Majoranas in condensed matter systems – PART I

Asked to talk about „Majorana fermions" covering especially semiconductor quantum wires & chains of magnetic adatoms; most experimentally active systems along with quantum spin Hall (→ C. Kane)

<u>Outline</u>

- Spinless p-wave SC in 1d
- non-abelian statistics

} lecture 1

- proximity coupled TI edge (→ Kane)
- " " semiconductor quantum wire
- chains of magn. adatoms

} lecture 2

- experimental signatures
- Majoranas & interactions

} lecture 3

Remarks:

* detailed discussion of experiment → Frolov, Yazdani
* focus on 1d (general TSC → E. A. Kim; see also N. Read for 2d)
* many omissions (disorder, beyond Majorana ....)

* Reviews: Beenakker arXiv: 1112:1950

Alicea Rep. Prog. Phys. 75, 076501 (2012)

Les Houches lecture notes → School webpage

* Goal: Show some basic arguments & calculations to make literature accessible

I) Majoranas:

Fermion: $c = \frac{1}{2}(\gamma_1 - i\gamma_2)$

hermitian antihermitian

γ_j are Majoranas: $\boxed{\gamma_j = \gamma_j^\dagger}$

$$\left.\begin{aligned} \{c, c^\dagger\} &= 1 \\ \{c, c\} = \{c^\dagger, c^\dagger\} &= 0 \end{aligned}\right\} \Rightarrow \boxed{\{\gamma_i, \gamma_j\} = 2\delta_{ij}}$$

* $\gamma_j^2 = (\gamma_j^\dagger)^2 = 1$ natural for particle = antiparticle

(cp. $c^{\dagger 2} = 0$ for fermions; for bosons $(b^\dagger)^2$ takes system to orthogonal Fock state)

Here : Majoranas as quasiparticle excitations

$$\boxed{\begin{array}{l}\underline{\text{Kitaev}}: \text{Majoranas possible as } E=0 \text{ excitations in} \\ \text{spinless p-wave superconductors}\end{array}}$$

Heuristic argument: see Les Houches notes

see also: Homework problem on Kitaev chain by C. Kane

II) Spinless p-wave SC in 1d (Kitaev)

* Continuum model & phase diagram

Many-body Hamiltonian (mean field)

$$\mathcal{H} = \int dx \left\{ \psi^\dagger(x) \left(\frac{p^2}{2m} - \mu \right) \psi(x) + \left[\Delta' \psi(x)\, \partial_x \psi(x) + h.c. \right] \right\}$$

p-wave pairing (needed by Pauli)

$$= \sum_k \left\{ \left(\frac{k^2}{2m} - \mu \right) c_k^\dagger c_k + i\Delta' k\, c_{-k} c_k - i\Delta' k\, c_k^\dagger c_{-k}^\dagger \right\}$$

Nambu representation $\quad \phi_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$

$$= \sum_{k>0} \phi_k^\dagger \underbrace{\begin{bmatrix} \frac{k^2}{2m} - \mu & -i\Delta' k \\ i\Delta' k & -\left(\frac{k^2}{2m} - \mu \right) \end{bmatrix}}_{H_k} \phi_k + \text{const.}$$

w/ BdG Hamiltonian

$$H_k = \xi_k \tau_z + \Delta' k \tau_y \qquad \left(\xi_k = \frac{k^2}{2m} - \mu\right)$$

