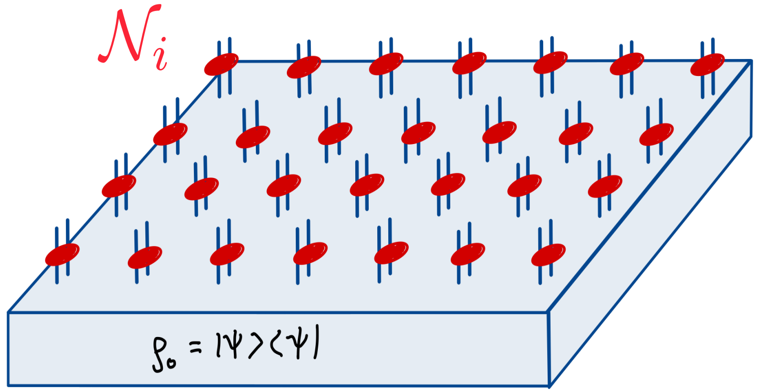
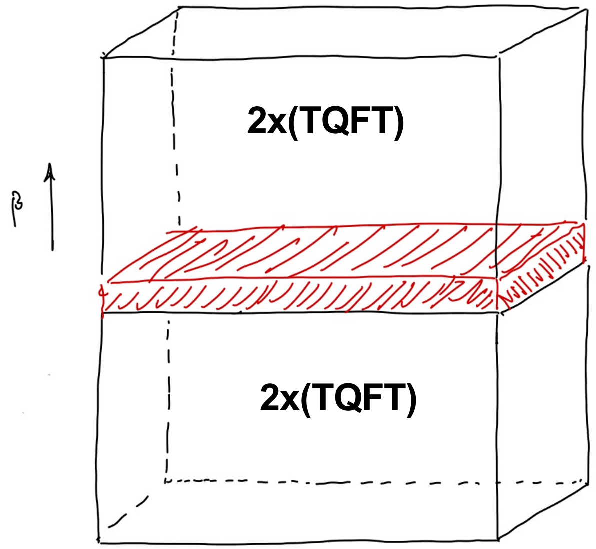
Lecture 3:

Decodability transition in error corrupted topological states

Decoherence



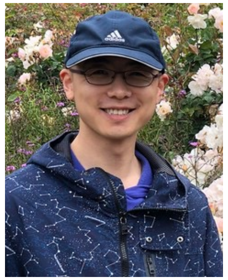
Map to boundary
anyon condensation



Y. Bao, R. Fan, A. Vishwanath, EA arXiv:2301.05687

R. Fan, Y.Bao, EA, A. Vishwanath arXiv:2301.05689

Yimu Bao

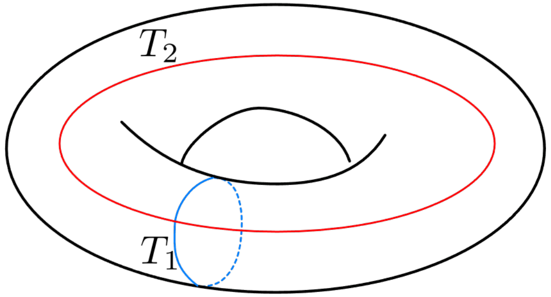


Ruihua Fan

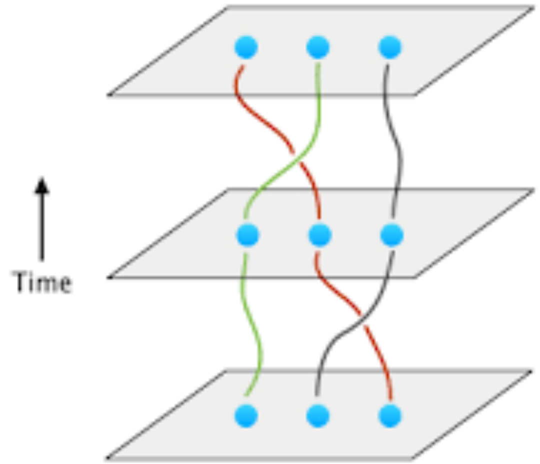
Ashvin
Vishwanath

Ground state topological order

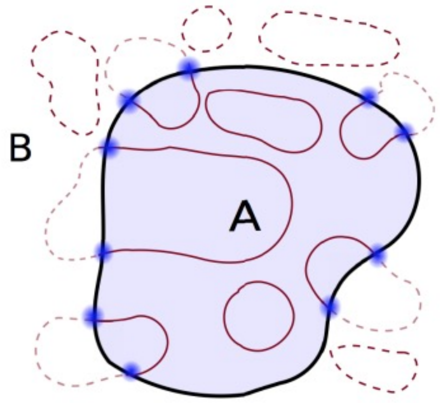
(i) Topological ground state degeneracy



(ii) Anyonic excitations



(iii) Topological entanglement entropy

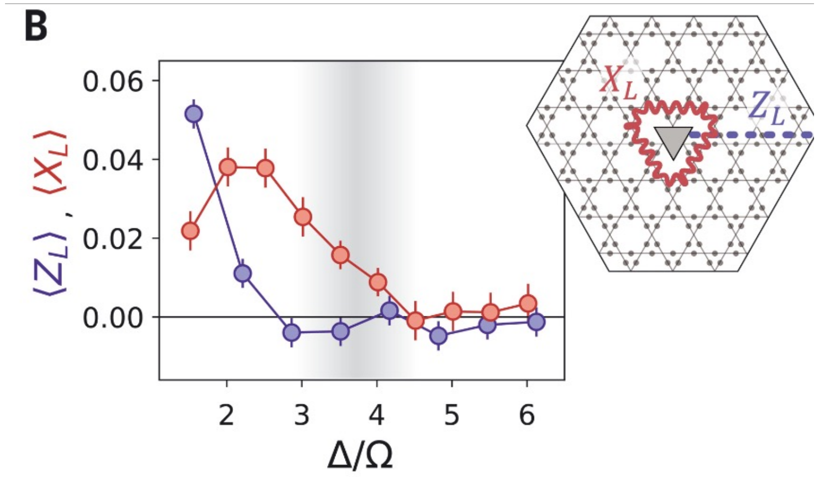


Recent demonstrations of topological order

1. Z_2 topological order in Rydberg atom arrays

Semeghini et al. *Science* 2021

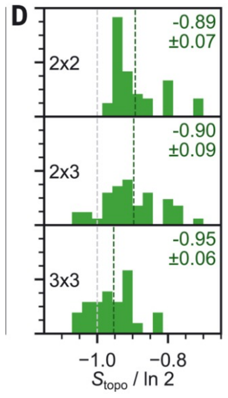
- Measured Wilson and t'Hooft lines (Open string order parameters)

