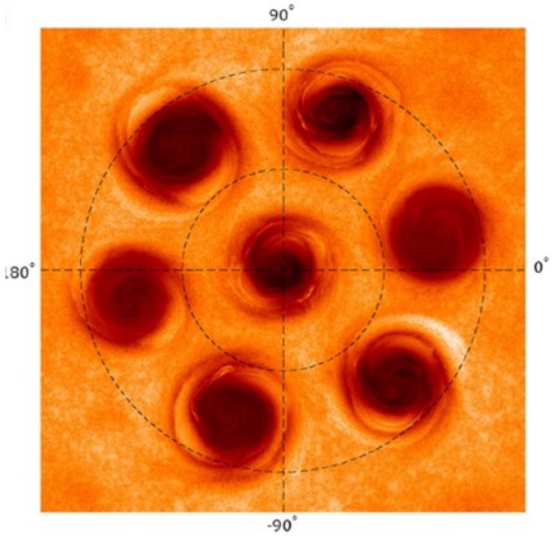
Stellmach,..,KJ. et al PRL 2014

Stellmach,..,KJ. et al PRL 2014,
Favier et al PoF 2014,
Guervilly et al JFM 2014
Favier & Knobloch JFM 2019
Guzman et al PRL 2020

GAFD ?

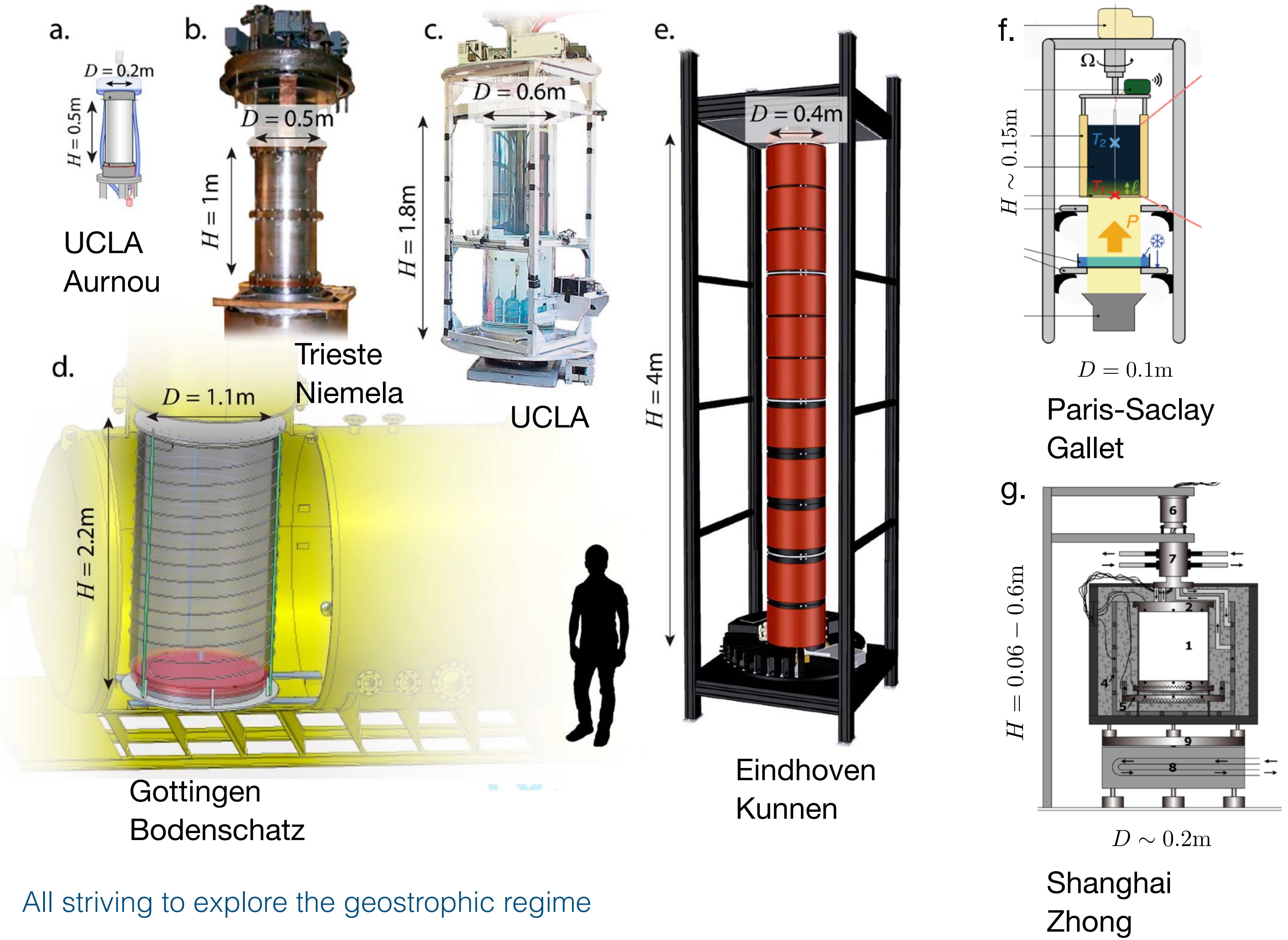


NASA/JPL-Caltech/SwRI/ASI/INAF/JIRAM)



Rotating Convection on γ-plane
DNS - Cai et al PSJ 2021,

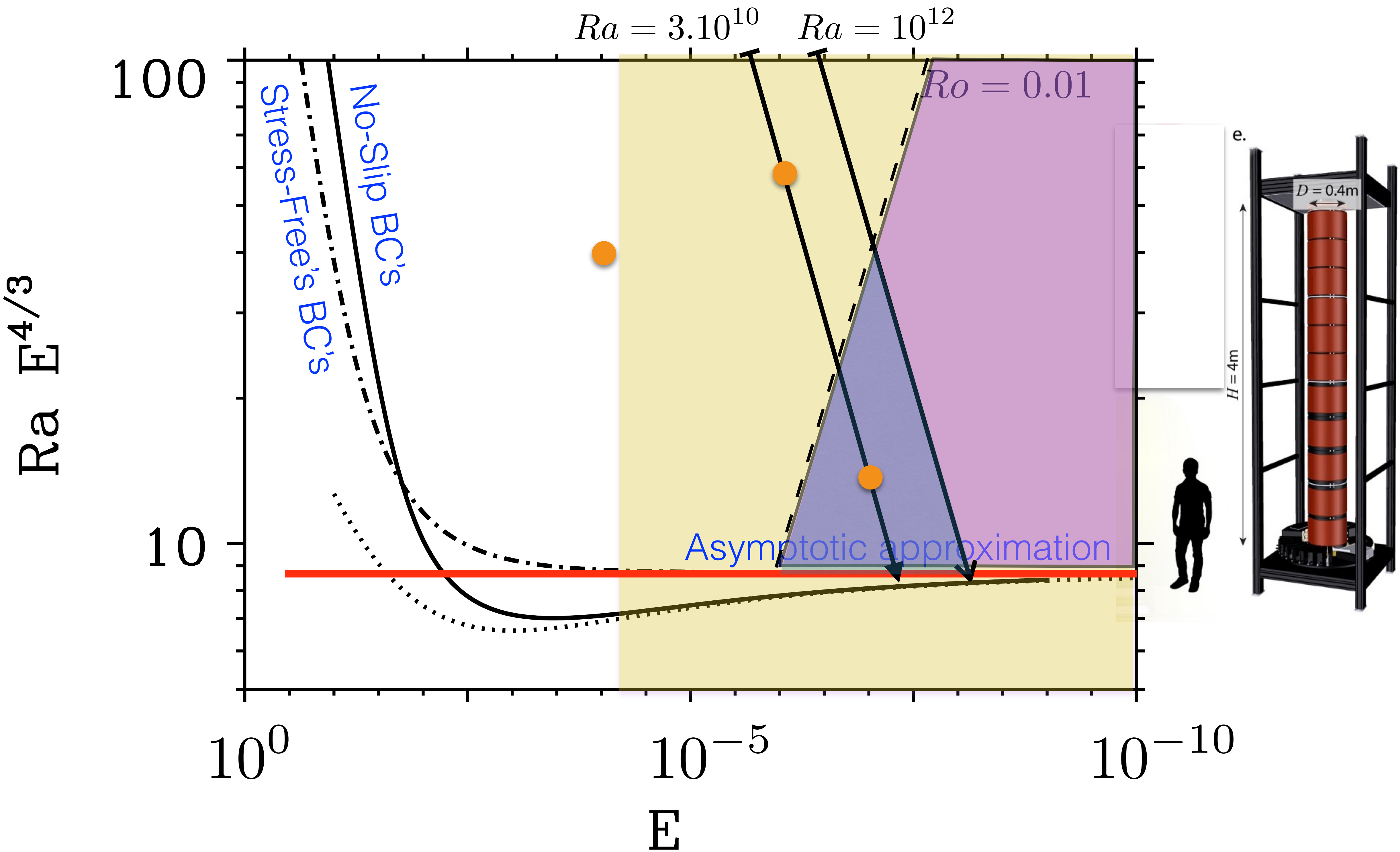
On-going Laboratory Experiments



All striving to explore the geostrophic regime

Ecke & Niemela PRL 2014

Geostrophic Regime - DNS & Lab. Experiments

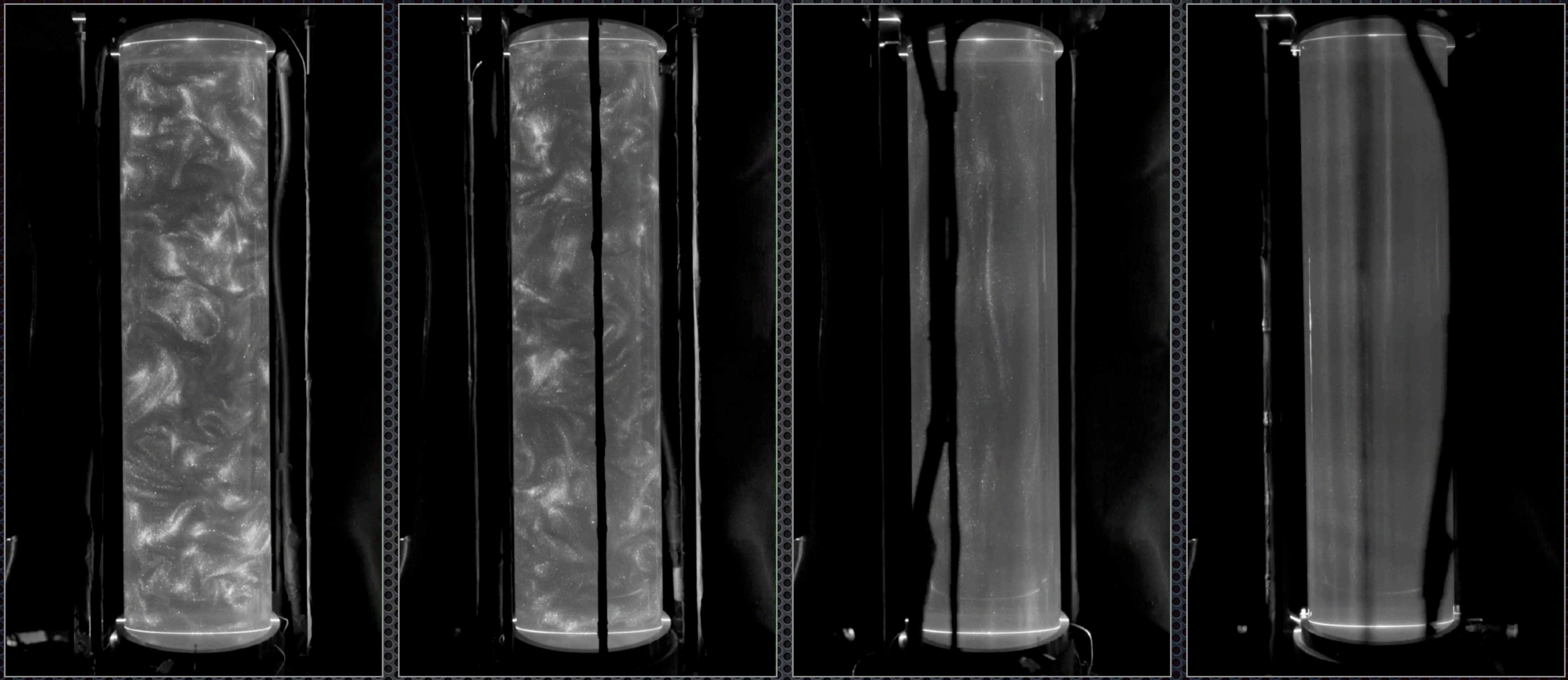


Extension of Reduced Model to No-Slip

M. Plumley, KJ et al JFM 2016

KJ. et al JFM 2016

RoMag: 80 cm tanks



E = inf.
$Ra \sim 2.7e10$
Ro_C = inf.