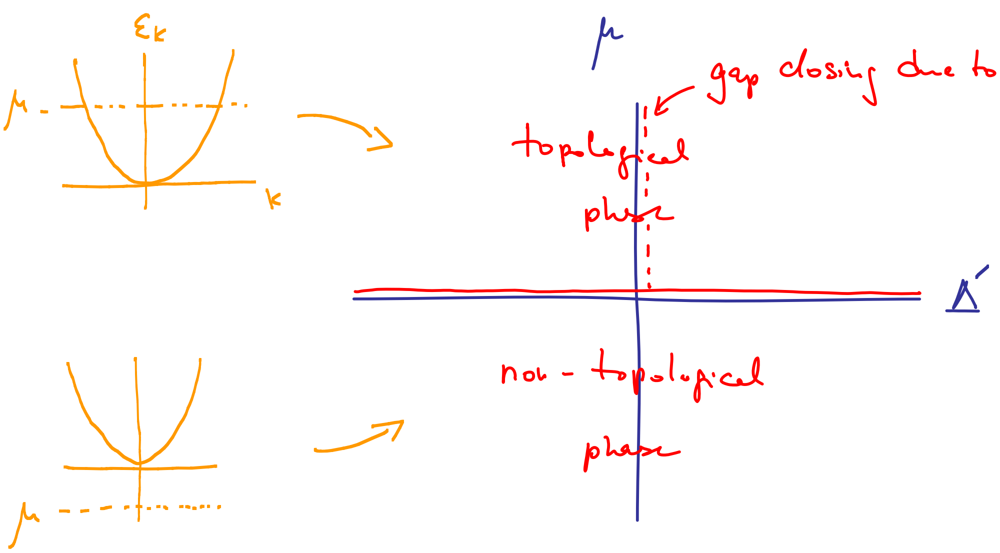
$$= \vec{b} \cdot \vec{\tau} \qquad \text{w/} \quad b_x = 0, \; b_y = \Delta' k, \; b_z = \xi_k$$

$\curvearrowright$ excitation spectrum

$$\boxed{E_k = \pm \sqrt{\xi_k^2 + (\Delta' k)^2}}$$

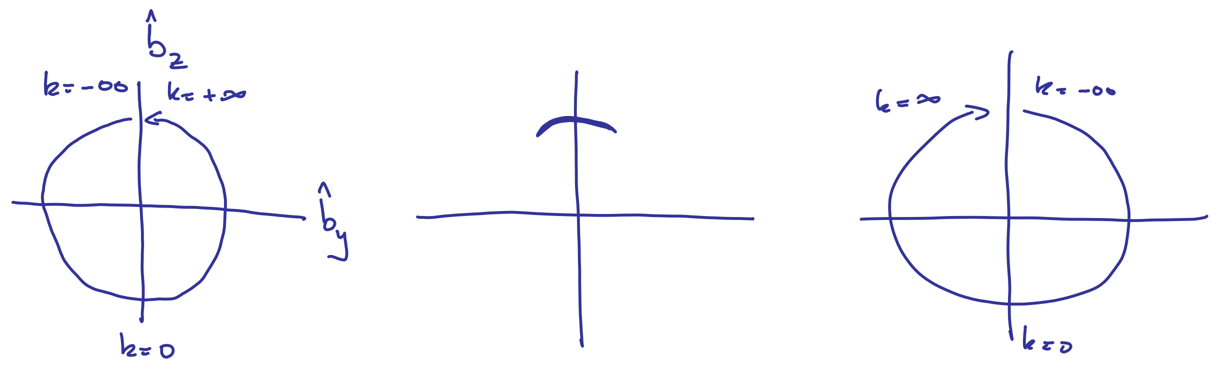
cp. $E_k = \pm\sqrt{\xi_k^2 + \Delta^2}$ for ordinary SC

fully gapped except when $\xi_k = 0$ for $k = 0$, i.e., for $\mu = 0$



phase diagram

Consider mapping $k \to \hat{b}_k$ $\rightsquigarrow$ winding number



$\mu > 0$, $\Delta' > 0$ $\qquad$ $\mu < 0$, Δ' arb. $\qquad$ $\mu > 0$, $\Delta' < 0$

(continuously connected to vacuum)

Symmetry class: BDI (H can be made real $\to$ generalized time reversal K)

$\rightsquigarrow$ topological $\mathbb{Z}$ index = winding # defined above

BDI $\to$ D if all three τ_i appear nontrivially (e.g., w/ winding order-parameter phase)

$$k \to \hat{b}_k \in S^2 \quad \text{w/} \quad \hat{b} = \begin{cases} \hat{z} & k \to \pm\infty \\ \pm\hat{z} & k = 0 \end{cases}$$

(no pairing for $k=0$!)