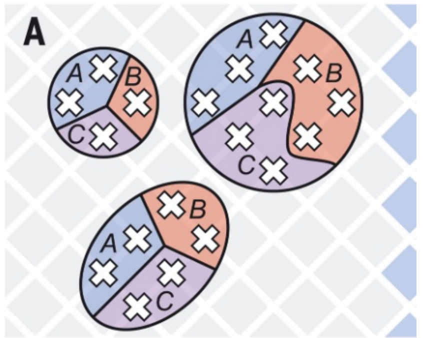


2. Toric code state realized in a SC qubit array

Google quantum AI Science 2021

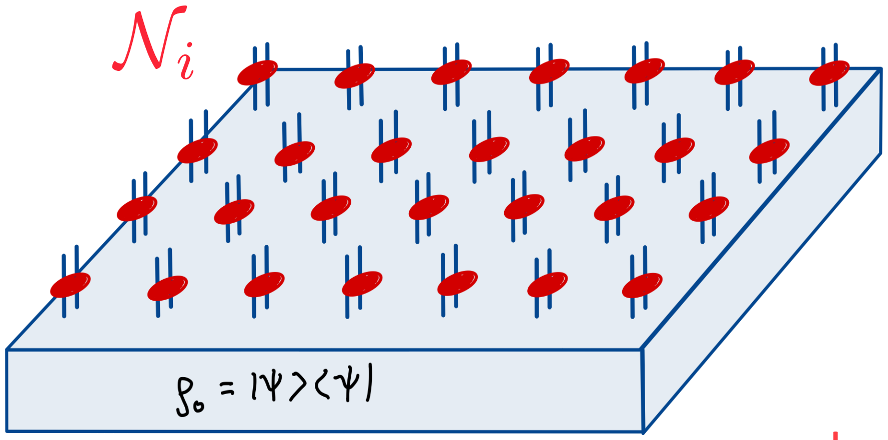
- Measured the topological entanglement entropy





But these systems are not prepared in their ground state!
They are mixed states due to decoherence.

Corrupted topological states



$$\rho = \prod_i \mathcal{N}_i[|\Psi_0\rangle\langle\Psi_0|]$$

$$\mathcal{N}_i[\cdot] = \sum_m \mathcal{K}_{m,i}\,(\cdot)\,\mathcal{K}_{m,i}^{\dagger}$$

Local channel

Kraus operator

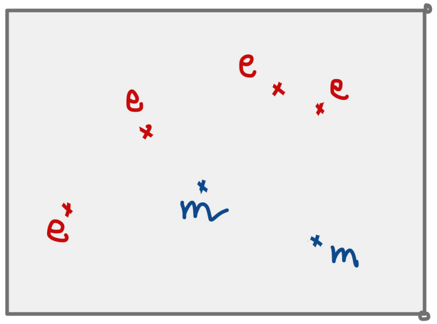
- Is topological order sharply defined in such corrupted mixed states?
- What are the possible phases?
- How to diagnose mixed topological states ?

Two conflicting perspectives on the fate of topological order in the corrupted mixed state

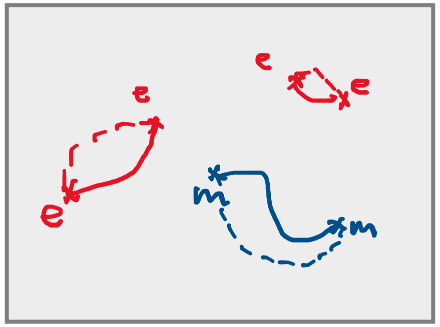
1. Existence of an error threshold → Topological transition in ρ ?

For Toric Code: Dennis et al., *J. Math. Phys.* 2002

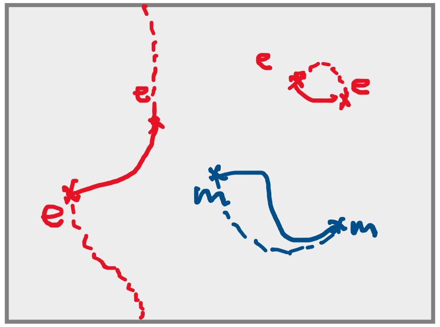
i) Detect error syndromes:
projects onto a configuration of anyons



ii) Attempt to correct errors by annihilating anyons in pairs



No logical error



“Logical error”!

Phase transition in the failure probability at a critical error rate. In principle can depend on the matching algorithm

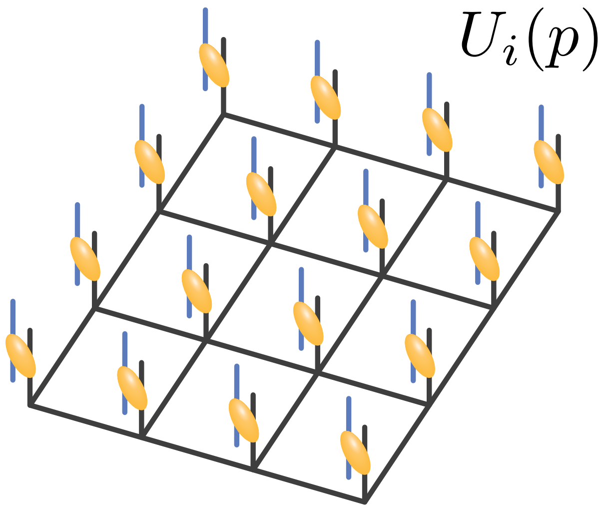
Intrinsic Topological transition in the underlying mixed state ρ ?

Two conflicting perspectives on the fate of topological order in the corrupted mixed state

2. Local decoherence = finite depth unitary circuit

Topologically ordered state → $|\Psi_0\rangle$; Ancillas → $(\otimes_i|0\rangle_i)$

$$|\Phi\rangle = \prod_i U_i(p)|\Psi_0\rangle \, (\otimes_i|0\rangle_i)$$



Cannot get singular change in any expectation value:

$$\langle\Phi|\hat{O}|\Phi\rangle = \mathrm{tr}(\rho\,\hat{O})$$

From this perspective, the mixed state remains topological for any finite decoherence strength.

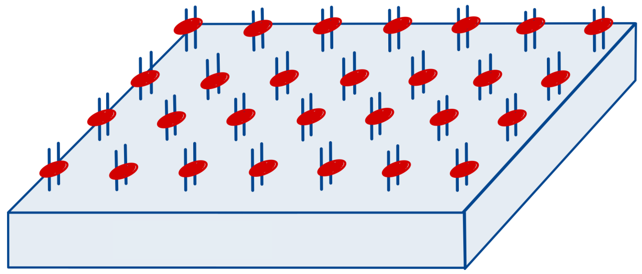
How to resolve the conflict ?

Resolution: transitions in an errorfield double

Treat the density matrix as a state vector in a doubled Hilbert space:

$$\rho_0 = |\psi_0\rangle\langle\psi_0| \;\rightarrow\; |\rho_0\rangle\!\rangle = |\psi_0\rangle \otimes |\psi_0\rangle$$

$$|\rho\rangle\!\rangle = \underbrace{\prod_i \sum_\ell K_{\ell,i}\,\overline{K}_{\ell,i}}_{\mathcal{N}_{\text{EFD}}} |\psi_0, \psi_0\rangle$$



Decoherence is a non-unitary transformation of the state vector.