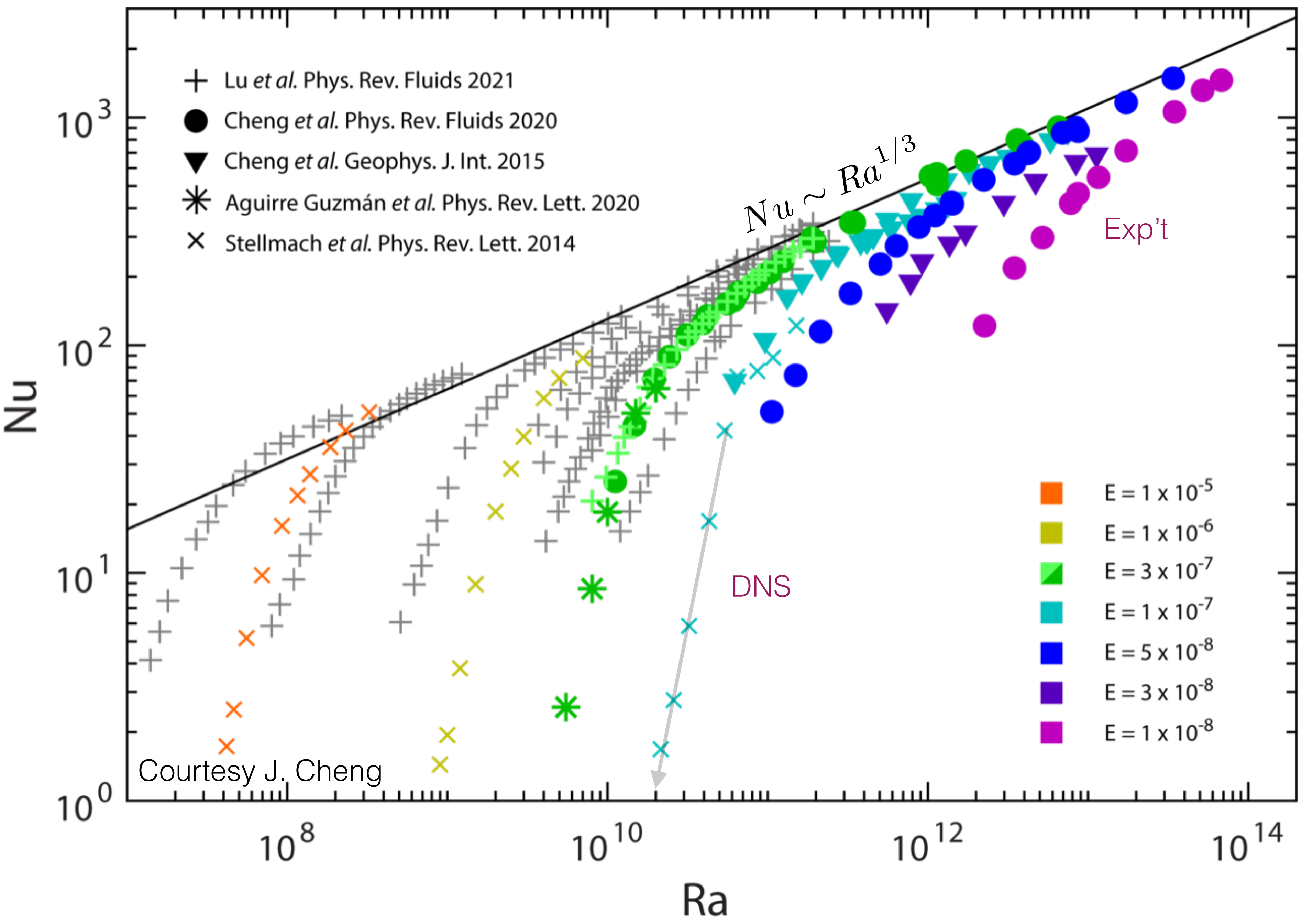
$E = 1.5e\text{-}5$
$Ra \sim 2.9e10$
$Ro_C \sim 0.97$
$RaE^{4/3}=1.1e3$

$E = 6.0e\text{-}7$
$Ra \sim 3.0e10$
$Ro_C \sim 0.039$
$RaE^{4/3}=152$

$E = 1.0e\text{-}7$
$Ra \sim 5.3e10$
$Ro_C \sim 0.0087$
$RaE^{4/3}=25$

Cheng et al. *GJI* 2015

Limitations as view by Heat Transport



$$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left(\frac{Ra}{Ra_c}\right)^{\beta}$$

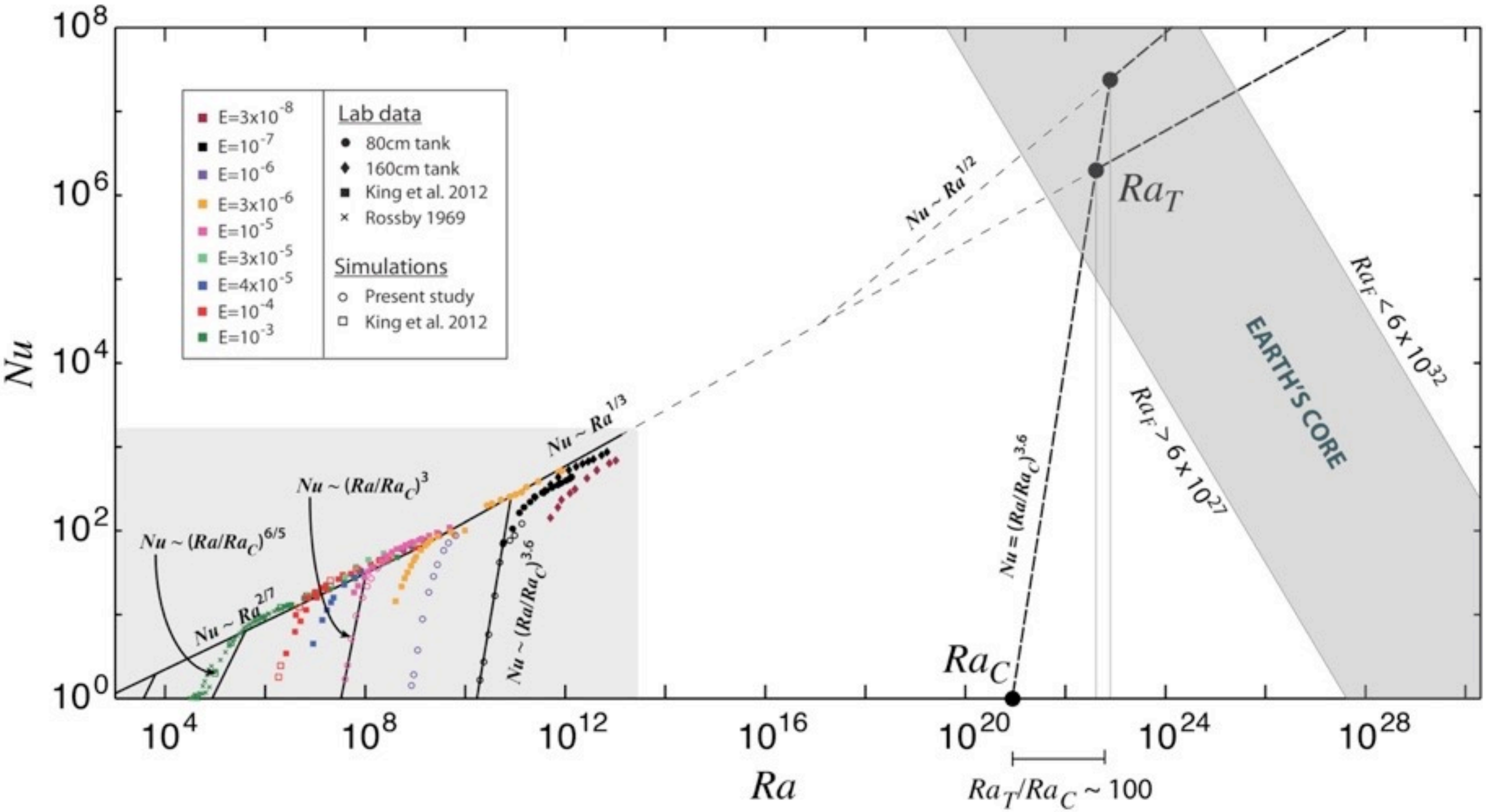
$\beta_\Omega > 1$

$\beta_0 \approx 1/3$

Characterization - Heat Transport
Low *Ro* branch characterized by steep branch in *Nu*-*Ra* space

$$Ro_{crit} \sim E^{1/3} < Ro < 1, \quad Ra \sim E^{-4//3}$$

UCLA Spin-Lab: *King et al Nature 2009*
Aurnou et al PEPI 2015



$$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left(\frac{Ra}{Ra_c}\right)^{\beta}$$

$$\beta_{rot} > 1$$

$$\beta_{norot} = 1/3 \to 1/2$$

Characterization - Heat Transport
Low *Ro* branch characterized by steep branch in *Nu*-*Ra* space

$Ro_{crit} \sim E^{1/3} < Ro < 1, \quad Ra \sim E^{-4//3}$

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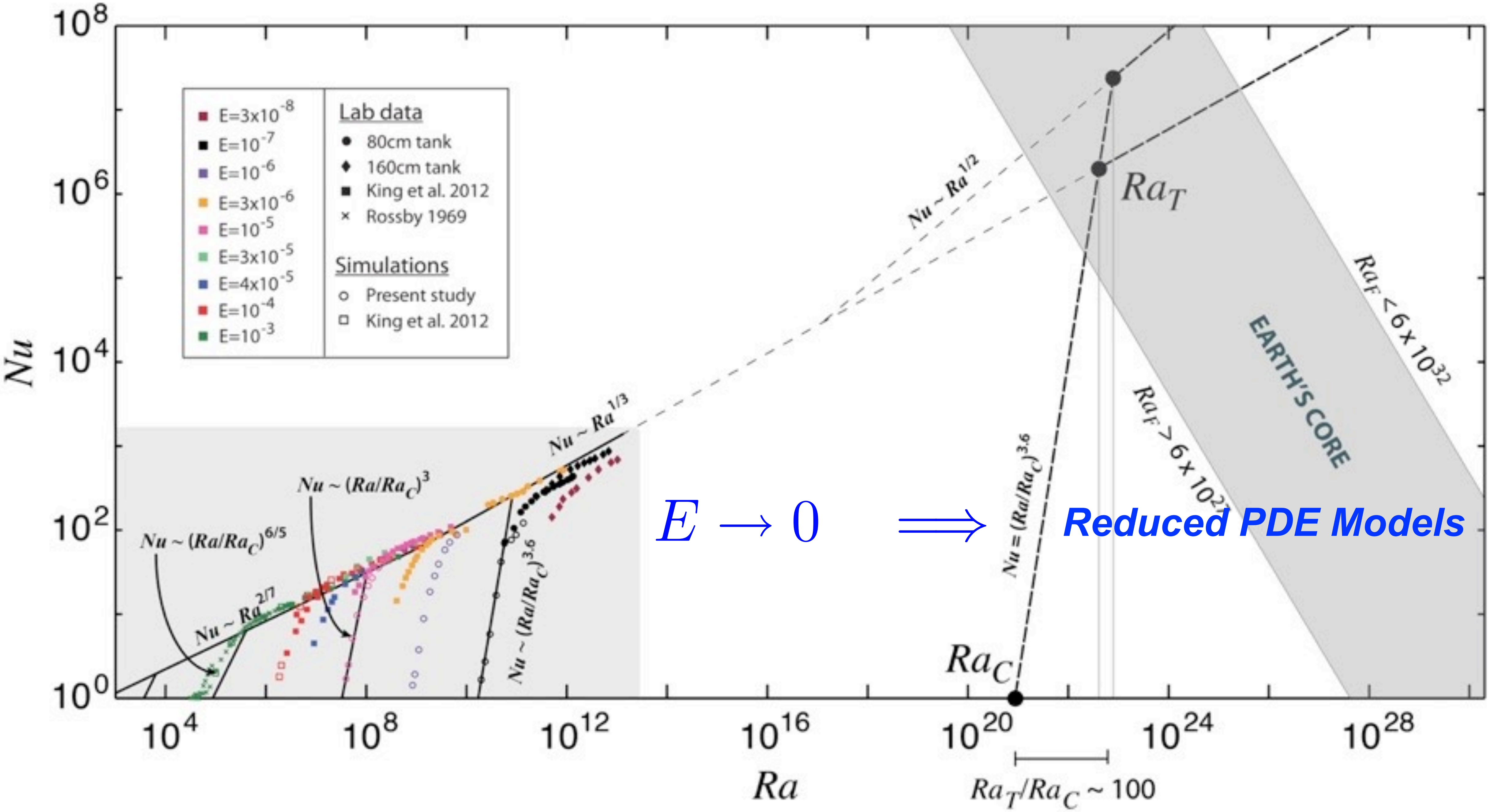


$E \to 0 \quad \Longrightarrow$ ***Reduced PDE Models***

$$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left(\frac{Ra}{Ra_c}\right)^{\beta}$$

$\beta_{rot} > 1$

$\beta_{norot} = 1/3 \to 1/2$

Turbulent Heat Transport

$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T} = \sigma \overline{w\theta} - \partial_z \overline{T}$$

$$Nu - 1 = C(\sigma)\left(Ra E^{4/3}\right)^{\alpha}$$

$$Q \propto \left(\frac{\kappa \Delta T}{H}\right)\left(\frac{\nu}{\kappa}\right)^{\gamma}\left(\frac{g\alpha_T \Delta T H^3}{\kappa\nu}\right)^{\alpha}\left(\frac{\nu}{2\Omega H^2}\right)^{4\alpha/3}$$