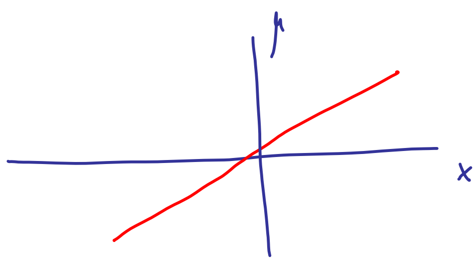
depends on phase $\mu \lessgtr 0$

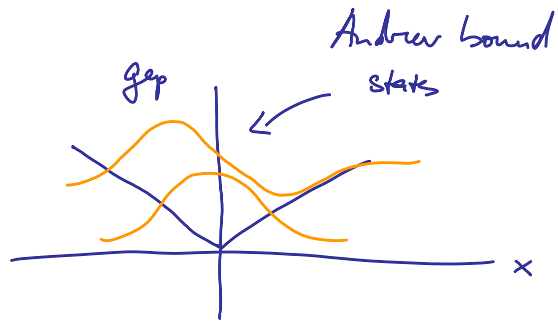
$\rightsquigarrow$ $\mathbb{Z}_2$ topological index $\hat{b}_\infty \cdot \hat{b}_0$

* domain wall and Majorana bound states

topological phase - terminated by gapless edge mode

→ study domain wall $\mu(x) = \alpha x$





start from BdG (now in real space)

$$H = \left(\frac{p^2}{2m} - \alpha x\right)\tau_z + \Delta' p \tau_y$$

$\mu = \alpha x$ f. domain wall

Simplification: Can neglect $\frac{p^2}{2m}$ for sufficiently smooth domain wall

w/ $\alpha \ll m^2 \Delta'^3$

EXERCISE : justify this statement

$\alpha x \sim \Delta p \rightsquigarrow$ characteristic length $\sqrt{\Delta'/\alpha}$

" momentum $\sqrt{\alpha/\Delta'}$

" energy $\sqrt{\Delta'\alpha}$

$\rightsquigarrow \frac{p^2}{2m} \sim \frac{\alpha}{m\Delta} \ll$ characteristic energy $\sqrt{\Delta'\alpha} \Rightarrow \alpha \ll m^2 \Delta'^3$

Then, H is just Dirac Hamiltonian

$$H \simeq -\alpha x \tau_z + \Delta' p \tau_y$$

w/ mass which changes sign

$\rightsquigarrow$ Zero-energy bound state localized at domain wall

(Jackiw-Rebbi $\longrightarrow$ Kane's lecture)

- $\psi(x) \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-(x/x_0)^2}$ w/ "oscillator length" $x_0 \propto \sqrt{\Delta'/\alpha}$

- Corresponding Bogoliubov quasiparticle

note $u(x) = v(x)$

$$\gamma = \int dx \left[u(x)\psi(x) + v(x)\psi^\dagger(x) \right] = \gamma^\dagger$$

$\Rightarrow \gamma$ is Majorana!

Exercise: For $\mu = \alpha x$, find complete spectrum of domain wall using the following observations:

* BdG Hamiltonian $\rightsquigarrow$ Spectrum symmetric under $E \to -E$

  $\rightsquigarrow$ Can consider spectrum of H^2

* use $\{\tau_i, \tau_j\} = 2\delta_{ij}$ to show that H^2 is just harmonic oscillator Hamiltonian.