➡ Can drive phase transitions in $|\rho\rangle\!\rangle$, diagnosed by super-observables:

$$\langle\!\langle \hat{O} \rangle\!\rangle \equiv \frac{\langle\!\langle \rho | O\,\bar{O} | \rho \rangle\!\rangle}{\langle\!\langle \rho | \rho \rangle\!\rangle} = \frac{\mathrm{tr}(\hat{O}\rho\,\hat{O}^\dagger \rho)}{\mathrm{tr}(\rho^2)}$$

Physical interpretation: overlap ~ distinguishability between the ensembles defined by the two mixed states ρ and $\hat{O}\rho\,\hat{O}^\dagger$

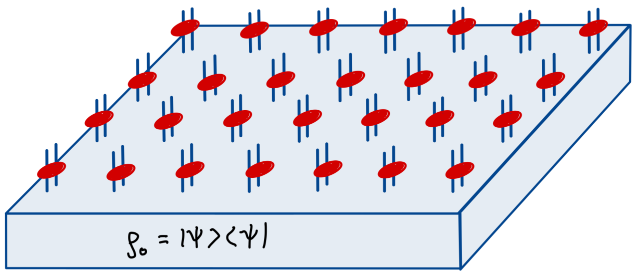
Decoherence induced anyon condensation

Example: bit-flip errors in the toric code

$$|\rho\rangle\rangle = \exp\left(\gamma \sum_i X_i \bar{X}_i\right) |\psi_0, \psi_0\rangle\rangle$$

$$= \exp\left(\gamma \sum_{\langle \ell\ell' \rangle} (m_\ell \overline{m}_\ell)(m_{\ell'} \overline{m}_{\ell'})\right) |\rho_0\rangle\rangle$$

$\rho \to (1-\gamma)\rho + \gamma\, X_i \rho X_i$

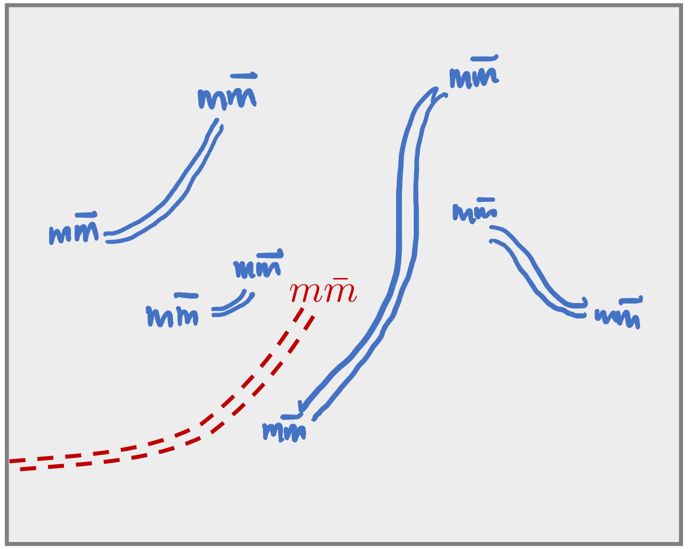


Drives condensation of $m\overline{m}$

Diagnosed by the open string order parameter (creation operator of $m\overline{m}$)

$$\widehat{O}_{m\overline{m}}(r) = O_m(r) \otimes O^*_m(r) = \prod_i^r X_i \overline{X}_i$$

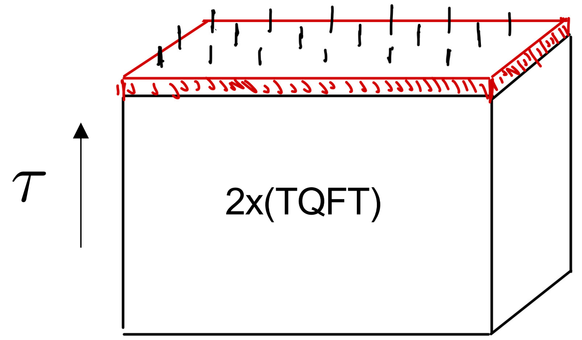
$$\langle\langle \widehat{O}_{m\overline{m}}(r) \rangle\rangle \neq 0$$



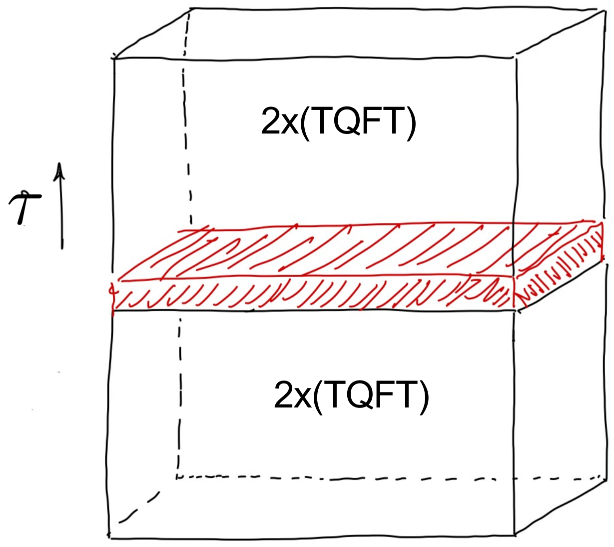
= Overlap between the corrupted state and the same corrupted state injected with a single m anyon. If vanishing, then the anyon excitation is well defined.

Mapping to a boundary transition

$$|\rho\rangle\rangle = \lim_{\beta\to\infty} \mathcal{N}_{\rm EFD}\, e^{-\beta(H+\bar{H})}|\rho_{\rm ref}\rangle\rangle$$