Boundary Layer vs Turbulent control

- marginally stable tbl's (Malkus, '54):
- depth independence (Priestley, '59):

$$\alpha = 3$$

- ultimate (dissipation-free) turbulent law (Kraichnan '63, Howard '63, Spiegel '71):

$$\alpha = \frac{3}{2}$$

$$\gamma = -\frac{1}{2}$$

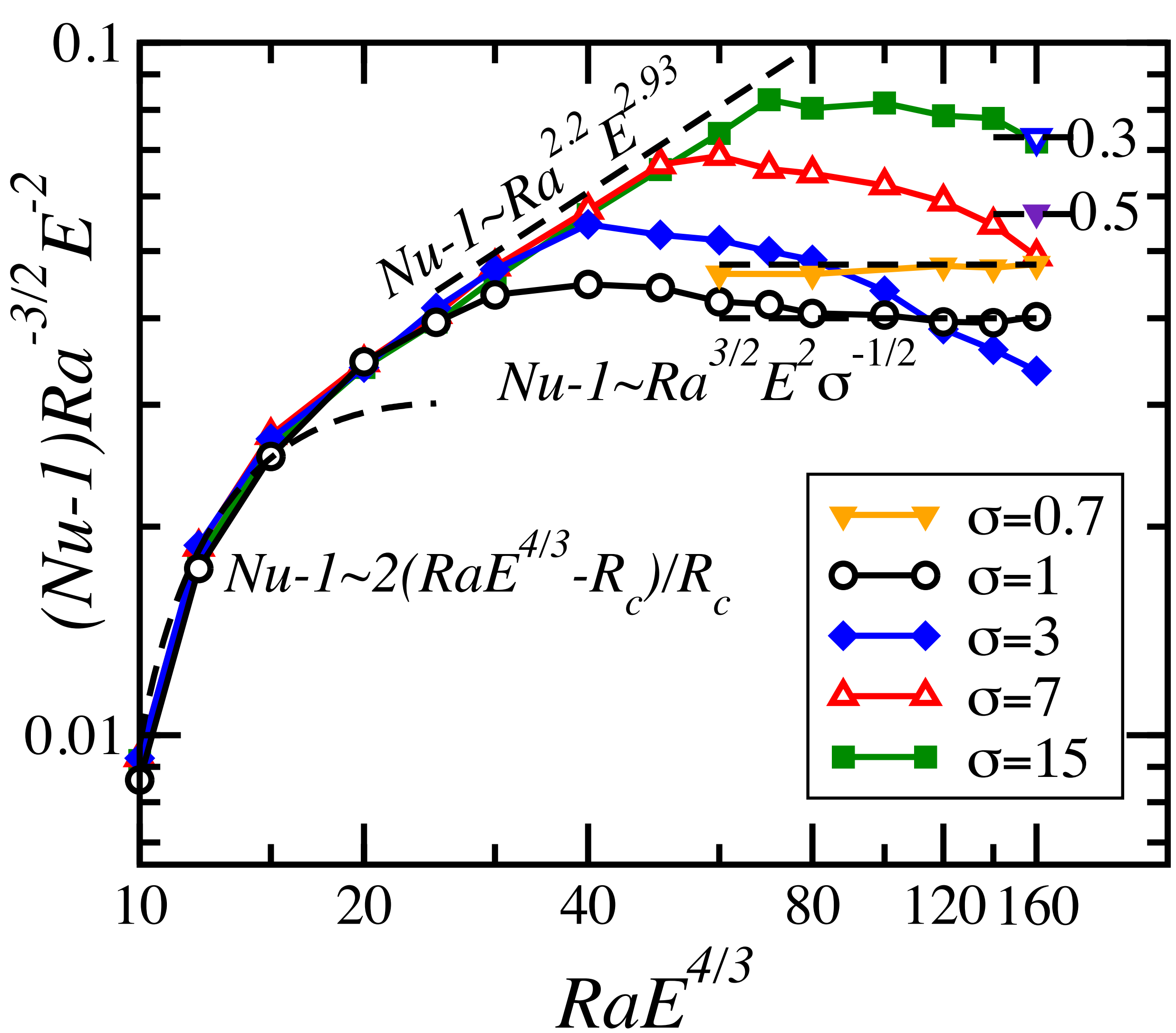
Turbulent control HX bottleneck

Greater potential for observation

Refs in Plumley & Julien ESS 2019

DNS RBC vs NH-QGE
Impenetrable Stress-Free Boundaries

KJ, et al PRL 2012



Good quantitative agreement

- *NH-QGE (Open Symbols)*
- *DNS (Closed Symbols)* $E = 10^{-7}$

$$Nu - 1 = \frac{1}{25}\sigma^{-\frac{1}{2}}\left(RaE^{\frac{4}{3}}\right)^{\frac{3}{2}}$$

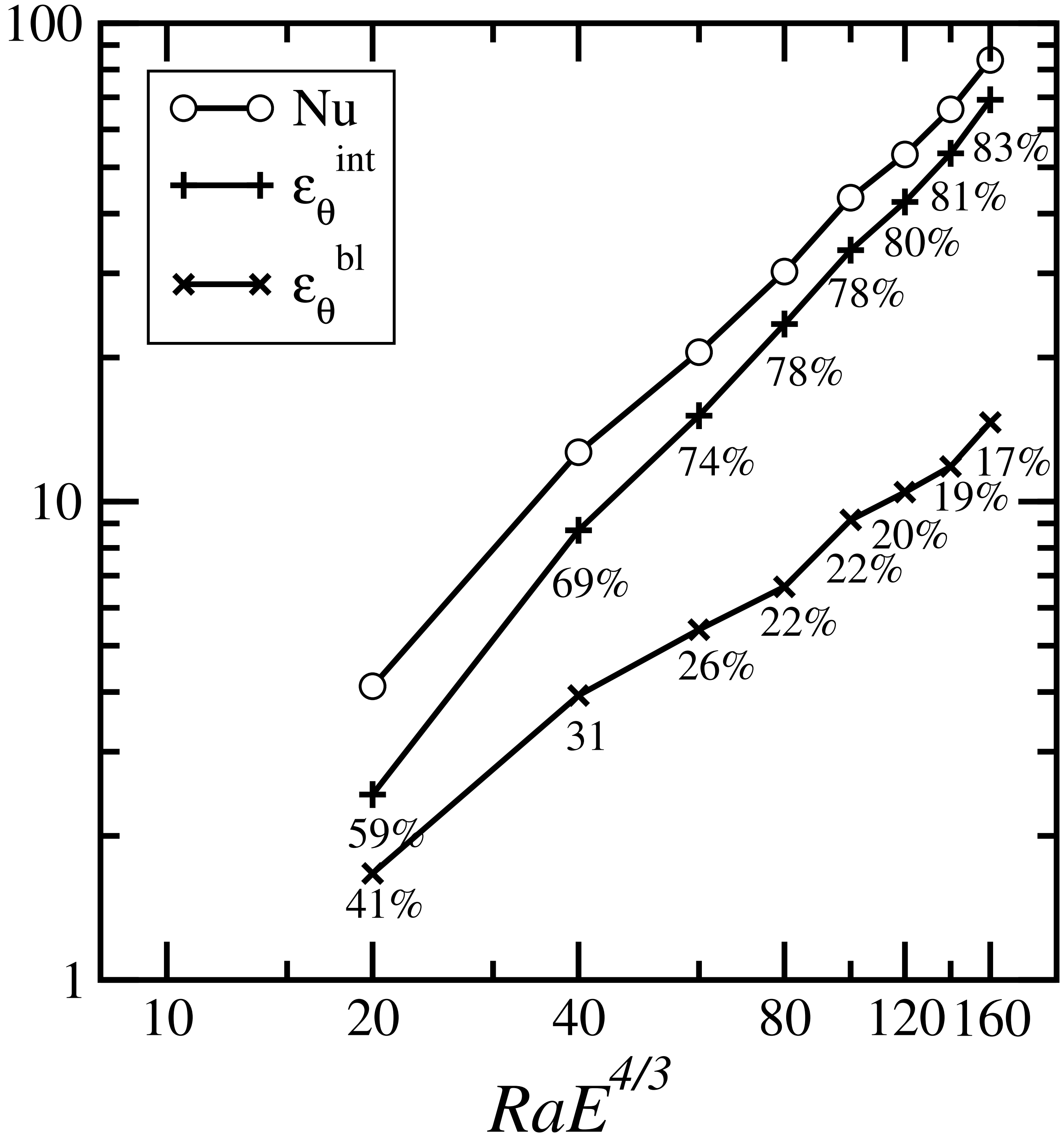
Dissipation-free Scaling Law

Flux bottleneck turbulent interior

- *Not thermal boundary layers*

$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g\alpha\Delta T H^3}{\nu\kappa}, \quad E = \frac{\nu}{2\Omega H^2}$$

Low Ro Heat Transfer: $Nu - 1 = \frac{1}{25}\sigma^{-\frac{1}{2}}\left(RaE^{\frac{4}{3}}\right)^{\frac{3}{2}}$



- turbulent interior controls heat transport (GL theory)

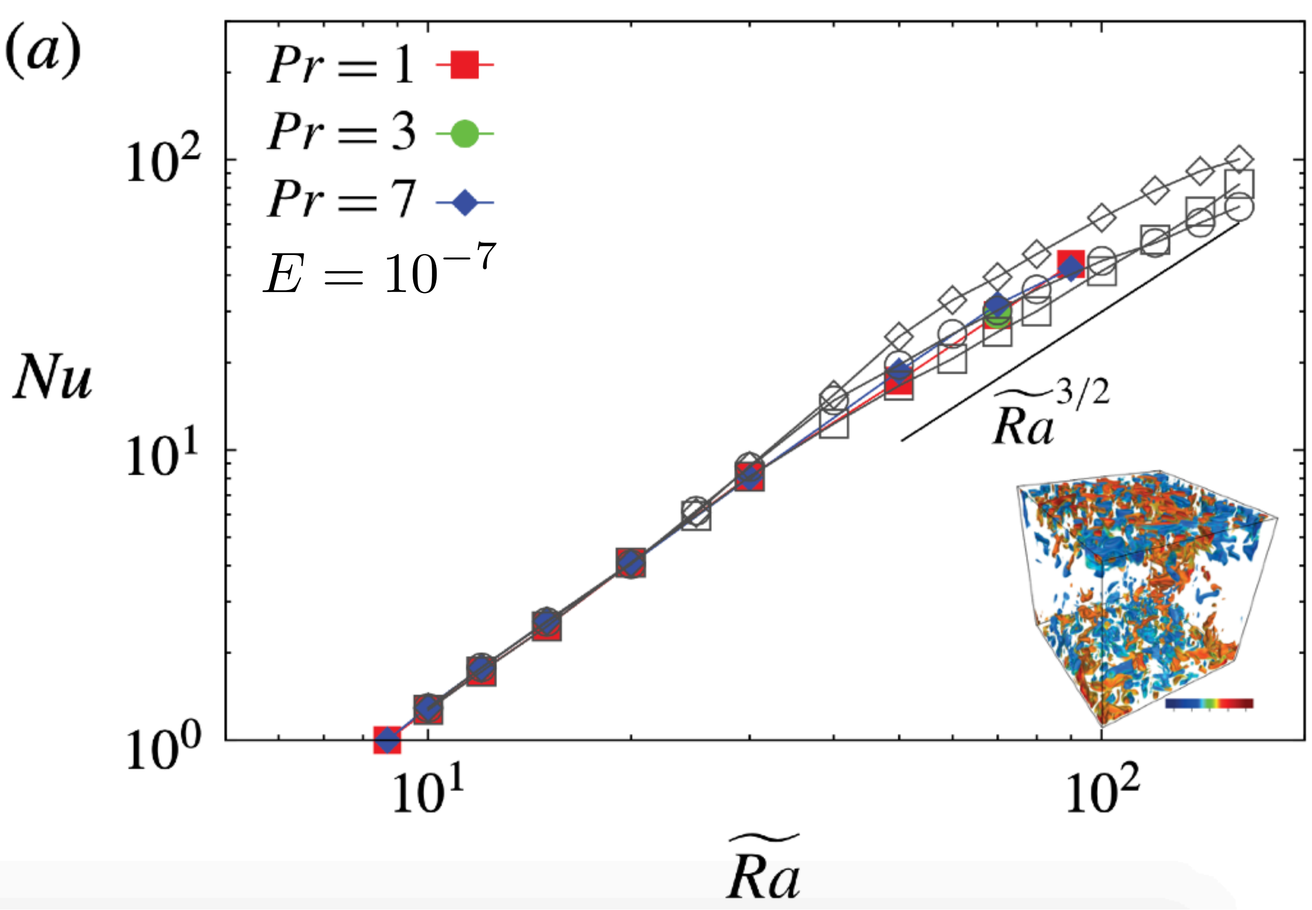
$$\mathcal{E}_\theta \approx \mathcal{E}_\theta^{int} = \left\langle \left|\partial_Z \overline{T}\right|^2 \right\rangle + \left\langle \left|\nabla_\perp \theta\right|^2 \right\rangle$$
$$\equiv Nu$$

Nondimensional #'s:

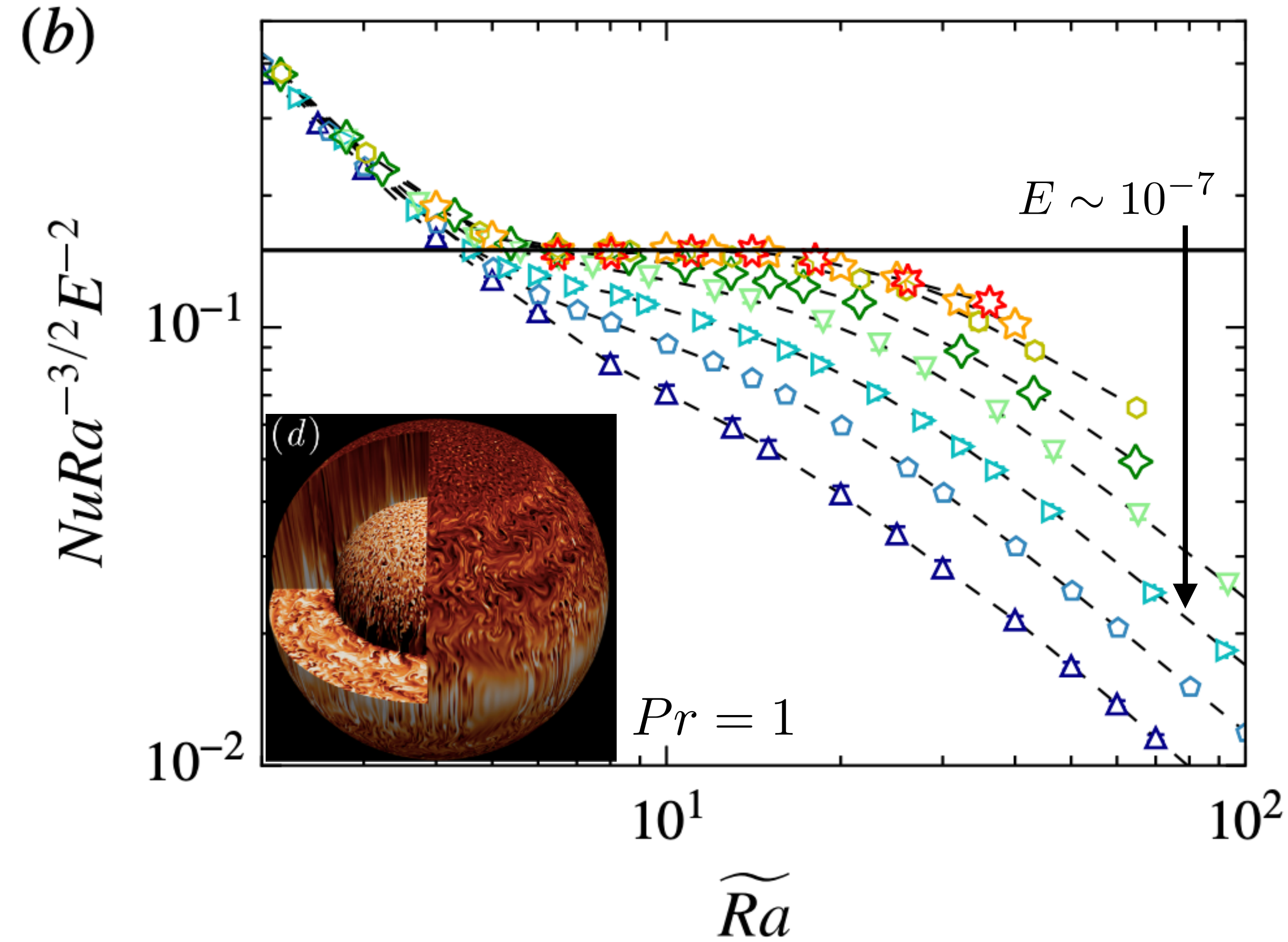
$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g\alpha \Delta T H^3}{\nu\kappa}, \quad E = \frac{\nu}{fH^2}$$
$$\sigma = \frac{\nu}{\kappa}$$