Solution: $H^2 = (\alpha x)^2 + (\Delta' p)^2 - \Delta'\alpha \underbrace{[x, p]}_{i} \underbrace{\tau_z \tau_y}_{-i\tau_x}$

$$= (\Delta' p)^2 + (\alpha x)^2 - \Delta' \alpha \, \tau_x$$

$\rightsquigarrow$ domain wall has shifted harmonic oscillator spectrum

$$\left(E_n^{\pm}\right)^2 = 2\Delta' \omega \left(n + \tfrac{1}{2}\right) \mp \Delta' \alpha$$

Note:

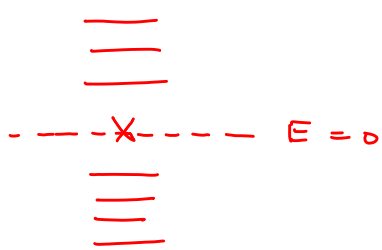
$$\left.\begin{aligned} \frac{1}{2m} &\Leftrightarrow \Delta'^2 \\ \tfrac{1}{2} m \omega^2 &\Leftrightarrow \alpha^2 \end{aligned}\right\} \quad \omega^2 \Leftrightarrow 4\Delta'^2 \alpha^2; \qquad l_{osc}^2 = \frac{1}{m\omega} \Leftrightarrow \Delta'/\alpha$$

$\rightsquigarrow$ E_0^+ is isolated zero-energy solution w/

wavefunction as quoted above.

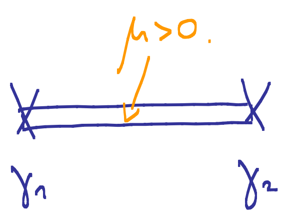
EXERCISE: Solve for the spectrum of an abrupt domain wall $\mu(x) = \mu_0 \,\mathrm{sgn}\, x$ by wavefunction matching, in the linearized model, the full quadratic model, or both. Show that you always find a zero-energy solution and find the corresponding wavefunction.

* Majorana zero mode strongly protected



isolated Majorana mode cannot move away from E = 0 due to particle-hole symmetry

* end of wire is special case of domain wall
(vacuum has $\mu \to -\infty$ and thus $\mu < 0$)



exponentially weak hybridization of Majorana zero modes
$\to$ exponentially small splitting $\propto e^{-L/\xi}$

Remark: gap function with arbitrary phase $|\Delta| e^{i\phi}$

$$\rightsquigarrow H = \left(\frac{p^2}{2m} - \mu\right)\tau_z - i|\Delta| e^{i\phi} p \tau_+ + i|\Delta| e^{-i\phi} p \tau_-$$

gauge transformation: $\mathcal{U} = e^{i\phi \tau_z/2}$

$\rightsquigarrow$ transformed Hamiltonian $\mathcal{U} H \mathcal{U}^\dagger \to H = \xi_p \tau_z + |\Delta'| p \tau_y$

transformed Majorana spinor:

$$u(x)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow u(x)\begin{pmatrix} e^{i\phi/2} \\ e^{-i\phi/2} \end{pmatrix}$$

and

$$\gamma \to \gamma = \int dx\, u(x)\left[e^{i\phi}\psi(x) + e^{-i\phi}\psi^\dagger(x)\right]$$

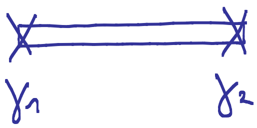
Note that:

$$\phi \to \phi + 2\pi \quad \text{implies} \quad \gamma \to -\gamma$$

useful to understand nonabelian statistics of Majoranas bound to vortices in 2D $(p_x + i p_y)$ SC (Ivanov PRL 2001)

III) Non-abelian statistics

* degeneracy of many-body ground state:



$$c = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

find ground state w/ additional condition $c|gs\rangle = 0$

→ $c^\dagger |gs\rangle$ is also a ground state

→ ground states have different fermion parity (good quantum number for pairing Hamiltonian, while particle # is <u>not</u> conserved)

low energy: fermion parity operator is

$$\boxed{P = (1 - 2c^\dagger c) = i\gamma_1\gamma_2}$$

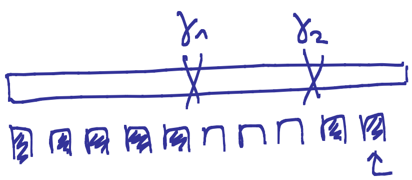
2N Majoranas:

