Biological flow networks: applications to other systems

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Abstract. These notes are intended as a brief introduction to flow networks, with motivation and emphasis in biological applications. In this third part we explore some of the things flow networks can teach us about mechanical networks and random walks on lattices.

1. Introduction

Flow networks are just a beautiful example of a general class of linear network systems. As such, there is a direct analogy with many other linear systems, such as mechanical networks, random walks on graphs etc. The following notes partially follow and adapt [1], where the interested reader is referred for a more complete discussion and citation list.

2. Mechanical networks

In Table 2 we list the one to one correspondence between flow, electrical and mechanical networks. The correspondence between the first two (flow and electrical) is rather obvious for steady state flows, but still listed for completeness.

Flow	**Electrical**	**Mechanical**
pressure p_i	voltage V_i	displacement x_i
pressure different Δp_{ij}	voltage drop ΔV_{ij}	elongation (strain) $\Delta xij = x_i - x_j$
fluid current Q_{ij}	electrical current	spring force F_{ij}
conductance C_{ij}	conductance	spring constant k
net current q_i	net current	Q_{ij}
pressure flow relationship	Ohm's law	$F = k\Delta x$
current conservation	current conservation	force conservation $\sum_i F_{ij} = f_j$
energy dissipation $Q_{ij}^2 C_{ij}$	electrical dissipation at resistors	mechanical energy $k\Delta x^2$
fluid storage	capacitance	
fluid inertia	inductance	

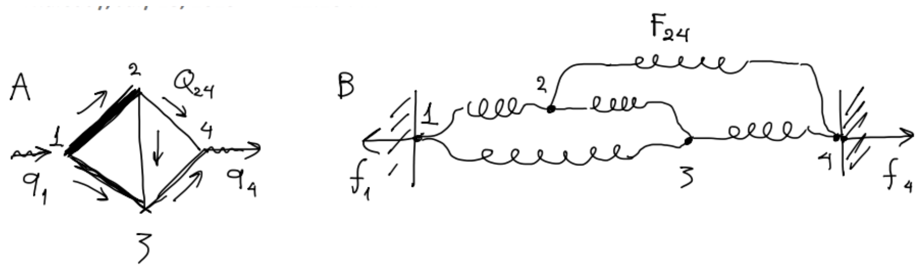

Figure 1: Flow network and its 1D mechanical analogue.

The direct analogue of the flow system Laplacian for mechanical networks is the stiffness matrix.

3. Random walks on graphs

3.1. Escape probability

Consider a random walked that when released, moves randomly along the edges of a graph. Let p_i be the probability that the random walked is present at node i. At each iteration of the random walk, the walker will transition to a neighbor vertex with probability $P_{ij} \equiv \frac{a_{ij}}{\sum_j a_{ij}}$. If k is the time step, then

$$\vec{p}[k] = \hat{D}^{-1}\hat{A}p[k] \tag{1}$$

where $\hat{D} = \text{diag}\{\sum_j a_{ij}\}$ and $\hat{A}$ is the graphs adjacency matrix with elements a_{ij}. What is the probability p_i that a random walker (drunken sailor) starting at i reaches node 1 before it reaches N? (perhaps 1 is the drunken sailor's home and N is a bar). It is easy to see that $p_1 = 1$, as the sailor starts the random walk already home, and $p_N = 0$, as the walker is already at the bar and has no chance to get home. Neighbor nodes to node 1 will have high p, close to N low p.

If we know the probability that the neighbors of node v_i send the walker to 1 before N, then we know that p_i is the sum of those probabilities weighted by the transition probability between the nodes.

$$p_i = \sum_j P_{ij} p_j \Rightarrow \vec{p} = \hat{D}^{-1}\hat{A}\vec{p} \Rightarrow (\hat{D} - \hat{A})\vec{p} = 0 \Rightarrow \hat{L}\vec{p} = 0 \tag{2}$$

Remember the nodes 0 and 1 are boundary conditions so the Laplacian $\hat{L}$ needs to be modified accordingly.

3.2. Travel time

How long (what is the expected number of steps) $t(v_i, 1) = t_i$ to go from node i to node 1? The time t_i is equal to the sum of the neighbors time weighted by the transition probability P_{ij} plus 1, for the extra time step.

$$\vec{t} = \hat{D}^{-1}\hat{A}\vec{t} + \vec{1} \Rightarrow (\hat{D} - \hat{A})\vec{t} = \hat{D}\vec{1} \Rightarrow \hat{L}\vec{t} = -\vec{q} \tag{3}$$

where $q_i = \sum_j a_{ij}$. Note that the dimension of $\hat{L}$ is $N - 1 \times N - 1$ as node 1 is not included in $\vec{t}$. The problem maps directly to that of a flow system with distributed net currents and a pressure boundary condition on node 1.

3.3. Beyond steady flow

So far we have considered exclusively steady, laminar flow. However the flow in most animals animals is pulsatile, produced either by a pump (the heart) or peristalsis of the vessels. The elastic arterial wall responds to the pulsating pressure waveform by expanding, and eventually changing the volume of the vessel. Modeling the complex interplay of fluid dynamics of the transported fluid and elasticity of the wall, as well as other non-linear effects is way beyond the scope of these notes. Instead, we abstractly discuss how one can incorporate delay and capacitance into the equations discussed in the previous paragraphs.

A capacitor in an electrical circuit models a circuit element that can store charge, and translates stored charge into a pressure gradient.

$$\Delta p_{ij} = \frac{\int Q_{ij} dt}{Z_{Cij}} \tag{4}$$

where Z_{Cij} is the capacitance of the vessel $\{i, j\}$. Similarly, parts of the circulatory system have the ability to store blood, by changing the vessel diameters (in animals), or directly storing water in the tissue (in plants).

An inductor element in a circuit introduces a “resistance” of the system to the change of current.

$$\Delta p_{ij} = Z_{Lij} \frac{dQ_{ij}}{dt} \tag{5}$$

where Z_{Lij} is the inductance of the vessel $\{i, j\}$. This is qualitatively similar to an inertial term in a circulatory system, where work is required to accelerate or decellerate the flow.

A simple pulsatile current source can be represented by a real boundary source vector $\vec{q}$ multiplied by a complex exponential $e^{i\omega t}$ where ω is the angular frequency of the oscillation. The observable quantities, pressure difference across the vessels and current, are now going to be the real parts of $\Delta p_{ij} e^{i\omega t}$ and $Q_{ij} e^{i\omega t}$. Note that in general this is not as simple as $\Delta p_{ij} \cos i\omega t$ as Δp_{ij} can be complex.

Equations 4 and 7 now translate to

$$\Delta p_{ij} = \frac{Q_{ij}}{i\omega Z_{Cij}} \tag{6}$$

$$\Delta p_{ij} = i\omega Z_{Lij} Q_{ij} \tag{7}$$

The inductor and capacitor introduce changes in the phase of the flow. With these equations we can generalize Ohm's law $\Delta p_{ij} = Z_{ij} Q_{ij}$ to include complex resistances (or impedances) Z_{ij}, and the derivations in the preceding paragraphs generalize with the introduction of a complex Laplacian.

Bibliography

[1] Leo J. Grady and Jonathan R. Polimeni. *Discrete calculus.* Springer, 2010.