Quasi-Geostrophic Heat Transport

DNS: Stellmach,..,KJ. et al PRL 2014



DNS: Gastine et al JFM 2016



Dissipation-free ultimate HT scaling law. $\propto \left(\frac{Ra}{Ra_c}\right)^{3/2} Pr^{-1/2}$ upheld KJ et al PRL '12

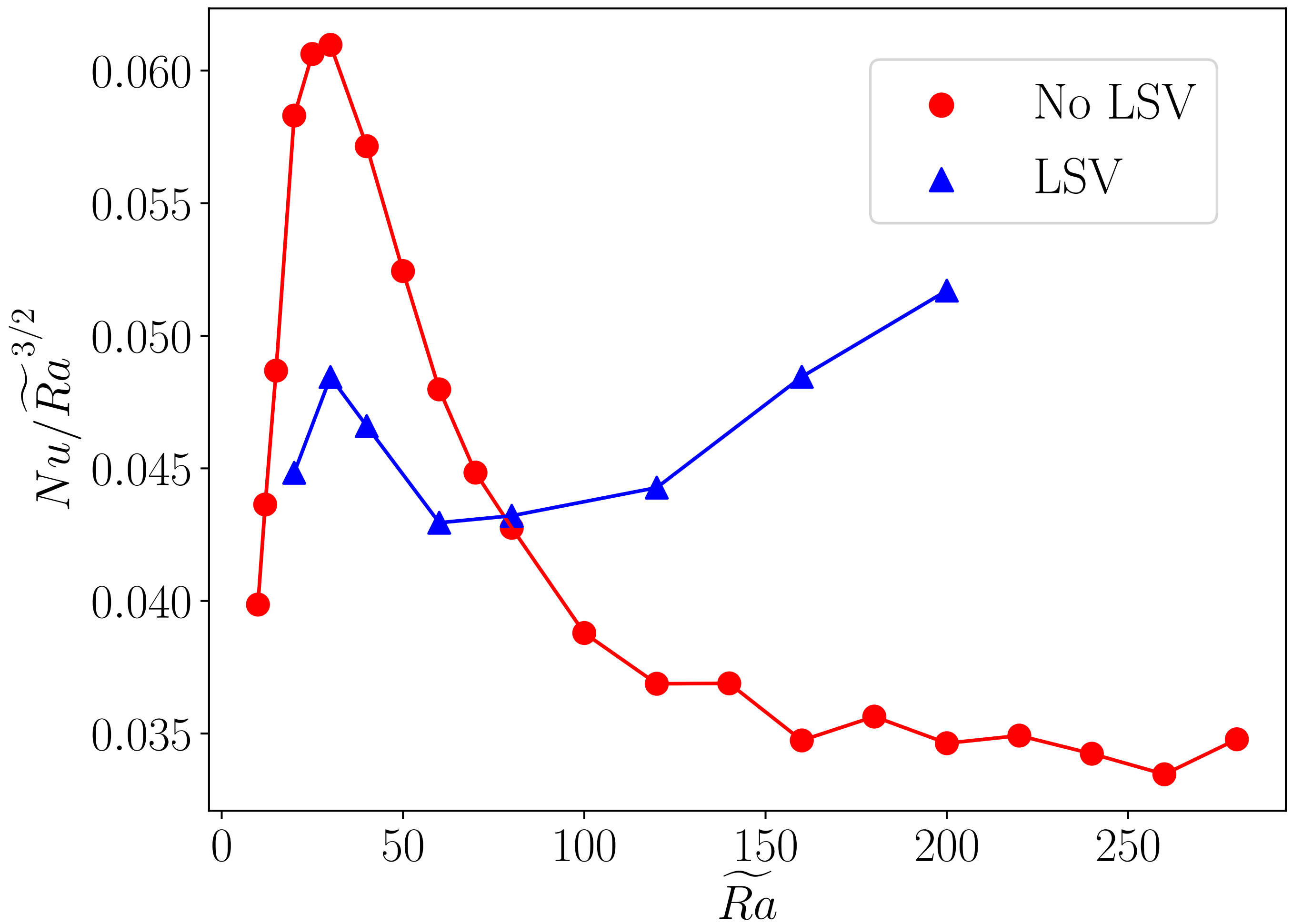
DNS in planar geometries Stellmach,..,KJ. et al PRL 2014

DNS in spherical geometries Gastine et al JFM 2016

Laboratory experiments

Challenges due to geometry and thermal control Ecke & Niemela PRL 2014

DNS RBC vs NH-QGE
Impenetrable Stress-Free Boundaries



LSV impacts heat transport

$$Nu - 1 = \frac{1}{25}\sigma^{-\frac{1}{2}}\left(RaE^{\frac{4}{3}}\right)^{\frac{3}{2}}$$

Dissipation-free Scaling Law

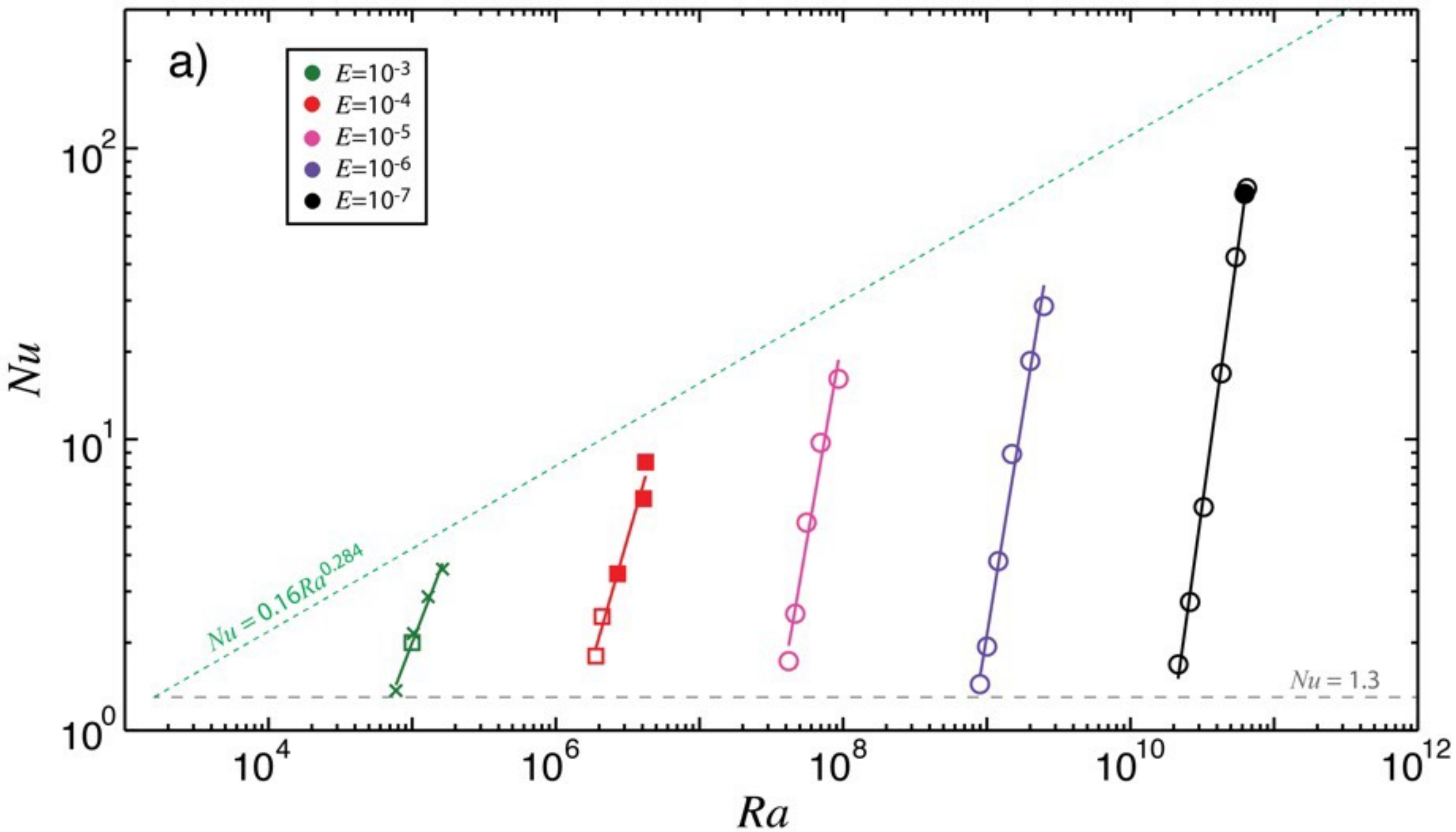
For underlying rotating turbulence

$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g\alpha\Delta T H^3}{\nu\kappa}, \quad E = \frac{\nu}{2\Omega H^2}$$

Tobias et al 2022 in prep

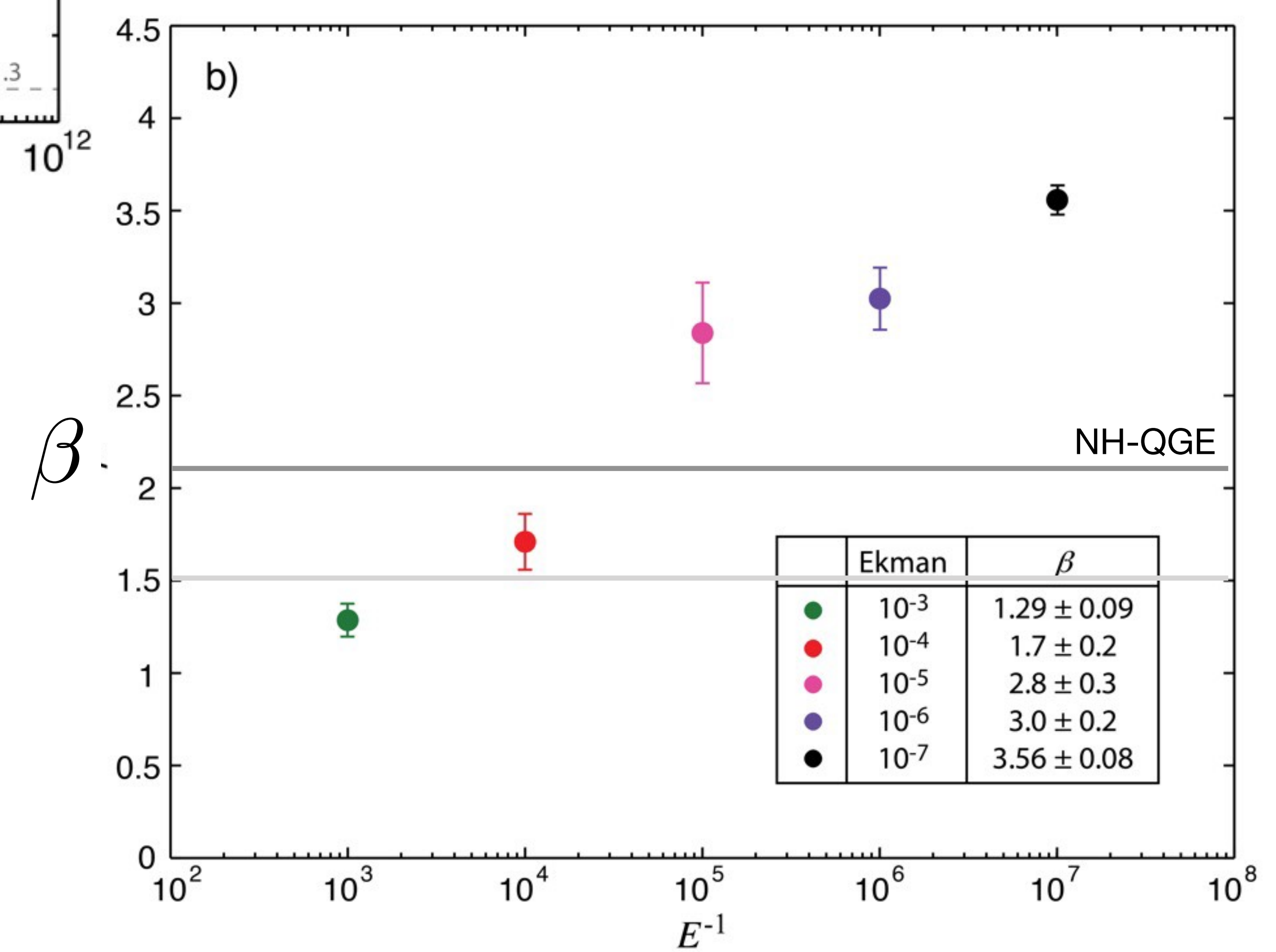
Heat Transport: Exponent vs Ekman number

$Nu \propto (Ra/Ra_c)^{\beta}$ nonconvergence: exponent appears to increase w/ decreasing *E* !