Path integral for the norm $\langle\langle\rho|\rho\rangle\rangle$

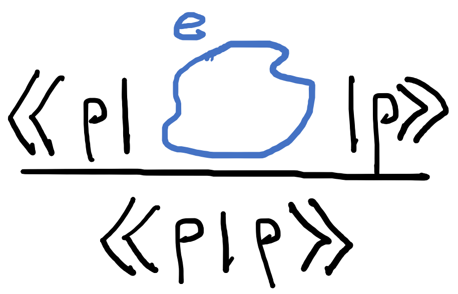


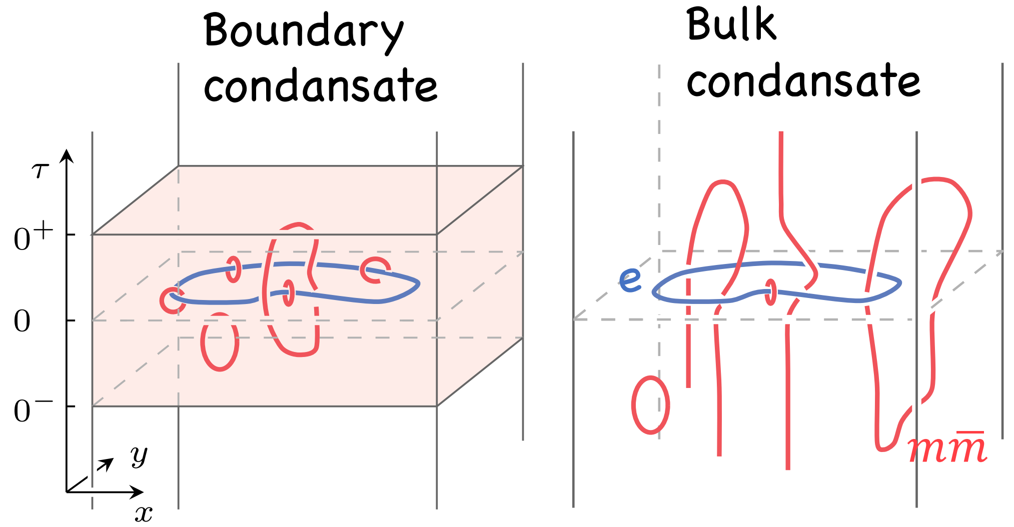
Two TQFTs in 2+1d space time coupled by the decoherece channel on the defect plane τ=0

Important distinction from bulk anyon condensation

Boundary condensation of $m\bar{m}$ doesn't lead to confinement of e

- Wilson loop

$\langle\langle W_e \rangle\rangle =$ 

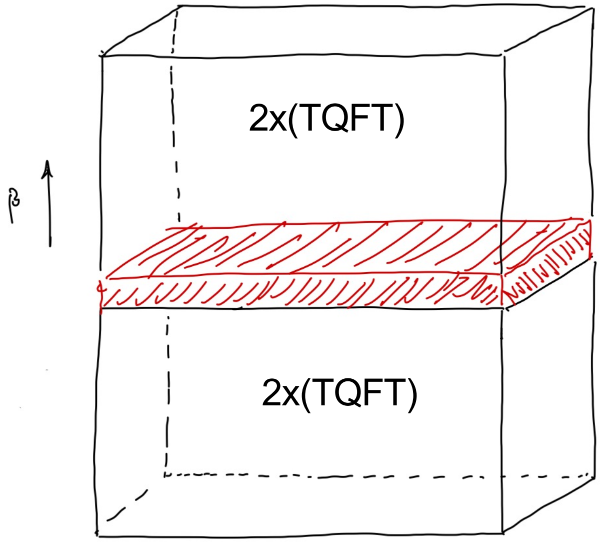


- Large world lines occur with decaying probability
- Wilson loop always follows a perimeter law
- Condensing $m\bar{m}$ on the boundary doesn't confine e

Pauli-X errors

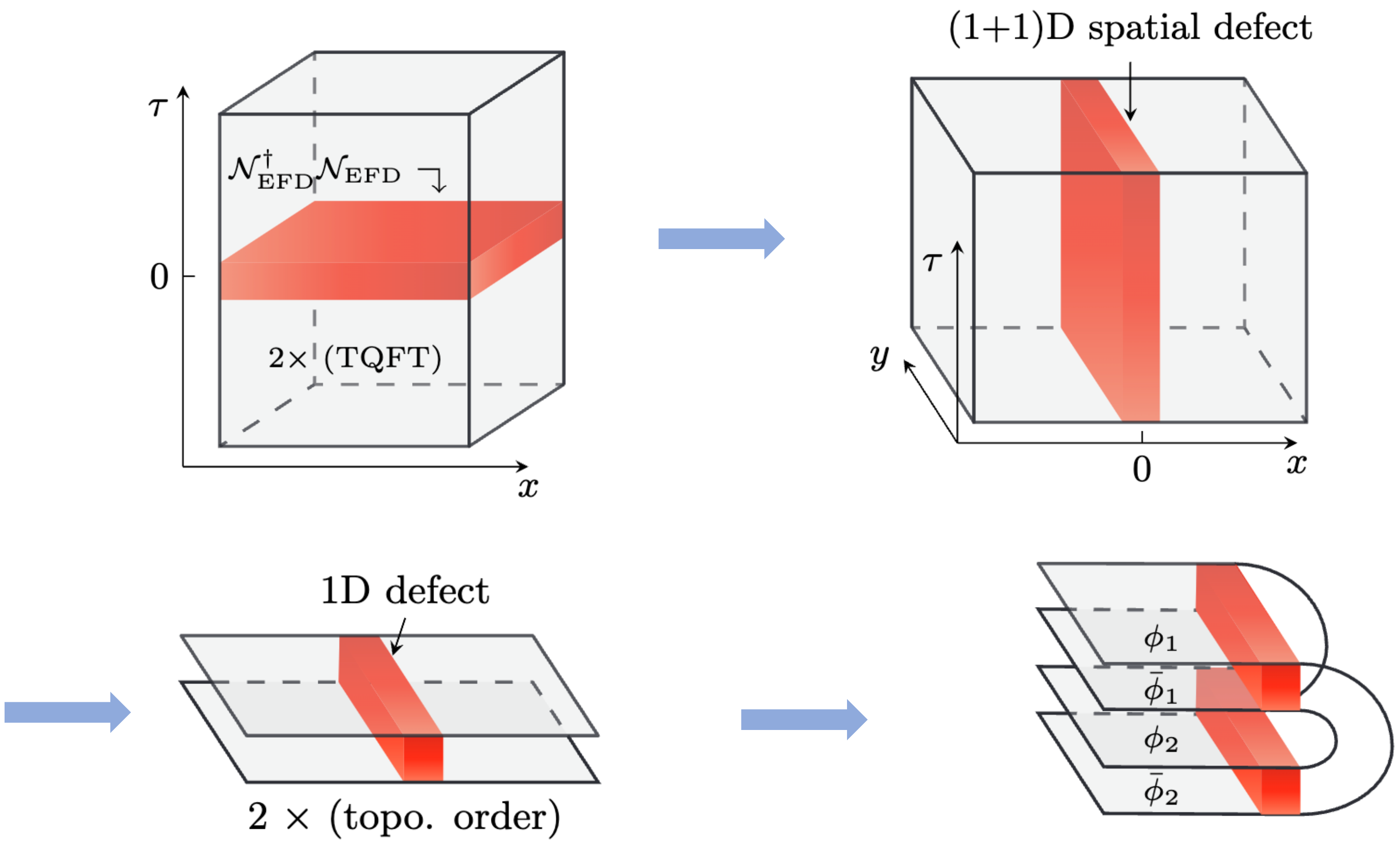
↓

Condensed phase of $m\bar{m}$



- Pauli-Y and -Z errors can give condensed phases of $e\bar{e}$ and $f\bar{f}$
- How to classify the possible error-induced phases?