* N fermion operators $c_j = \frac{1}{2}(\gamma_{2j-1} - i\gamma_{2j})$
* 2^N fold degeneracy of ground-state manifold.
* 2^{N-1} even/odd states under fermion parity

w/ $\boxed{P = \prod_{j=1}^{N} i\gamma_{2j-1}\gamma_{2j}}$

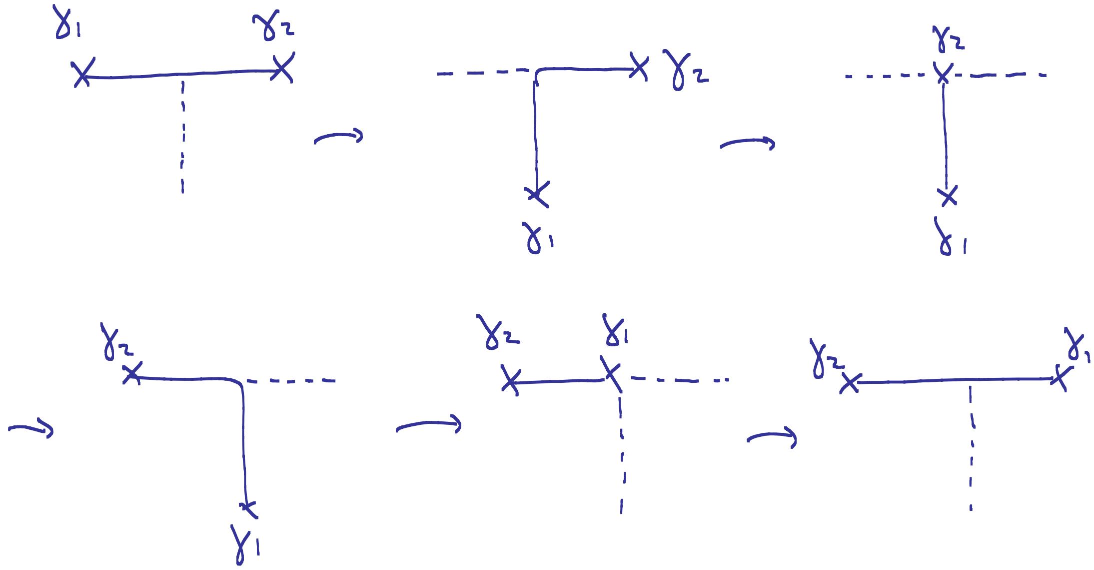
* exchanging Majoranas: need 2d system or wire network

OPTION 1: physically move Majoranas (by moving domain walls)



gate electrodes control μ

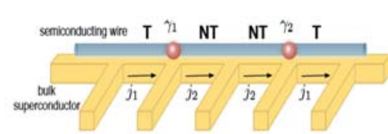
* move Majoranas
* create pairs of Majoranas
* fuse pairs of Majoranas



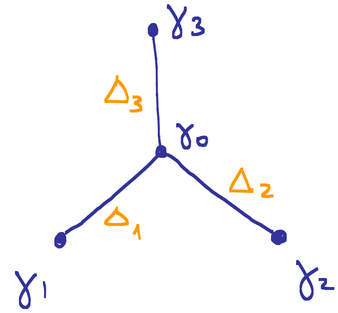
explicit proof of nonabelian statistics: Alicea et al. (2011).

remark: alternative strategy to move Majoranas – supercurrent through proximity-providing s-wave SC

Romito et al. PRB (2012)



OPTION 2: variation of coupling between Majoranas



low-energy Hamiltonian:

$$\boxed{H = i \sum_{j=1}^{3} \Delta_j \gamma_0 \gamma_j}$$

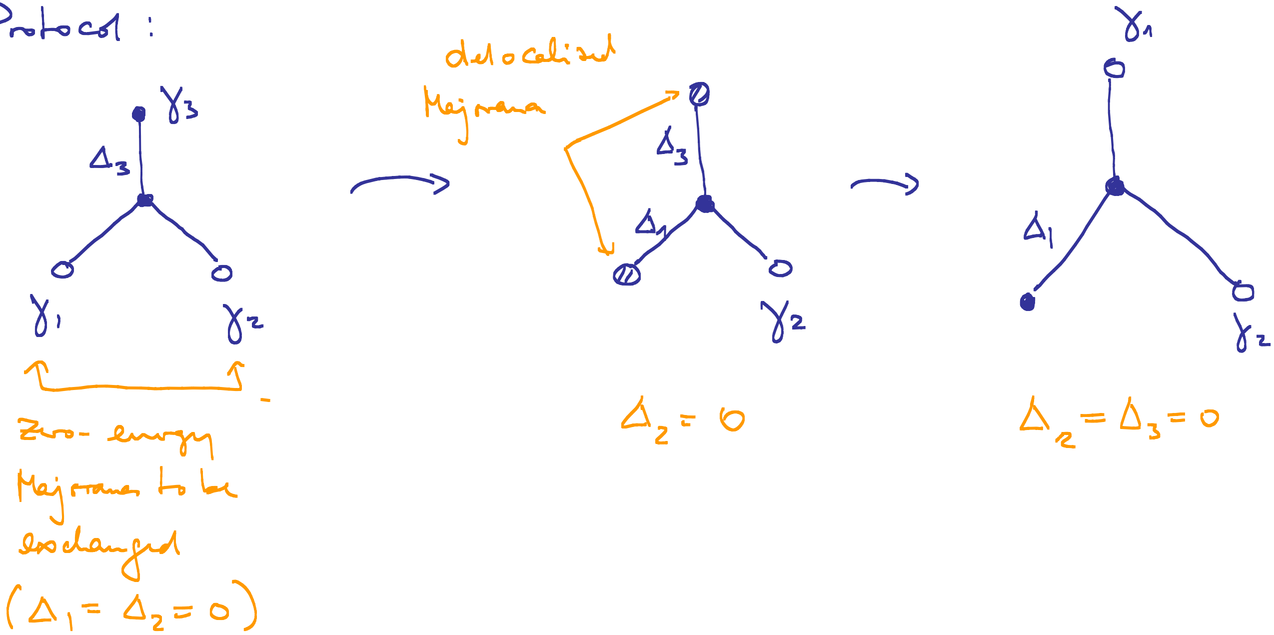
couplings of neighboring Majoranas

$$H = iE\gamma_0\gamma_\Sigma \quad \text{w/} \quad \gamma_\Sigma = \frac{1}{E}\sum_{j=1}^{3}\Delta_j\gamma_j \quad \& \quad E = \sqrt{\Delta_1^2+\Delta_2^2+\Delta_3^2}$$

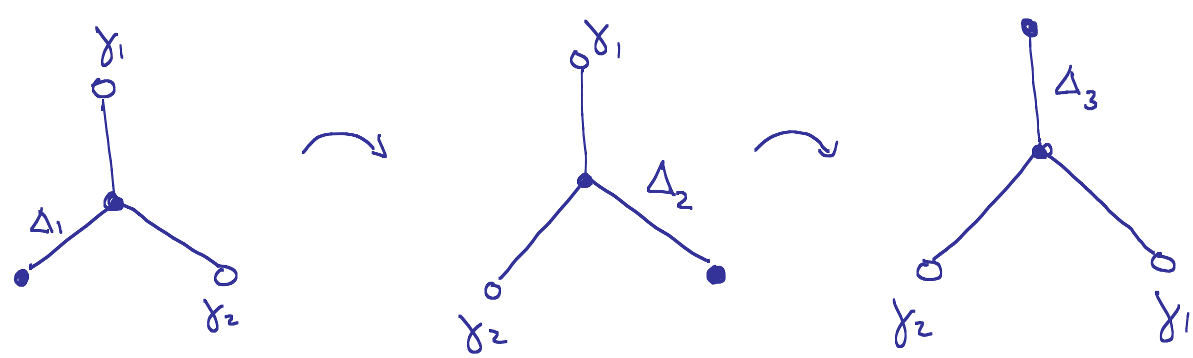
- γ_0 and γ_Σ hybridize into an energy-E Bogoliubov quasiparticle
- two orthogonal linear combinations of $\gamma_1, \gamma_2, \gamma_3$ remain zero-energy Majorana for any Δ_j's
- spectrum has four states, w/ two degenerate states at $\pm E$.
- problem of adiabatic approximation for degenerate systems (Wilczek & Zee PRL 1984), except that degenerate states differ by fermion parity (conserved under adiabatic time evolution) $\rightsquigarrow$ ordinary Berry phase for 2-state system suffices:

Berry phase = half the solid angle subtended by the "magnetic field".