Conclusion:

- $\mathcal{O}(E^{1/6}H)$ EBL's having leading order affect on heat transport
- Not captured in asymptotic reduced model



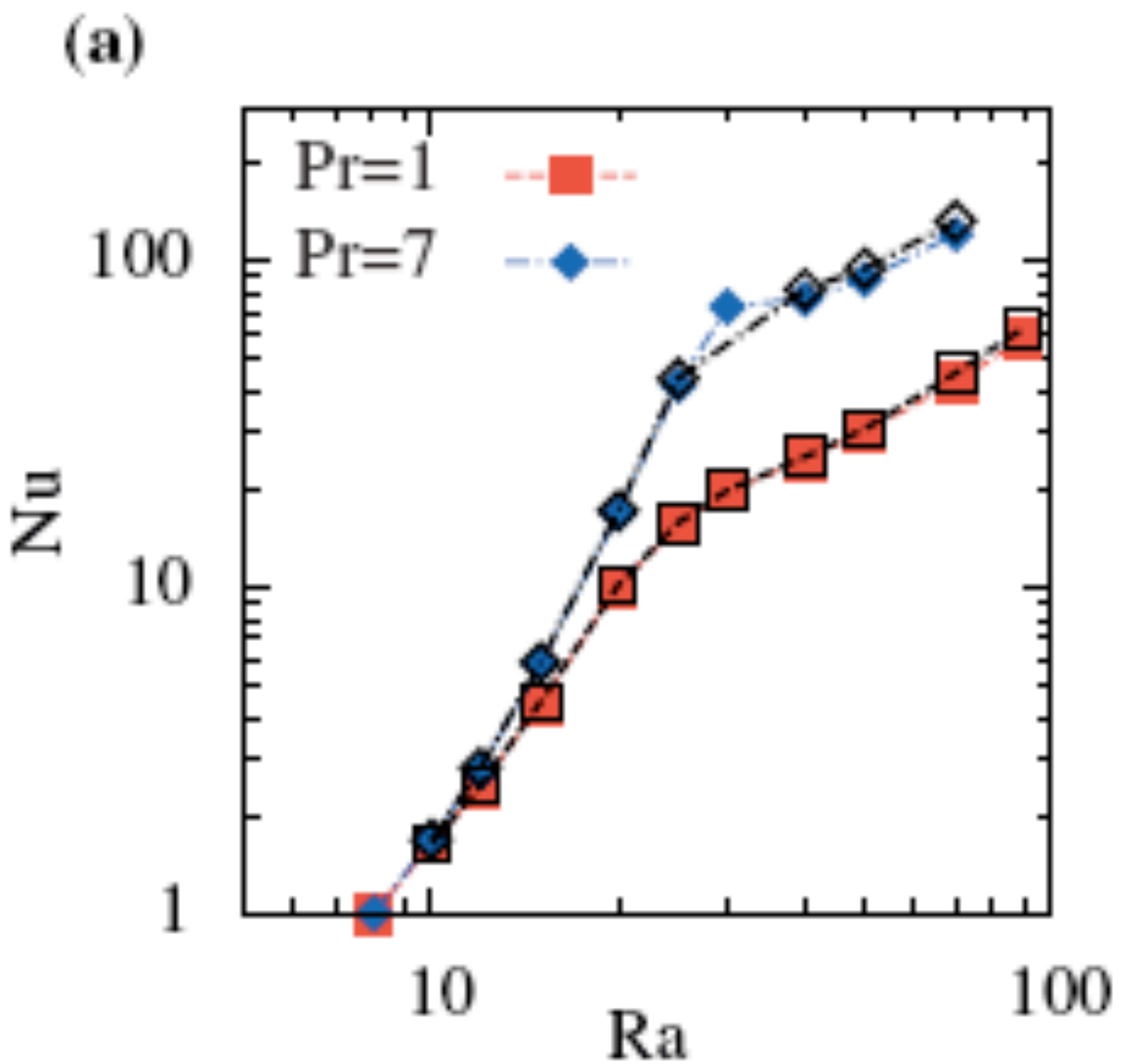
	Ekman	β
● (green)	10^{-3}	1.29 ± 0.09
● (red)	10^{-4}	1.7 ± 0.2
● (pink)	10^{-5}	2.8 ± 0.3
● (purple)	10^{-6}	3.0 ± 0.2
● (black)	10^{-7}	3.56 ± 0.08

Results from UCLA SpinLab

Courtesy Jon Aurnou, Jon Cheng

DNS RBC vs DNS with Parameterized Pumping

$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp \frac{E^{1//6}}{\sqrt{2}} \zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$



- *Filled symbols (DNS with No-Slip BC's)*
- Open symbols (DNS w/ Pumping BC's)

Conclusion:

- *Ekman pumping responsible for enhanced HT*
- Either *E* too large

  as *E ⇒ 0, DNS ⇒ NHQGE*

OR

- Pumping remains important as *E ⇒ 0*

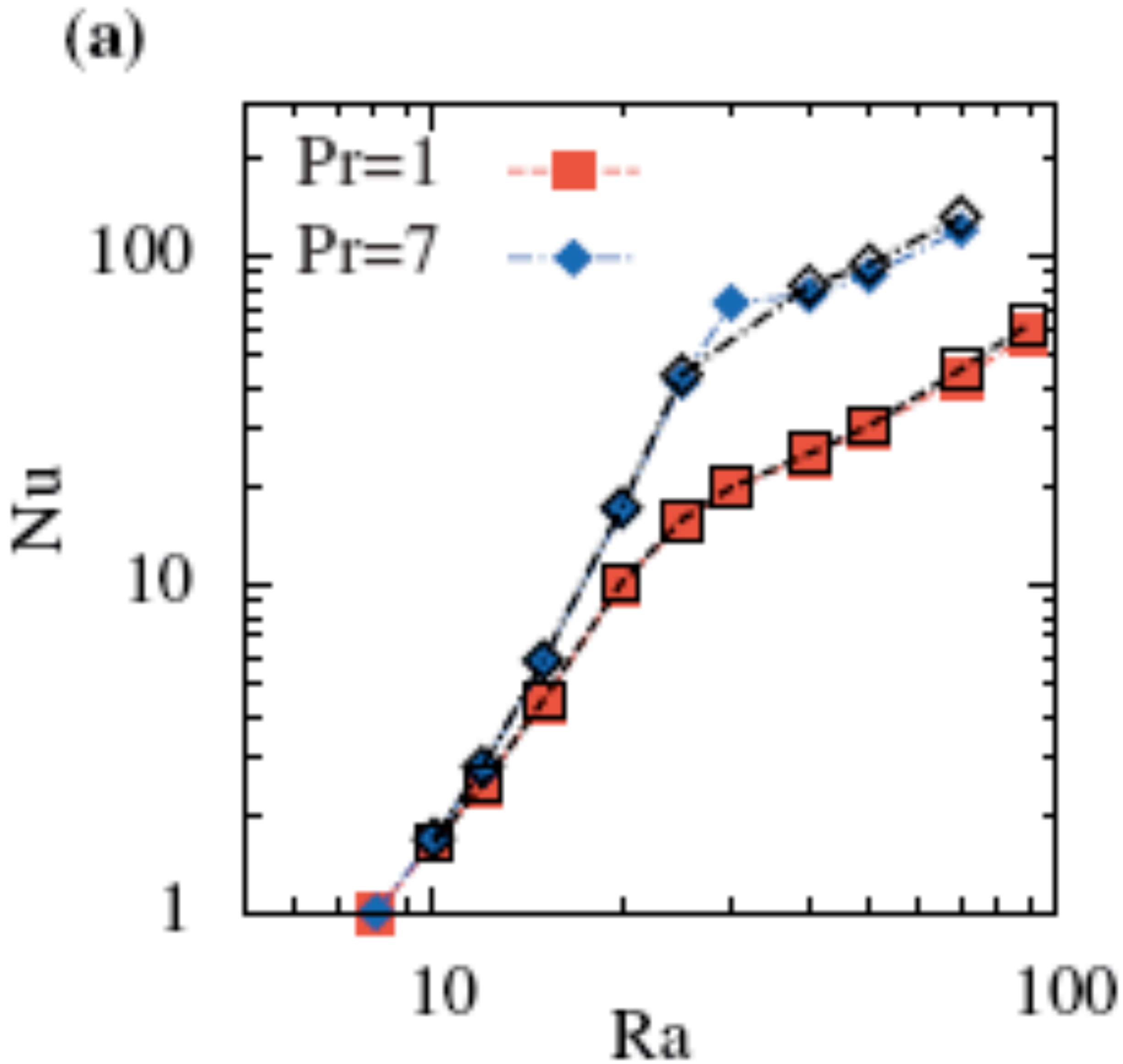
  as *E ⇒ 0, non-convergence to SF bc's*

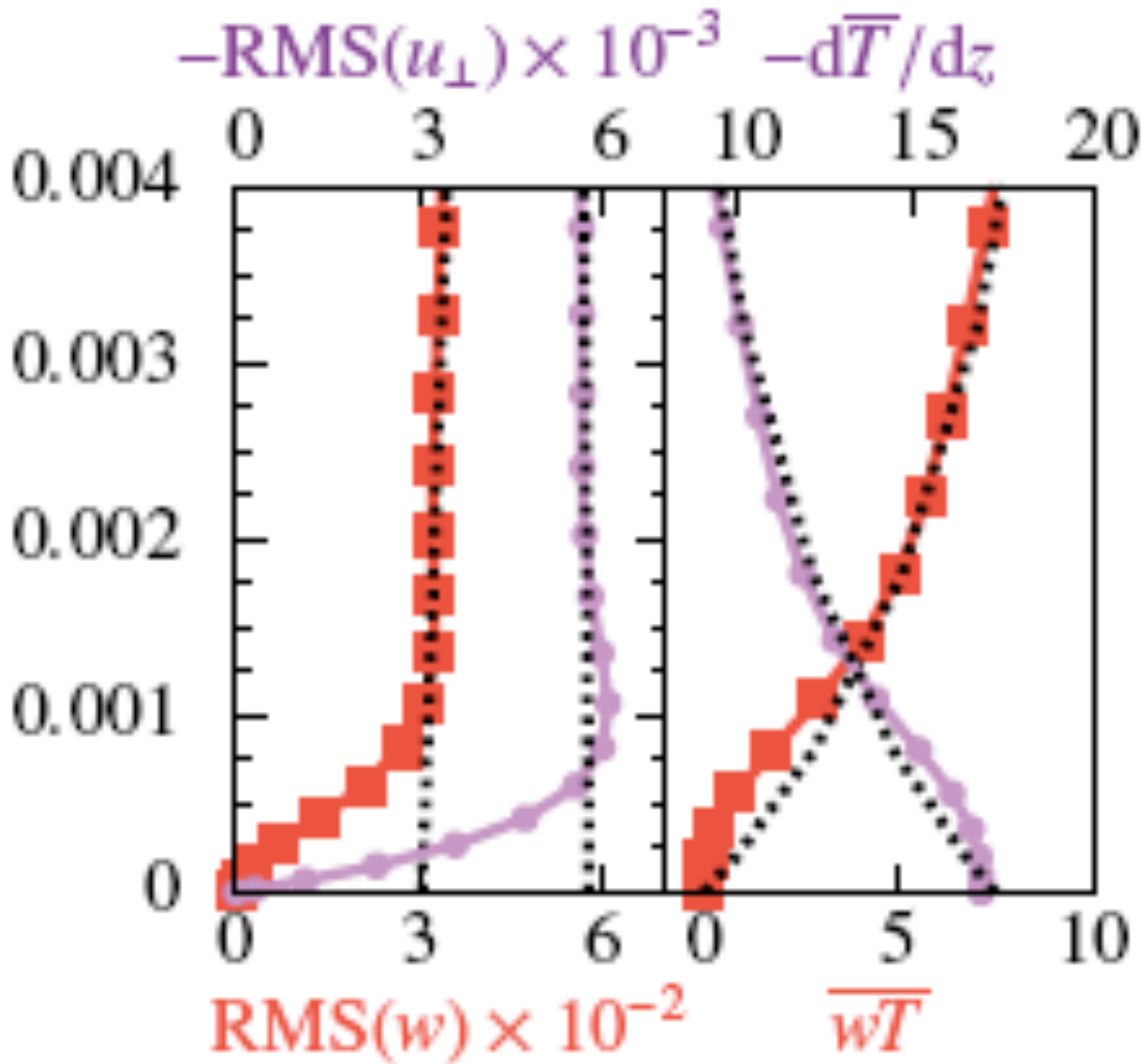
Stellmach ..,J,.., Aurnou, PRL '14

DNS RBC vs DNS with Parameterized Pumping

$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp \frac{E^{1//6}}{\sqrt{2}} \zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