Mapping to one dimensional edge states

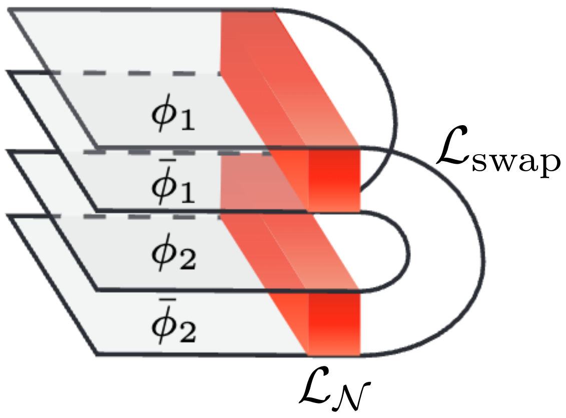


Edge of quadruple topological order

K-matrix classification for Abelian topological order

Generalized Chern-Simons edge theory:

$$\mathcal{L}[\phi] = \frac{1}{4\pi}\sum_{I,J} \mathbb{K}_{IJ}\partial_\tau\phi^I\partial_y\phi^J - \mathbb{V}_{IJ}\partial_y\phi^I\partial_y\phi^J$$
$$+\mathcal{L}_{\text{swap}} + \mathcal{L}_{\mathcal{N}}$$



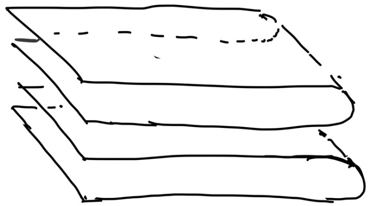
Restrict to incoherent errors (don't create superpositions of different anyon pairs) $K_i = \alpha_r \alpha_{r'}$

Different ways to gap the edge: Restrict to incoherent errors

topological

→ intermediate states →

trivial



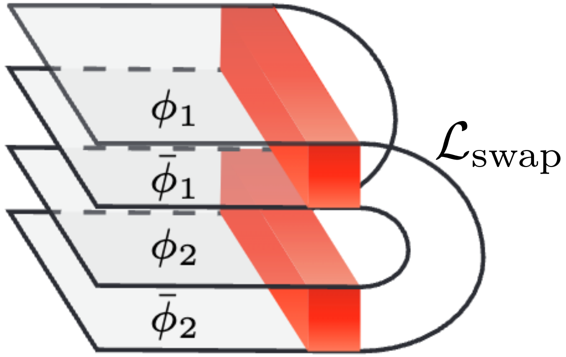
Generally classified by "Lagrangian subgroups" of anyon condensates

Levin PRX 2013, Barkeshli et al, PRB 2013

K-matrix classification for Abelian topological order

Generalized Chern-Simons edge theory:

$$\mathcal{L}[\phi] = \frac{1}{4\pi} \sum_{I,J} \mathbb{K}_{IJ} \partial_\tau \phi^I \partial_y \phi^J - \mathbb{V}_{IJ} \partial_y \phi^I \partial_y \phi^J$$
$$+ \mathcal{L}_{\text{swap}} + \mathcal{L}_{\mathcal{N}}$$

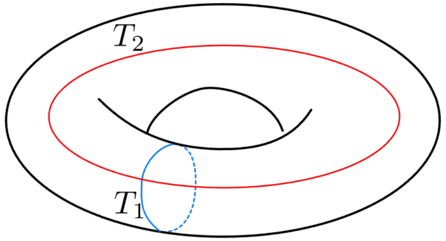


Model		Memory	Edge condensate (generators of Lagrangian subgroup)	Error that realizes the phase
Toric code	I	Quantum	$e_1\bar{e}_2, \bar{e}_1 e_2, m_1\overline{m}_2, \overline{m}_1 m_2$	No error
	II	Classical	$e_1\bar{e}_1, e_2\bar{e}_2, e_1\bar{e}_2, m_1\overline{m}_1 m_2\overline{m}_2$	Incoherent e error
	III	Classical	$m_1\overline{m}_1, m_2\overline{m}_2, m_1\overline{m}_2, e_1\bar{e}_1 e_2\bar{e}_2$	Incoherent m error
	IV	Classical	$f_1\bar{f}_1, f_2\bar{f}_2, f_1\bar{f}_2, e_1\bar{e}_1 e_2\bar{e}_2$	Incoherent f error
	V	Trivial	$e_1\bar{e}_1, e_2\bar{e}_2, m_1\overline{m}_1, m_2\overline{m}_2$	Incoherent e and m error
Double semion	I	Quantum	$m_{a,1}\overline{m}_{a,2}, \overline{m}_{a,1}m_{a,2}, m_{b,1}\overline{m}_{b,2}, \overline{m}_{b,1}m_{b,2}$	No error
	II	Quantum	$m_{a,1}\overline{m}_{a,1}, m_{a,2}\overline{m}_{a,2}, m_{b,1}\overline{m}_{b,2}, \overline{m}_{b,1}m_{b,2}$	Incoherent m_a error
	III	Quantum	$m_{b,1}\overline{m}_{b,1}, m_{b,2}\overline{m}_{b,2}, m_{a,1}\overline{m}_{a,2}, \overline{m}_{a,1}m_{a,2}$	Incoherent m_b error
	IV	Quantum	$b_1\bar{b}_1, b_2\bar{b}_2, b_1\bar{b}_2, m_{a,1}\overline{m}_{a,1}m_{a,2}\overline{m}_{a,2}$	Incoherent b error
	V	Trivial	$m_{a,1}\overline{m}_{a,1}, m_{a,2}\overline{m}_{a,2}, m_{b,1}\overline{m}_{b,1}, m_{b,2}\overline{m}_{b,2}$	Incoherent m_a and m_b error

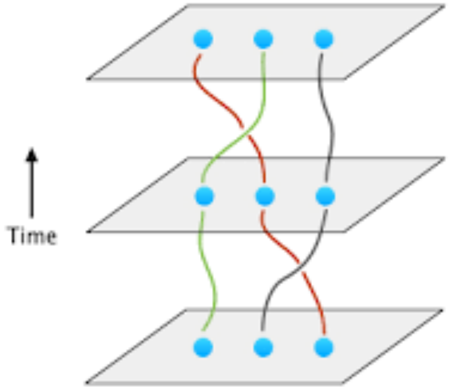
- Because of the doubling this applies also to chiral topological states
- Can be generalized to chiral non abelian states

Ground state topological order

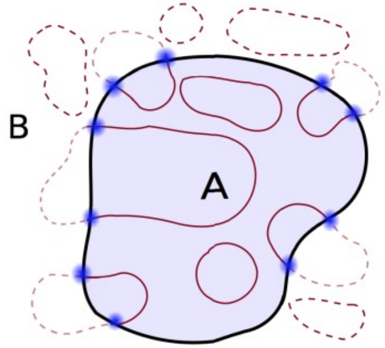
(i) Topological ground state degeneracy



(ii) Anyonic excitations



(iii) Topological entanglement entropy

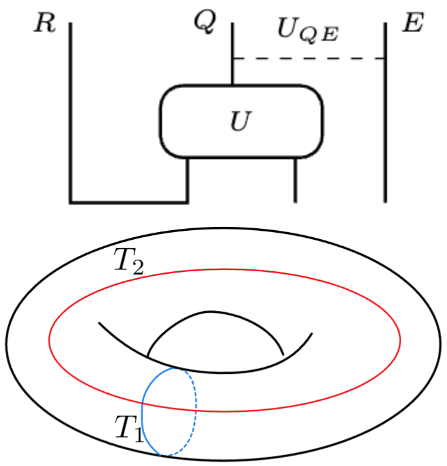


How to generalize these defining properties to corrupted mixed states?