Protocol :



+ two similar moves :



Explicit Berry phase calculation:

introduce conventional fermions: $c_1 = \frac{1}{2}(\gamma_1 - i\gamma_2)$

$$c_2 = \frac{1}{2}(\gamma_0 - i\gamma_3)$$

w/ inverse

$$\gamma_1 = c_1 + c_1^\dagger \,;\; \gamma_2 = i(c_1 - c_1^\dagger)\,;\; \gamma_3 = i(c_2 - c_2^\dagger)\,;\; \gamma_0 = c_2 + c_2^\dagger$$

write Hamiltonian in basis $\{|00\rangle, |11\rangle, |10\rangle, |01\rangle\}$

w/ $|11\rangle = c_1^\dagger c_2^\dagger |00\rangle \qquad |10\rangle = c_1^\dagger |00\rangle \qquad |01\rangle = c_2^\dagger |00\rangle$

& $c_1|00\rangle = c_2|00\rangle = 0$

which yields

$$H = \left(\begin{array}{cc|cc} \Delta_3 & i\Delta_1 - \Delta_2 & 0 & 0 \\ -i\Delta_1 - \Delta_2 & \Delta_3 & 0 & 0 \\ \hline 0 & 0 & \Delta_3 & -i\Delta_1 - \Delta_2 \\ 0 & 0 & i\Delta_1 - \Delta_2 & -\Delta_3 \end{array}\right)$$

block structure reflects conservation of fermion parity

$$P = -\gamma_0\gamma_1\gamma_2\gamma_3$$

$\rightarrow$ $H_{even} = \Delta_3 \tau_z - \Delta_1 \tau_y - \Delta_2 \tau_x$ ← Paulis within even or odd subspace

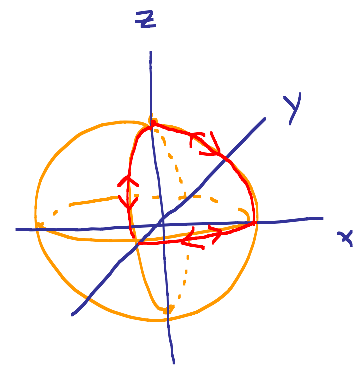
$H_{odd} = \Delta_3 \tau_z + \Delta_1 \tau_y - \Delta_2 \tau_x$

or, w/ Pauli matrices π_j in even-odd subspace

$$H = \Delta_3 \tau_z - \Delta_1 \tau_y \pi_z - \Delta_2 \tau_x .$$

$\rightarrow$ $B_{even} = (-\Delta_2, -\Delta_1, \Delta_3)$

$B_{odd} = (-\Delta_2, \Delta_1, \Delta_3)$



braiding path of $\hat{\Delta}$

$\rightarrow$ B_{even} & B_{odd} enclose a solid angle of $\pm \frac{\pi}{2}$

$\rightarrow$ $U_{12} = \exp\left\{ \frac{i\pi}{4} \tau_z \pi_z \right\}$ Berry phase due to braiding.

finally note:

$$i\gamma_1\gamma_2 = \tau_z \pi_z$$

This yields:

$$\boxed{U_{12} = \exp\left\{-\frac{\pi}{4}\gamma_1\gamma_2\right\} = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)}$$

$$\boxed{\begin{aligned} \gamma_1 &\to U_{12}\,\gamma_1\,U_{12}^{-1} = -\gamma_2 \\ \gamma_2 &\to U_{12}\,\gamma_2\,U_{12}^{-1} = \gamma_1 \end{aligned}}$$