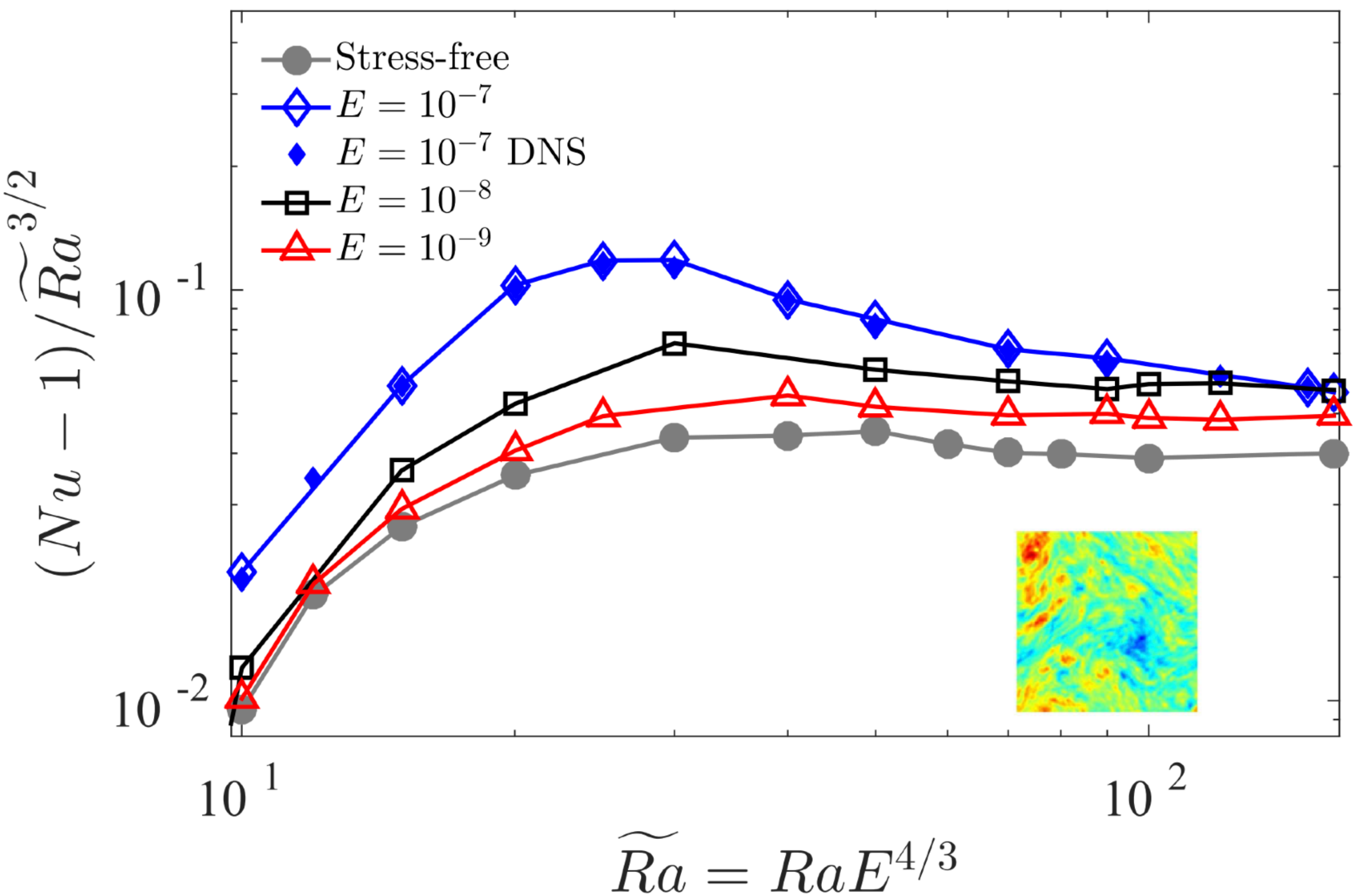
$Ek = 10^{-7}$





- *Filled symbols (DNS with No-Slip BC's)*
- Open symbols (DNS w/ Pumping BC's)
- *Convergence outside Ekman layers*
- Ekman layers can be filtered

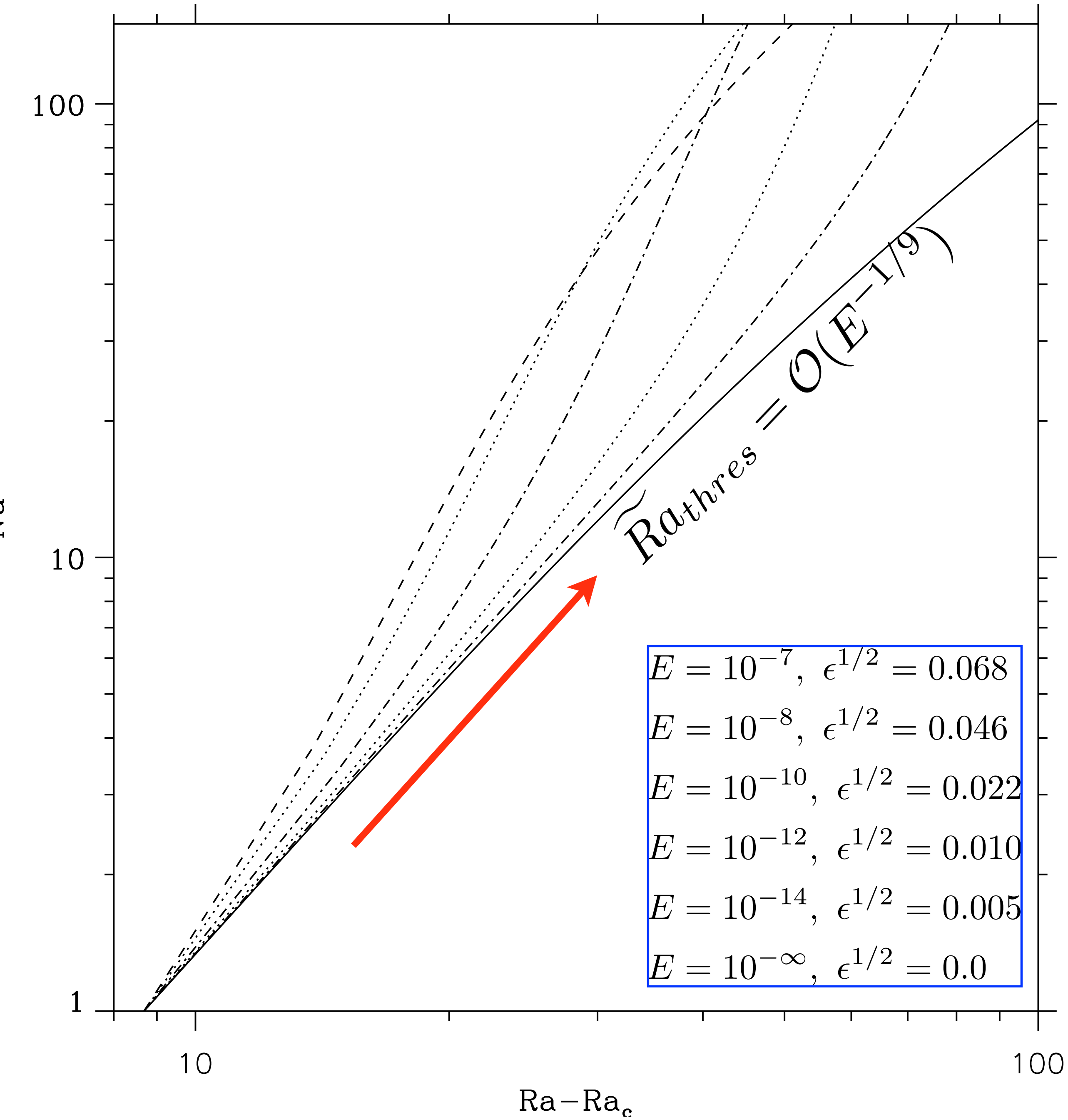
NH-QGE Heat Transport (Pumping)

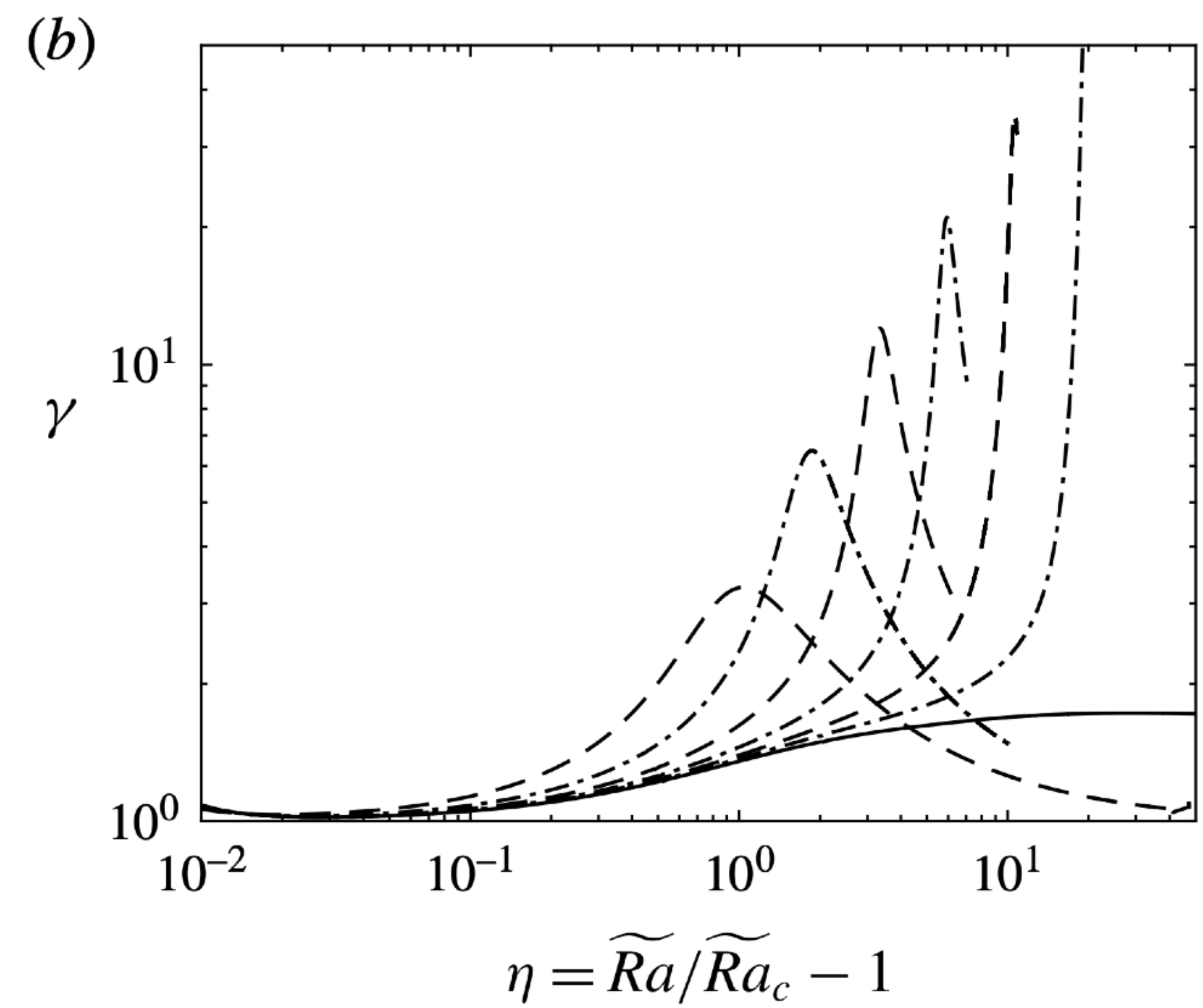


$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp \frac{E^{1//6}}{\sqrt{2}} \zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

Plumley et al JFM 2016

Single-Mode Results with Ekman pumping
Nu vs Ra



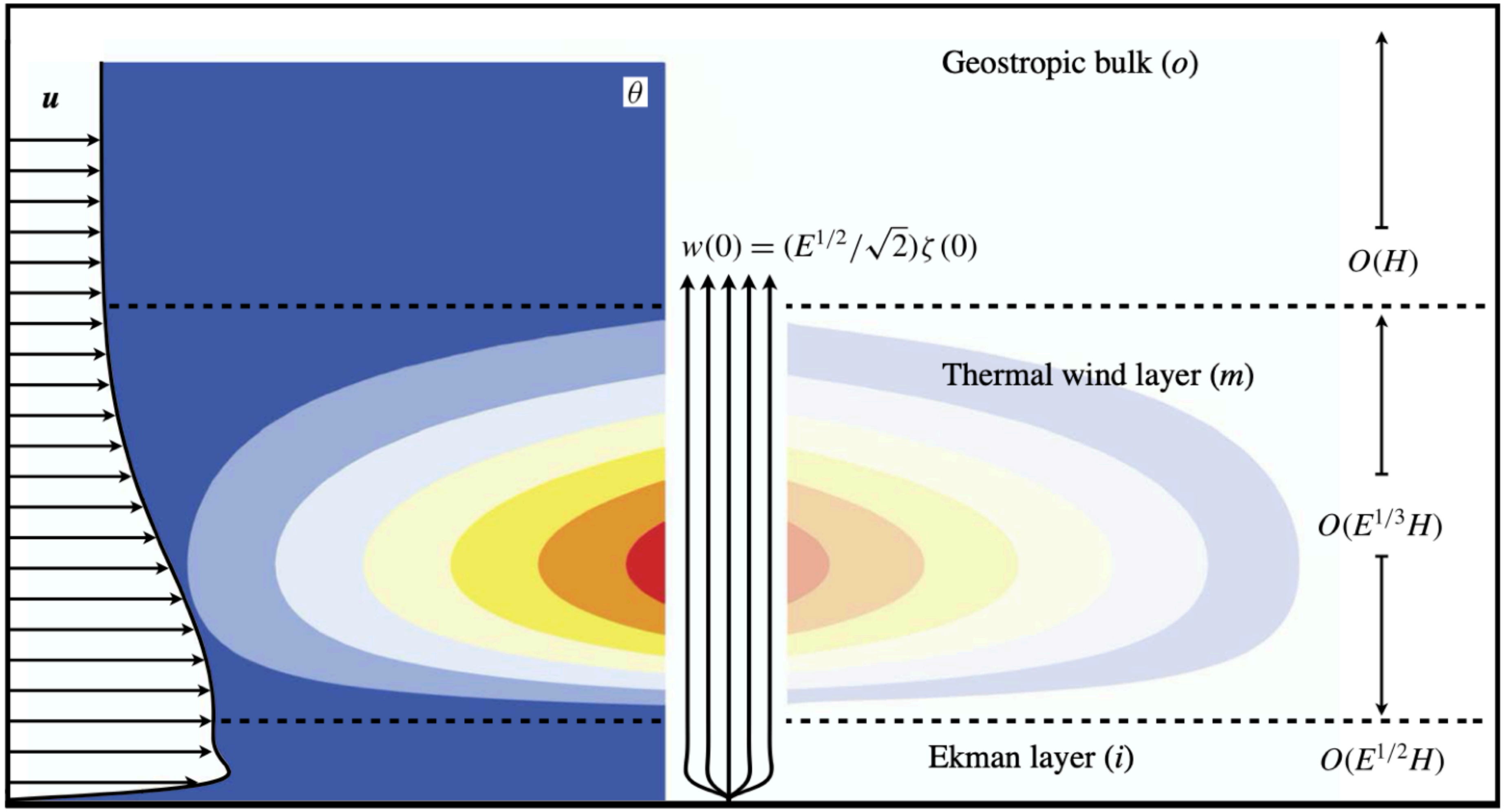


$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp \frac{E^{1//6}}{\sqrt{2}} \zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