Information theoretic diagnostics

(i) Coherent information

- Encode information into the degenerate ground state
- Apply decoherence channel
- How much of the information can be recovered?



$$I_c(R\rangle Q) := S_Q - S_{QR}.$$

- For successful quantum error correction need $I_c = S_R$

Information theoretic diagnostics

(ii) Quantum relative entropy

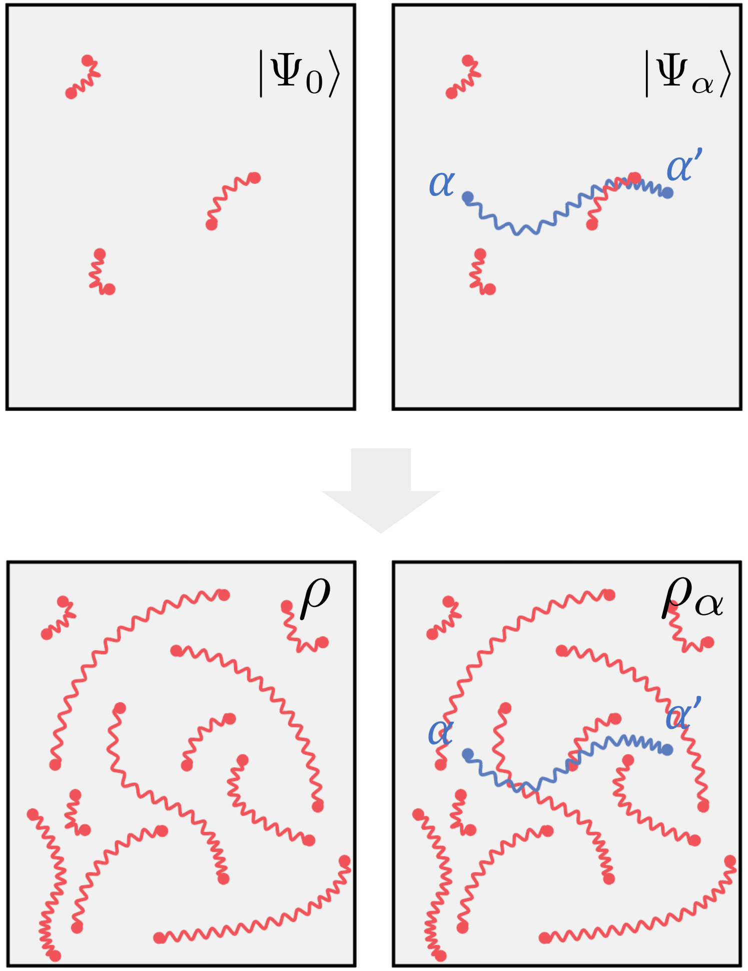
Doubled string order parameter

↓

Information theoretic measure of the integrity of anyon excitations

Distinguishability between the corrupted state and the with an extra anyon.

$$D(\rho||\rho_\alpha) := \mathrm{tr}\rho\log\rho - \mathrm{tr}\rho\log\rho_\alpha$$



Information theoretic diagnostics

(iii) Topological entanglement entropy

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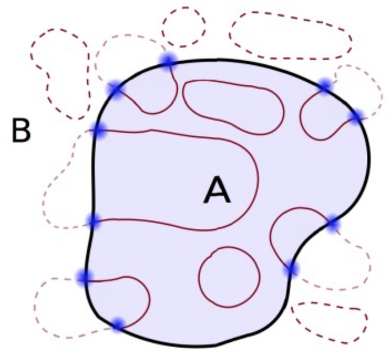
Sub-system logarithmic negativity



A measure of quantum entanglement in a mixed state

$$\mathcal{E}_A(\rho) := \log \| \rho^{T_A} \|$$
$$= c|\partial A| - \gamma_N$$

Conjectured topological term

Example: TC+incoherent errors

- Toric code with incoherent errors, i.e. bit-flip and phase errors (generalizations to Zn Toric code with certain errors are straightforward)

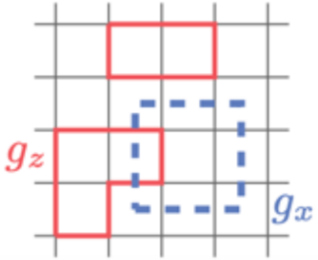
$$A_s = \prod_{\ell \in \text{star}(s)} X_\ell\,, \quad B_p = \prod_{\ell \in \text{plaq(p)}} Z_\ell$$

$$\rho_0 \to (1-p_x)\rho_0 + p_x X_\ell \rho_0 X_\ell$$

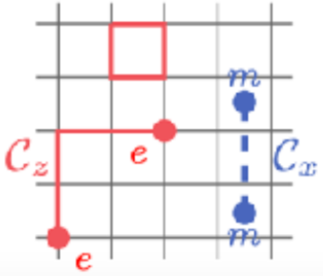
$$\rho_0 \to (1-p_z)\rho_0 + p_z Z_\ell \rho_0 Z_\ell$$

- There are two representations of the error-corrupted state

$$\rho \propto \sum_{g_x, g_z} (1-2p_x)^{|g_z|}(1-2p_z)^{|g_x|} g_x g_z$$



$$\rho = \sum_{C_x, C_z} p(C_x)p(C_z)\, X^{C_x} Z^{C_z} \rho_0 Z^{C_z} X^{C_x}$$



The toric code as a test case

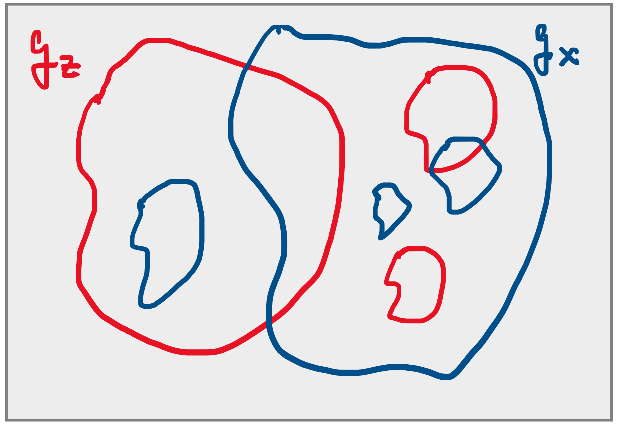
Pauli-X and Z errors

$$\mathcal{N}_X[\rho] = (1-p_x)\rho + p_x X_i \rho X_i$$
$$\mathcal{N}_Z[\rho] = (1-p_z)\rho + p_z Z_i \rho Z_i$$

