Boulder Theoretical Biophysics 2019

Neuroscience Mini-course: Exercise Set 1 **solutions**

1. Solution: The maximum entropy weighing has three equiprobable outcomes: either the balance will be even because the counterfeit coin is in the held out pile, or the balance will have the left side high, or the balance will have the right side high. We therefore compute

$$S = -\sum p \log(p)$$

with 3 states, all with probability 1/3. That is $\log_2(3) = 1.585$ bits.

2. Solution:

This answer will depend on what you decided to do for the second weighing. Here is an example answer: After the first weighing, the result was "uneven", so we have 4 true weight coins (let's label them G), 4 coins that each have equal probability of being heavy (let's call those H) and 4 that might be light (let's label those L). We could go with this weighing next:

$$\text{side A} = LGGG \qquad \text{vs} \qquad \text{side B} = LLLH$$

Here are the possible outcomes and their probabilities:

A = B, $p = 3/8$ (This means one of the three potentially heavy coins we left out was in fact heavy. The final weighing involves simply picking two of the those three H coins and weighing them against each other.)

A < B, $p = 1/4$ (This means either L on side A, or H on side B were counterfeit. Next pick either one and weigh against a G.)

A > B, $p = 3/8$ (This means that one of the 3 L's on side B was actually light. Pick any two and weigh them against each other.)

The entropy of this distribution is: $-\sum_{k=1}^{3} p_k \log_2 p_k = 1.56$ bits

3. Solution:

The possible first weighings of 16 coins are 1 versus 1, 2 versus 2, etc..., and 8 vs 8. The entropies of these weighings are:

1 v 1, $p(\text{left=right}) = 14/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 1/16$, $S \approx 0.67$ bits
2 v 2, $p(\text{left=right}) = 12/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 2/16$, $S \approx 1.06$ bits
3 v 3, $p(\text{left=right}) = 10/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 3/16$, $S \approx 1.34$ bits
4 v 4, $p(\text{left=right}) = 8/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 4/16$, $S \approx 1.5$ bits
5 v 5, $p(\text{left=right}) = 6/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 5/16$, $S \approx 1.58$ bits
6 v 6, $p(\text{left=right}) = 4/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 6/16$, $S \approx 1.56$ bits
7 v 7, $p(\text{left=right}) = 2/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 7/16$, $S \approx 1.42$ bits
8 v 8, $p(\text{left=right}) = 0/16$, $p(\text{left}<\text{right}) = p(\text{left}>\text{right}) = 8/16$, $S \approx 1$ bit

So, we should pick the 5 v 5 weighing, since it has the largest entropy. If we could get equiprobable distributions of weighing outcomes, we could obtain about 1.585 bits per weighing (the entropy of the uniform distribution with probabilities 1/3, 1/3 and 1/3), but we can't quite reach that with 16 discrete coins. We need to determine which coin is counterfeit ($\log_2 16 = 4$ bits) plus whether it is heavy or light (1 bit of information) for a total of 5 bits of information. If we can get about 1.5 bits per weighing, we should be able to solve the problem in 4 weighings.

Footnote: There exists a 3-weighing solution that you can pre-register, meaning that you can determine what you put on the balance in each weighing ahead of time and readout the information about the coins afterwards. It's a fun solution!