- Limit *E* ⇒ *0* pumping displays sharp transition region prior to saturation

Julien et al JFM 2016

Mechanism - Boundary Layer Analysis



BL balance

$$\Rightarrow w^{(o)}\theta^{(o)} \sim \partial_Z \overline{T}^{(o)} \sim Nu$$

Pumping Heat Flux

$$\Rightarrow \quad w^{(i)}\theta^{(m)} \sim \partial_Z \overline{T}^{(o)} \; ?$$

Estimate Thermal fluctuations

$$\Rightarrow \quad \theta^{(m)} \sim w^{(i)}\partial_Z \overline{T}^{(o)}$$

$$\Downarrow$$

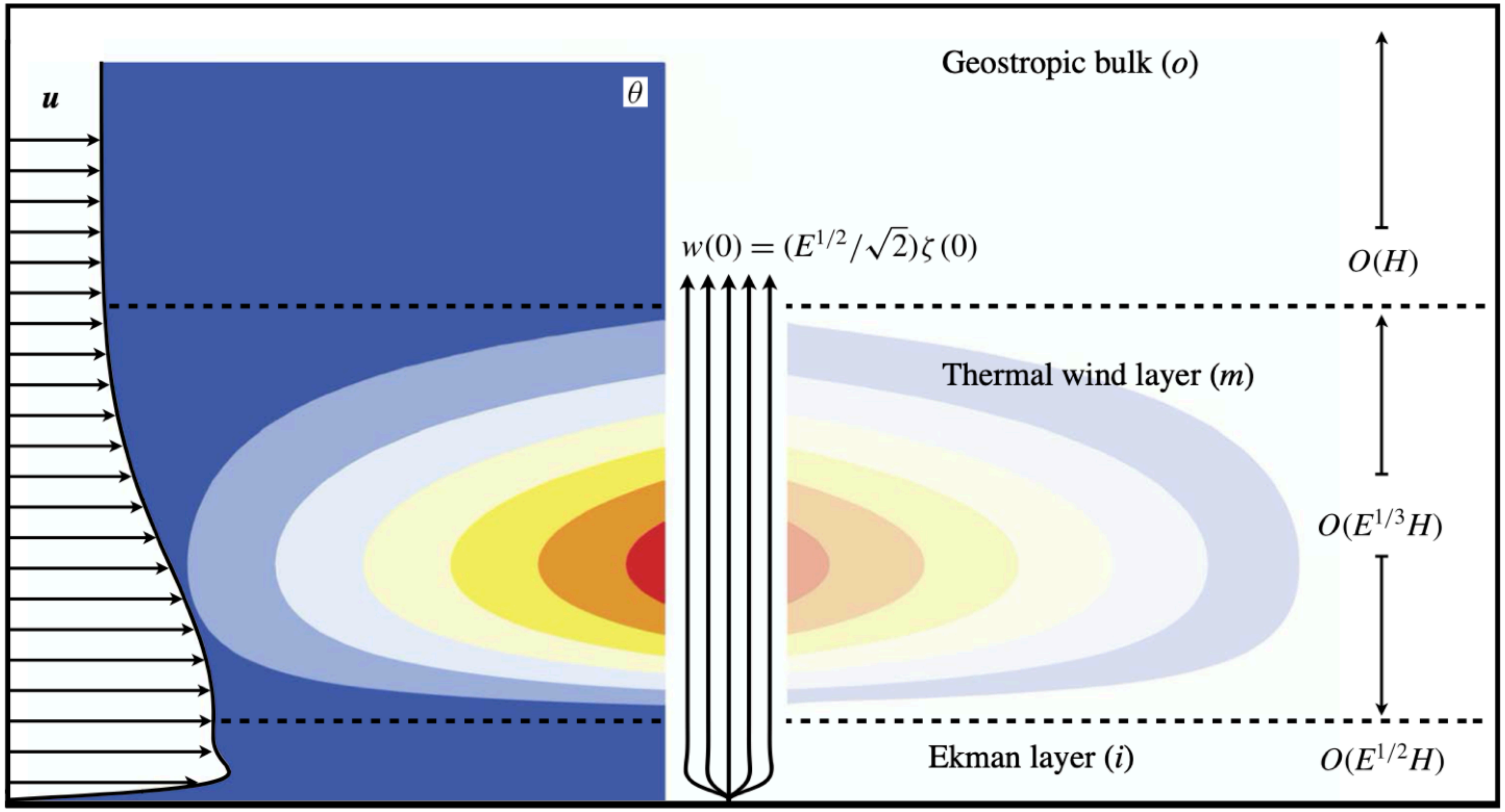
$$E^{-1/6} < \zeta^{(o)} = o(E^{-1/3})$$

Criteria for Pumping importance in HX

$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp \frac{E^{1//6}}{\sqrt{2}} \zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

Julien et al JFM 2016

Mechanism - Boundary Layer Analysis



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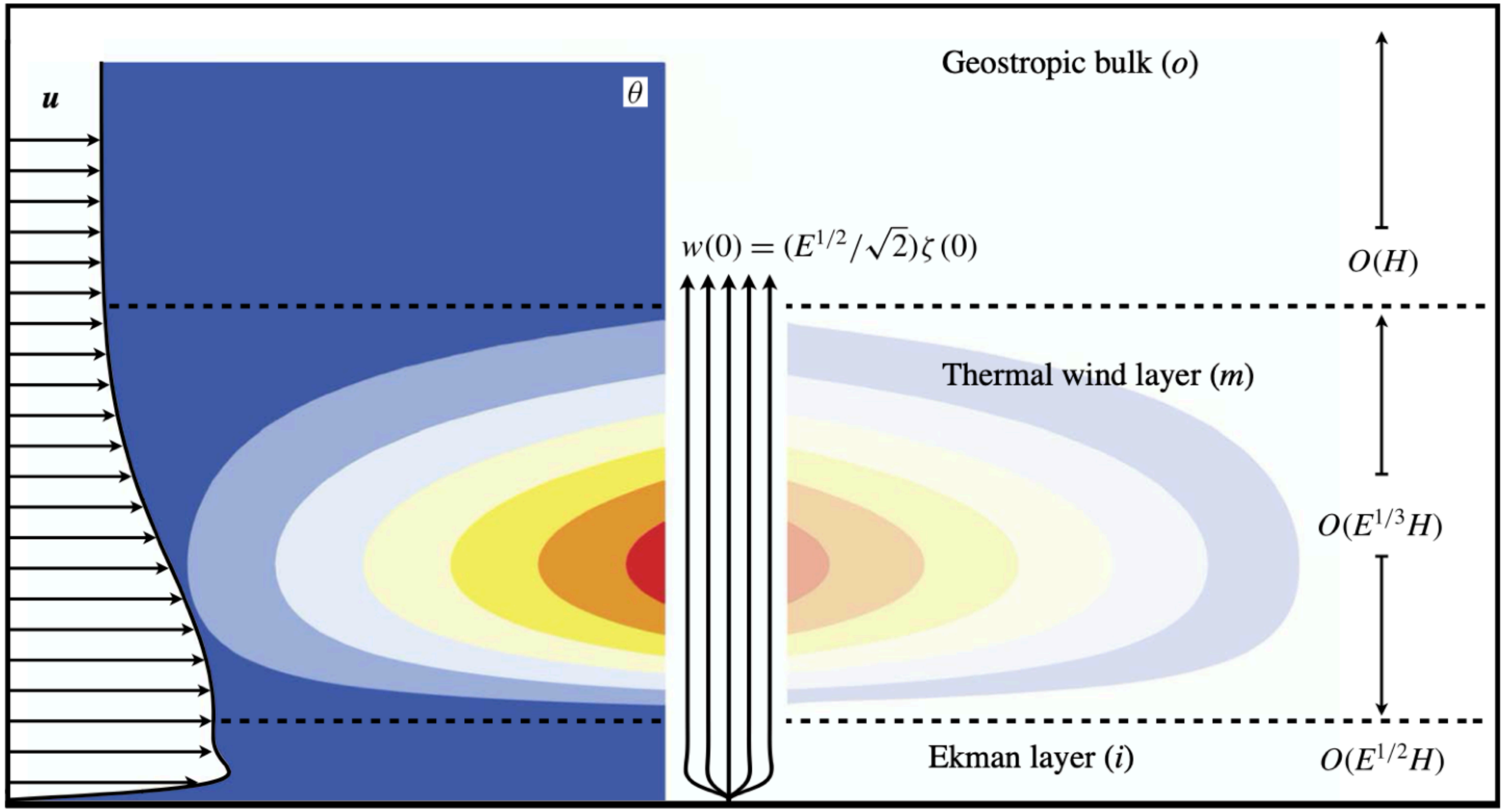
Julien et al JFM 2016

Mechanism - Boundary Layer Analysis



BL balance

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Pumping Heat Flux

$$\Rightarrow \quad w^{(i)}\theta^{(m)} \sim \partial_Z \overline{T}^{(o)} \text{ ?}$$

Estimate Thermal fluctuations

$$\Rightarrow \quad \theta^{(m)} \sim w^{(i)}\partial_Z \overline{T}^{(o)}$$

$$\Downarrow$$

$$E^{-1/6} < \zeta^{(o)} = o(E^{-1/3})$$

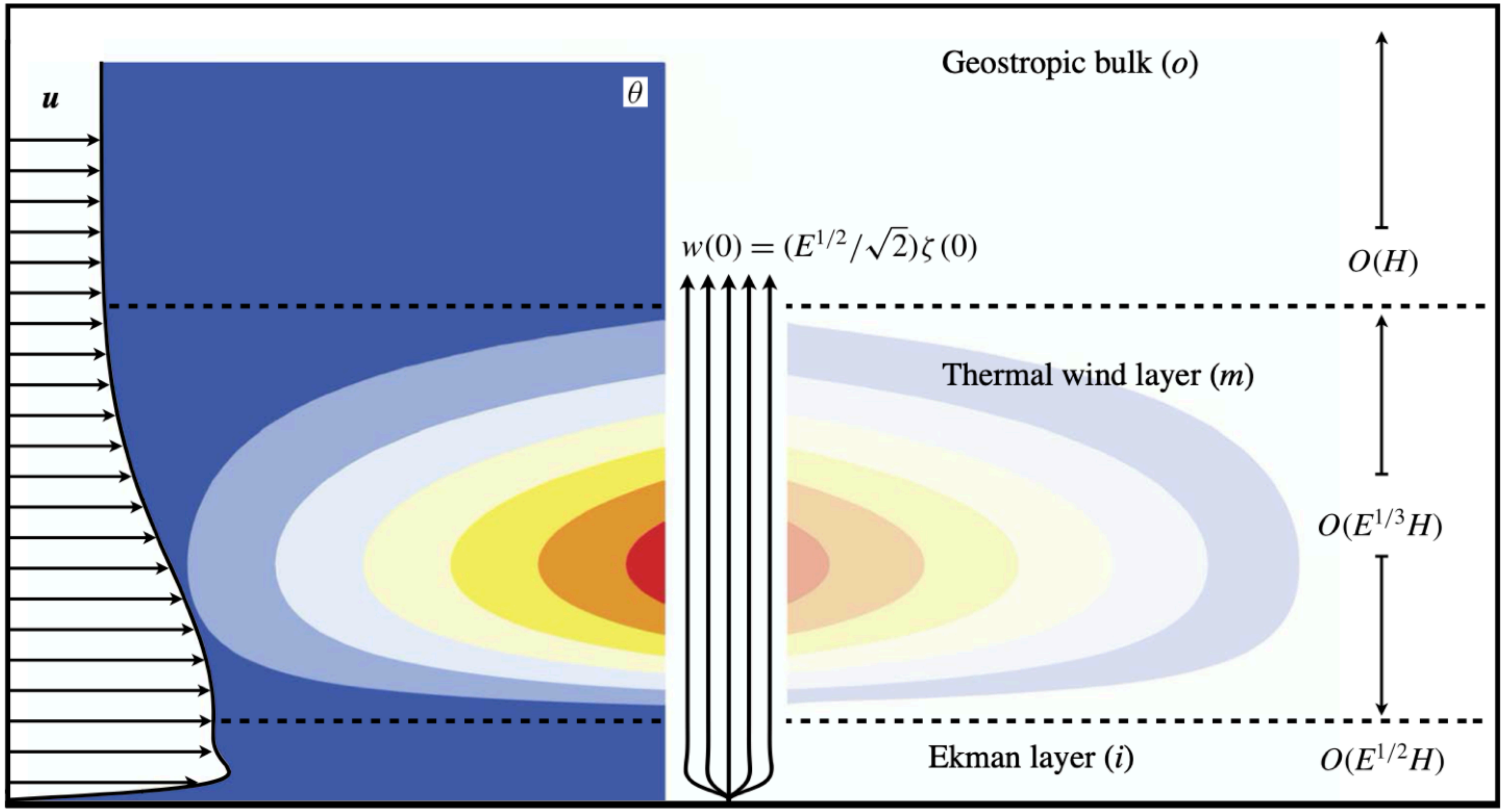
Criteria for Pumping importance in HX

Boundary layer scaling (analyze NH-QGE):

$$Nu \sim \partial_Z \overline{T}_0 \sim \theta_1 \sim \widetilde{Ra}^{\beta}, \ \partial_Z \sim \widetilde{Ra}^{(1+\beta)/2}$$
$$w_0 \sim \widetilde{Ra}^{0}, \ \zeta_0 \sim \psi_0 \sim \widetilde{Ra}^{(1+\beta)/2}$$