Density matrix as an effective loop model:

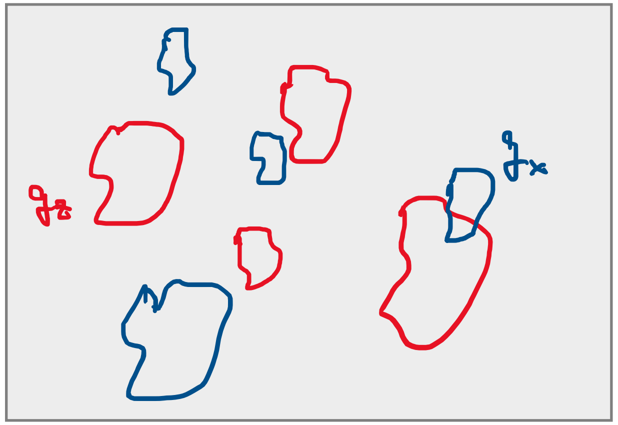
No errors:

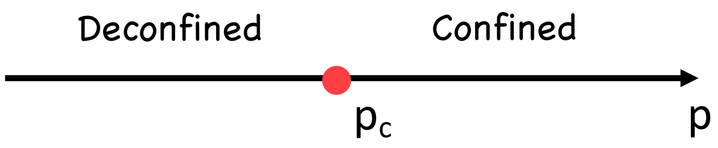
$$\rho_0 = \prod_s \frac{1+A_s}{2} \prod_p \frac{1+B_p}{2} = \sum g_x g_z$$



Errors add loop tension:

$$\rho = \sum_{g_x, g_z} e^{-\mu_x |g_x| - \mu_z |g_z|} g_x g_z$$

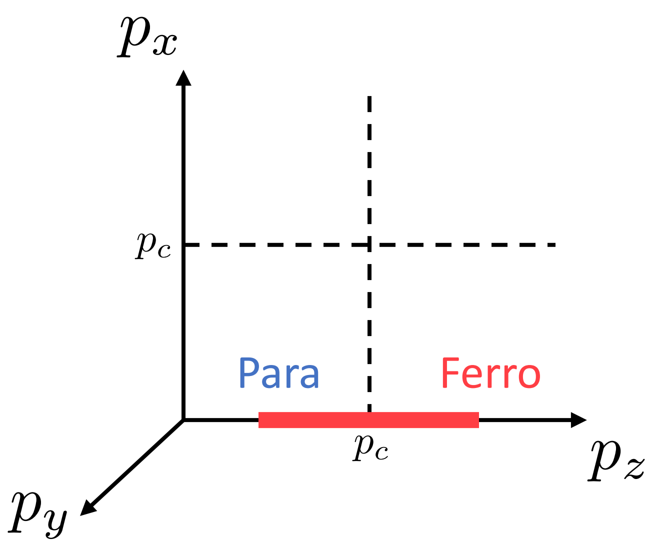




The toric code as a test case

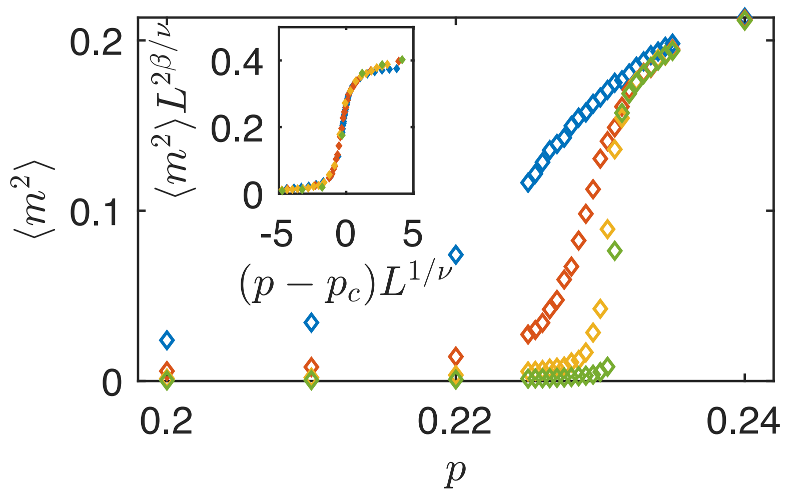
Information theoretic quantities derived from the stat-mech model for

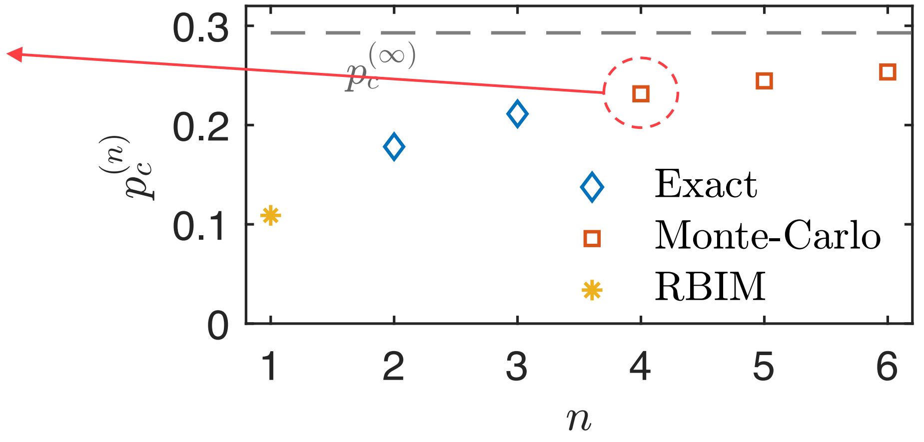
$$\mathrm{tr}\rho^n = \mathcal{Z}_n$$



n flavors of loops (with one constraint)
= n-1 flavor Ising model

$$H_{n,a} = -J_a \sum_{\langle i,j\rangle} \left(\sum_{s=1}^{n-1} \sigma_i^{(s)} \sigma_j^{(s)} + \prod_{s=1}^{n-1} \sigma_i^{(s)} \sigma_j^{(s)} \right),$$





Information theoretic diagnostics in the stat-mech model

All three diagnostics detect ferromagnetic ordering and undergo transition simultaneously