Indeed:

$$U_{ij}\,\gamma_i\,U_{ij}^{-1} = \frac{1}{\sqrt{2}}(1+\gamma_i\gamma_j)\,\gamma_i\,\frac{1}{\sqrt{2}}(1-\gamma_i\gamma_j)$$

$$= \frac{1}{2}(1+\gamma_i\gamma_j)(\gamma_i-\gamma_j) = \frac{1}{2}(\gamma_i-\gamma_j-\gamma_j-\gamma_i)$$

$$= -\gamma_j$$

$$U_{ij}\,\gamma_j\,U_{ij}^{-1} = \frac{1}{2}(1+\gamma_i\gamma_j)(\gamma_j+\gamma_i)$$

$$= \frac{1}{2}(\gamma_j+\gamma_i+\gamma_i-\gamma_j) = \gamma_i\,.$$

- statistics is nontrivial

$$u_{ij}^2 = \frac{1}{2}(1+\gamma_i\gamma_j)(1+\gamma_i\gamma_j)$$

$$= \frac{1}{2}(1+2\gamma_i\gamma_j+\gamma_i\gamma_j\gamma_i\gamma_j)$$

$$= \frac{1}{2}(1+2\gamma_i\gamma_j-1)$$

$$\Rightarrow \boxed{u_{ij}^2 = \gamma_i\gamma_j \neq \mathbb{1}.}$$

- statistics is nonabelian:

$$[u_{12}, u_{23}] = \gamma_1\gamma_3$$

Since

$$u_{12}u_{23} = \frac{1}{2}(1+\gamma_1\gamma_2)(1+\gamma_2\gamma_3) = \frac{1}{2}(1+\gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_1\gamma_3)$$

$$u_{23}u_{12} = \frac{1}{2}(1+\gamma_2\gamma_3)(1+\gamma_1\gamma_2) = \frac{1}{2}(1+\gamma_1\gamma_3+\gamma_2\gamma_3-\gamma_1\gamma_3)$$

Check defining relation of braid group!

shorthand notation $\quad U_i = U_{i,i+1}$

- $U_i U_j = U_j U_i \quad$ for $|i-j| \geq 2$

checks trivially

- $U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}$

Since

$$U_1 U_2 U_1 = \frac{1}{2\sqrt{2}} (1+\gamma_1\gamma_2)(1+\gamma_2\gamma_3)(1+\gamma_1\gamma_2)$$

$$= \frac{1}{2\sqrt{2}} \Big(1 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_2\gamma_3$$

$$+ \gamma_1\gamma_2\gamma_1\gamma_2 + \gamma_2\gamma_3\gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_2\gamma_3\gamma_1\gamma_2\Big)$$

$$= \frac{1}{2\sqrt{2}} \Big(1 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_2 + \gamma_1\gamma_3$$

$$-1 - \gamma_1\gamma_3 + \gamma_2\gamma_3\Big)$$

$$= \frac{1}{\sqrt{2}} (\gamma_1\gamma_2 + \gamma_2\gamma_3)$$

$$u_2 u_1 u_2 = \frac{1}{2\sqrt{2}} (1+\gamma_2\gamma_3)(1+\gamma_1\gamma_2)(1+\gamma_2\gamma_3)$$

$$= \frac{1}{2\sqrt{2}} \big(1+\gamma_2\gamma_3+\gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_2\gamma_3\gamma_1\gamma_2$$

$$+\gamma_2\gamma_3\gamma_2\gamma_3+\gamma_1\gamma_2\gamma_2\gamma_3+\gamma_2\gamma_3\gamma_1\gamma_2\gamma_2\gamma_3\big)$$

$$= \frac{1}{2\sqrt{2}} \big(1+\gamma_2\gamma_3+\gamma_1\gamma_2+\gamma_2\gamma_3-\gamma_1\gamma_3$$

$$-1+\gamma_1\gamma_3+\gamma_1\gamma_2\big)$$

$$= \frac{1}{\sqrt{2}} (\gamma_1\gamma_2+\gamma_2\gamma_3)$$