Julien et al JFM 2016

Mechanism - Boundary Layer Analysis



BL balance

$$w^{(o)}\theta^{(o)} \sim \partial_Z \overline{T}^{(o)} \sim Nu$$

Pumping Heat Flux

$$\Rightarrow \quad w^{(i)}\theta^{(m)} \sim \partial_Z \overline{T}^{(o)}$$

Estimate Thermal fluctuations

$$\Rightarrow \quad \theta^{(m)} \sim w^{(i)}\partial_Z \overline{T}^{(o)}$$

$$\Downarrow$$

$$E^{-1/6} < \zeta^{(o)} = o(E^{-1/3})$$

Boundary layer scaling (analyze NH-QGE):

$$Nu \sim \partial_Z \overline{T}_0 \sim \theta_1 \sim \widetilde{Ra}^{\beta}, \ \partial_Z \sim \widetilde{Ra}^{(1+\beta)/2}$$

$$w_0 \sim \widetilde{Ra}^{0}, \ \zeta_0 \sim \psi_0 \sim \widetilde{Ra}^{(1+\beta)/2}$$

$$\zeta_{\pm} = \mathcal{O}(E^{-1/6}) = o(E^{-1/3})$$

$$\widetilde{Ra}_{thres} = E^{-1/3(1+\beta)}$$

$\uparrow$

validity limit of NH-QGE

Julien et al JFM 2016

Asymptotic Development
Estimates of transitional values

$$\zeta_0(\pm) = \mathcal{O}(\epsilon^{-1/2}) = o(\epsilon^{-1}), \quad \epsilon = E^{1/3}$$

$$\widetilde{Ra}_{thres} = E^{-1/3(1+\beta)} = E^{-1/9}, \quad \beta \approx 2$$

$$\widetilde{Ra}_c = 8.7$$

DNS, Lab →

$$E = 10^{-7},\ \epsilon^{1/2} = 0.068,\ \widetilde{Ra}_t \approx 6.0$$

$$E = 10^{-8},\ \epsilon^{1/2} = 0.046,\ \widetilde{Ra}_t \approx 7.7$$

$$E = 10^{-10},\ \epsilon^{1/2} = 0.022,\ \widetilde{Ra}_t \approx 12.9$$

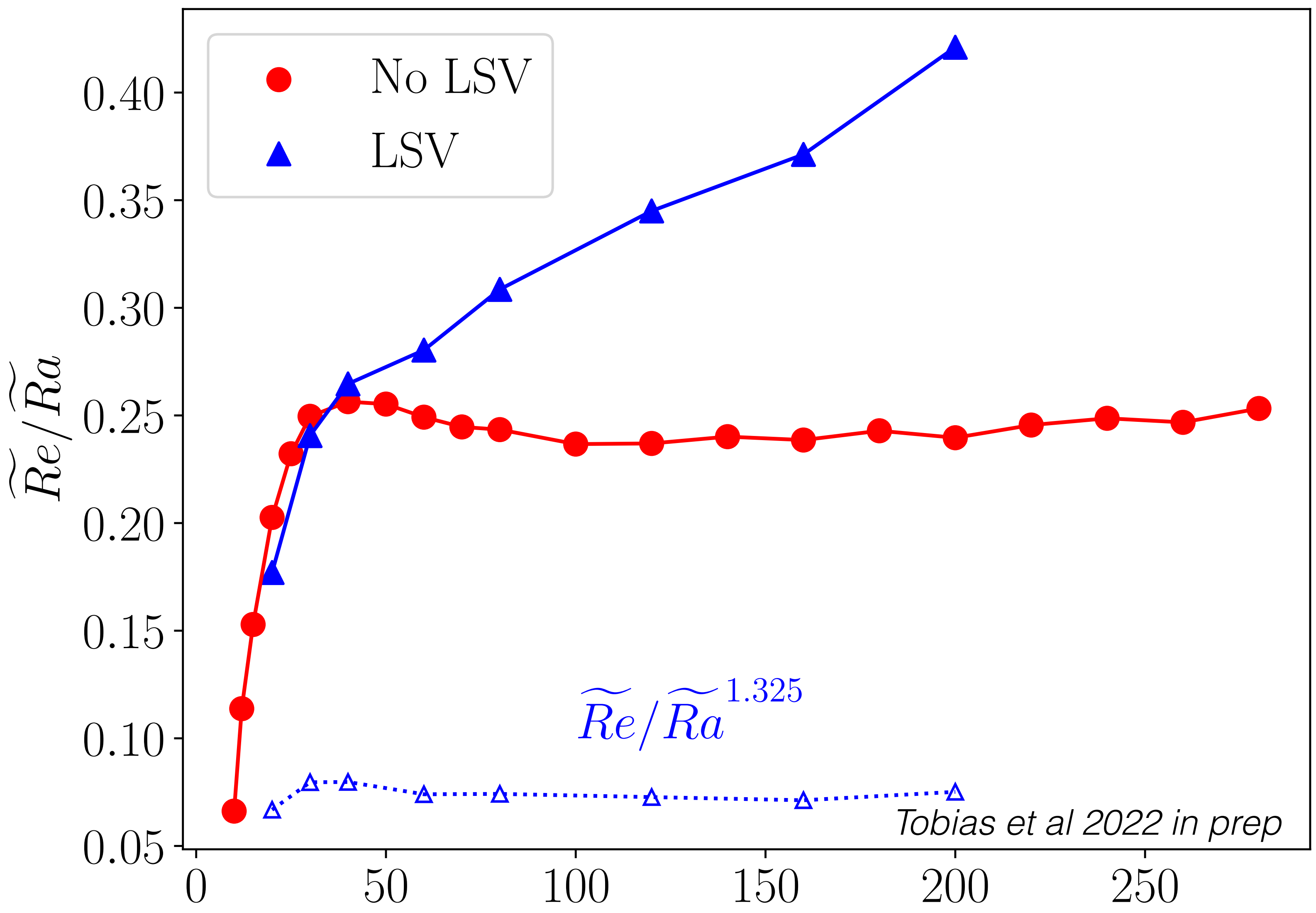
$$E = 10^{-12},\ \epsilon^{1/2} = 0.010,\ \widetilde{Ra}_t \approx 21.5$$

$$E = 10^{-14},\ \epsilon^{1/2} = 0.005,\ \widetilde{Ra}_t \approx 35.9$$

Earth's core →

$$E = 10^{-15},\ \epsilon^{1/2} = 0.003,\ \widetilde{Ra}_t \approx 46.4$$

NH-QGE Momentum Transport



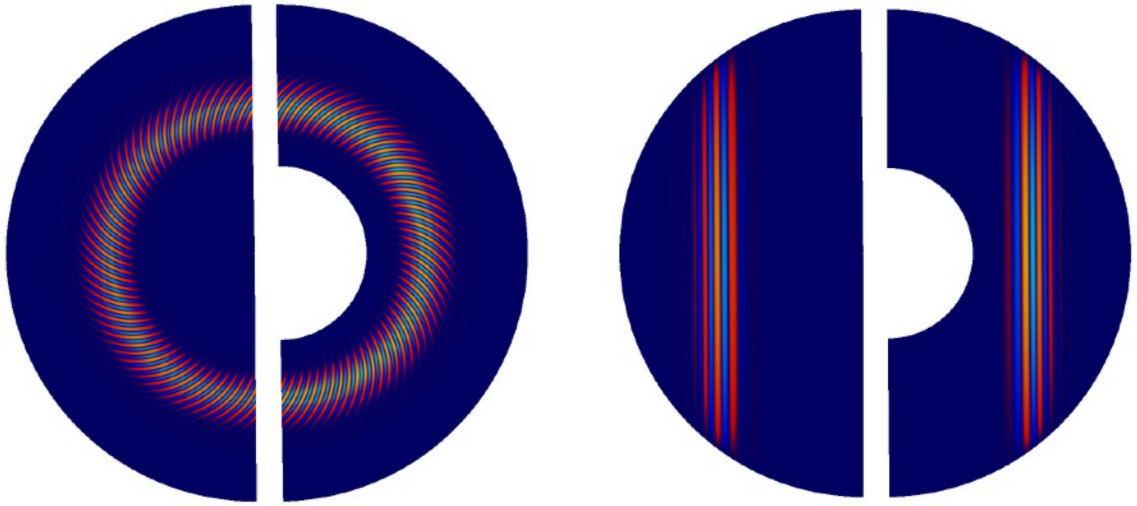
CIA balance

$$U_{\Omega ff} = \frac{g\alpha\Delta_T}{2\Omega} = Ro_c U_{0ff}$$

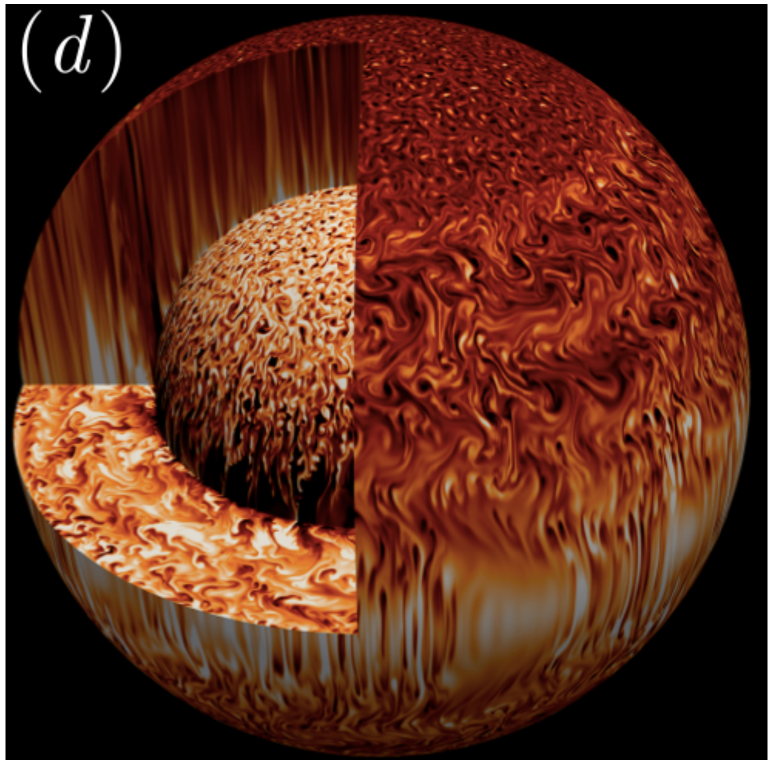
$$Re_H = \frac{Ra}{Pr}E \quad \Longrightarrow \quad Re_\ell = \frac{\widetilde{Ra}}{Pr}$$

Spherical Convection

Asymptotic Linear Theories (*Roberts P.Trans RSL '68, Busse jFM '70, JFM Jones et al JFM 2000*)

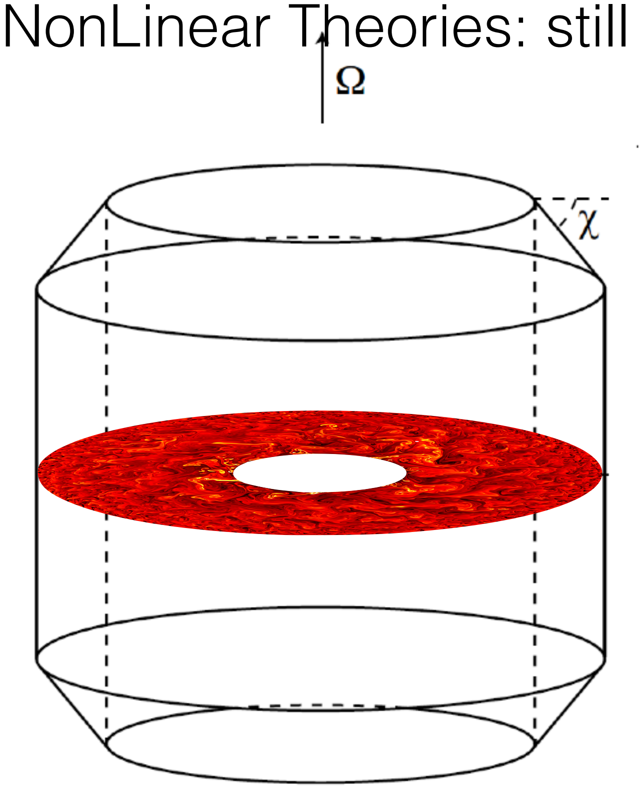


Marti et al G3 2017

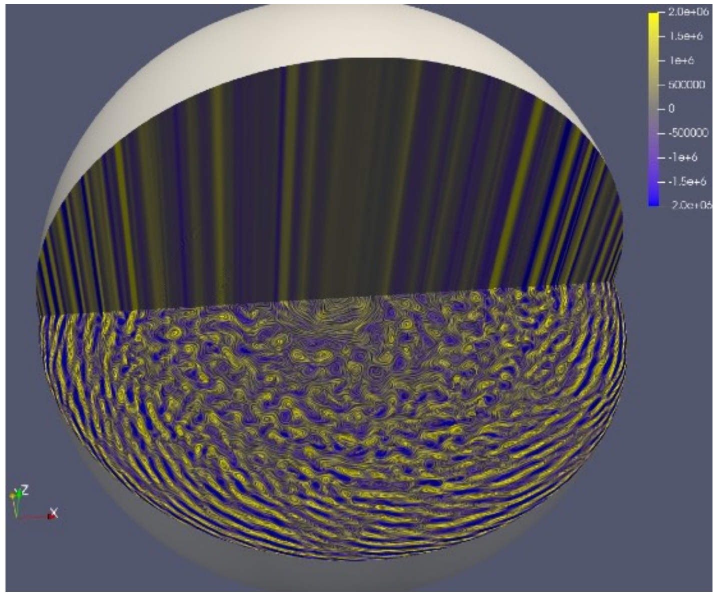


DNS: Gastine et al JFM 2016

NonLinear Theories: still local or non-asymptotic



Annulus: *Busse JFM'70, Calkin, J, Marti JFM 2013*)



Guervilly Cardin JFM 2016)

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