- Relative entropy is order parameter correlation

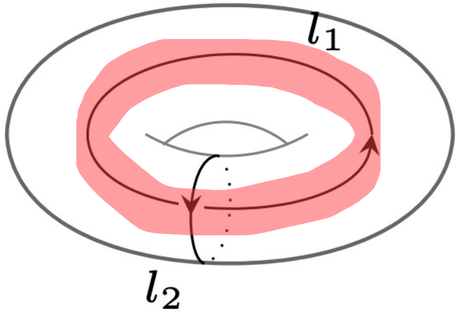
$$D^{(n)} = \frac{1}{1-n} \log \left\langle \sigma_{i_l}^{(1)} \sigma_{i_r}^{(1)} \right\rangle$$

Ferro: O(1)

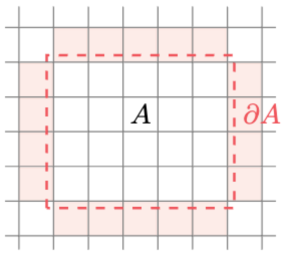
Para: $\propto |i_l - i_r|$

- Coherent info is related to inserting large defects

$$I_c^{(n)} = \frac{1}{n-1} \sum_{a=x,z} \log \left(\sum_{\mathbf{d}_{a1}\mathbf{d}_{a2}} e^{-F_{n,a}^{(\mathbf{d}_{a1},\mathbf{d}_{a2})}} \right) - 2\log 2$$



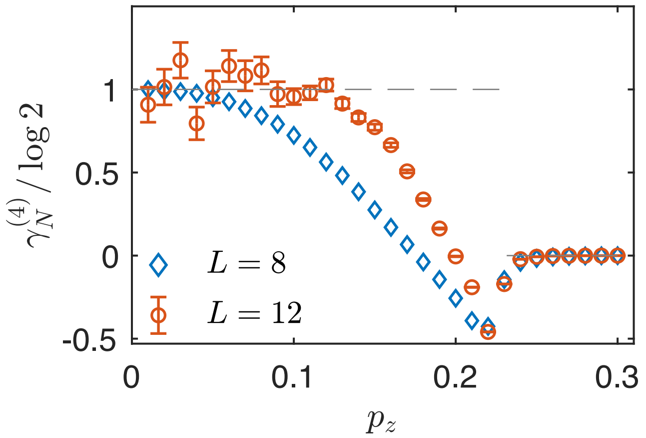
- Negativity is excess free energy for spin pinning



Ferro: $\frac{|\partial A|}{\xi}$

Para: $\frac{|\partial A|}{\xi} - \log 2$

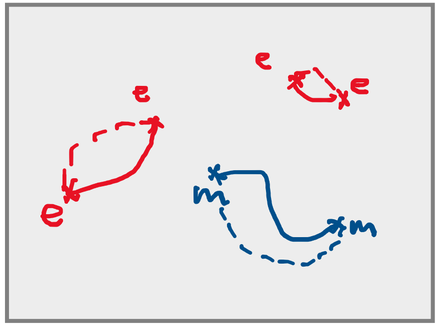
Topo. negativity



Connection to quantum error correction

Quantum error correction algorithm:

- Detect anyons by syndrome measurements
- Annihilate anyons in pairs



non-unitary, partial loss of coherent info.

$$p_c \geqslant p_{c,\text{algo}}$$

Dennis et. al. 2002: Optimal algorithm for correcting incoherent errors in the toric code. Maps to critical point of random bond Ising model (RBIM)

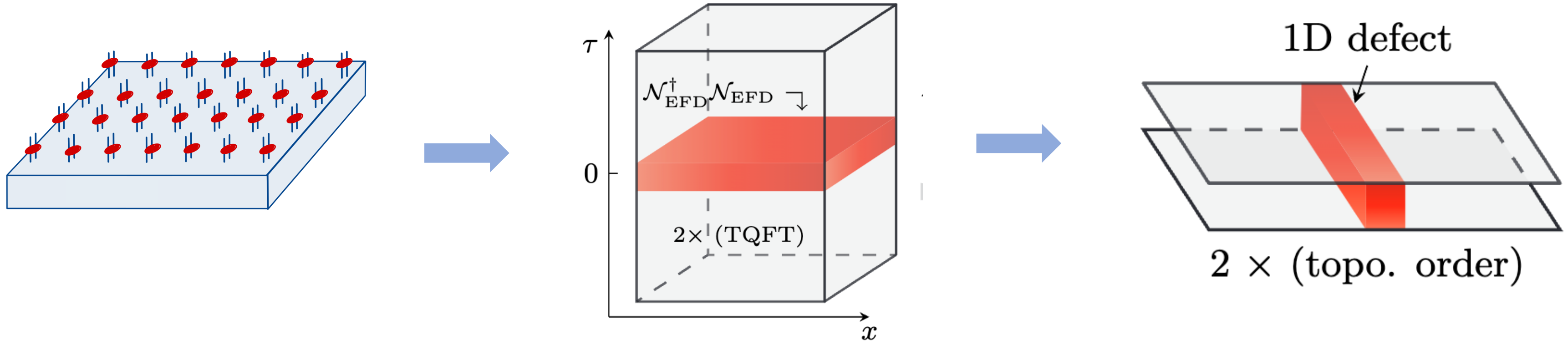
Indeed we can show an exact duality between the n-flavor Ising and the replicated RBIM

Error configuration picture (low-T expansion)

$$\rho = \sum_{\mathcal{C}_x, \mathcal{C}_z} P(\mathcal{C}_x, \mathcal{C}_z) X^{\mathcal{C}_x} Z^{\mathcal{C}_z} |\Psi_0\rangle\langle\Psi_0| Z^{\mathcal{C}_z} X^{\mathcal{C}_x} \qquad \mathrm{tr}\rho^n = \mathcal{Z}_{\mathrm{RBIM}}$$

Summary

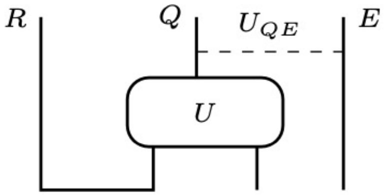
- Decoherence induced topologically ordered phases map to boundary anyon condensation

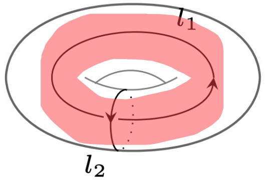


- Characterized by information theoretic measures

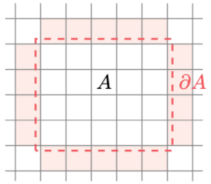
(i) Quantum relative entropy $D(\rho||\rho_\alpha) := \mathrm{tr}\rho \log \rho - \mathrm{tr}\rho \log \rho_\alpha$

(ii) Coherent information $I_c(R\rangle Q) := S_Q - S_{QR}.$





(iii) Topological entanglement negativity $\mathcal{E}_A := \log \|\rho^{T_A}\|_1 = a|\partial A| - \gamma_N$



Outlook

- Quantization of log negativity demonstrated only for Toric code. More general understanding?
- Generalize the edge theory to information theoretic quantities (replica limit of n-copy CS theory)
- Characterize non abelian states subject to dissipation
- Intrinsic error thresholds for quantum computation with non-abelian anyons.