Claim: assume there are no nodes with only inductances

Then $C$ is invertible if (and only if) the circuit has a spanning tree which consists of capacitive branches only.

(Other branches can also be capacitive)

$C$ has no zero eigenvalues: no $\{ \vec{\Phi}_n = V_n \text{ node voltage} \}$ with zero energy, except for $\forall n V_n = 0$ (all equal to ground)

$T = \frac{1}{2} \sum_{\text{bcT}} \frac{1}{C_b} + \sum_{\text{other}} \ldots \geq 0$

Let some $V_n \neq 0$, then on path (over spanning tree) to ground some $(V_k - V_j) > 0$, so $T \geq C_{k \to j} (V_k - V_j) > 0$
So, e.g. Hamiltonian for

\[ H = \frac{-i}{2C} \frac{\partial^2}{\partial \phi^2} + \frac{\hbar^2}{2L} \frac{\partial^2}{\partial \phi^2} - \omega_0 \frac{\hbar^2}{2L} \phi (\frac{\partial^2}{\partial \phi^2}) + \frac{1}{2L} \hbar^2 \frac{\partial^2}{\partial \phi^2} + \frac{\hbar^2}{L} \]

Highly non-harmonic potential!

Linear term (*) \( \Rightarrow \) characteristic "phase singularity" - characteristic of superconductivity (not true of "perfect conductor")

MORE GENERAL THEORY FOR DERIVING CIRCUIT HAMILTONIAN:

Network Graph Theory.
Applications:

1. Phase qubit - Eq. \( \square \) on p. 3.
   - use metastable levels - initialize by ramping \( \frac{B_y}{B} \)
   - read out by pumping to metastable state \( |2\rangle \)
   - observe “escape”

2. Flux qubit
   3D Hamiltonian

\[
\mathcal{H}(\Phi) = \frac{\Phi_0}{2} \sum_{j=1}^{3} L_3 \cos \gamma_j + \frac{1}{2L^2} (\Phi_1 + \Phi_2 + \Phi_3)^2 + \frac{1}{2} (\Phi_1 + \Phi_2 + \Phi_3) \cos \theta
\]

\( <111> \) direction very tightly confined.
- Potential in plane \( \perp <111> \):

[Diagram of potential well]

\( \rightarrow \) very controllable double-well potential
Hamiltonian of a superconducting qubit

Flux [a.u.]

Energy [a.u.]

Optimal choice of $E_J/E_C$?

\[
H = \frac{\hat{Q}^2}{2C} - E_J \cos \phi \frac{2\pi \hat{\Phi}}{\Phi_0} = \frac{(2e)^2}{2C} (\hat{n} - n_g)^2 - E_J \cos \phi = 4E_C(\hat{n} - n_g)^2 - E_J \cos \phi
\]
\[ H = \frac{1}{2(C+C_G)} (Q - C_G V)^2 - \left( \frac{E_0}{2\pi} \right)^2 \frac{1}{L J} \cos \frac{\phi}{2} \]

\[ = \left( \frac{2e^2}{2(C+C_G)} \right) \left( \hat{N} - \frac{C_G V}{2e} \right)^2 - \left( \frac{E_0}{2\pi} \right)^2 \frac{1}{L J} \cos \frac{\phi}{2} \]

\[ H = 4E_c (\hat{N} - n_{\text{offset}})^2 - E_J \cos \phi \]

**NOTE** \( \phi \) is only defined on interval \( 0 \leq \phi < 2\pi \).

Periodic boundary conditions if \( n_{\text{offset}} = 0 \)

\( n_{\text{offset}} \neq 0 \) acts like Aharonov-Bohm flux

For \( E_J = 0 \), eigenfunctions are plane waves:

\( |\text{14}\rangle = e^{im\phi} \)

\( m = 0, \pm 1, \pm 2, \ldots \)

\[ E = 4E_c (m - n_{\text{offset}})^2 \]
H = \frac{1}{2} \frac{\dot{q}^2}{C} + \frac{1}{2} \frac{\ddot{q}^2}{L}

Euler-Lagrange: \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0

I = \frac{1}{d} (A - C)

Q = \frac{1}{\sqrt{2\pi a^2}} (a \cdot q + q^2)

Introduce \quad \hat{a} = \frac{i \hat{A}}{\sqrt{2 \pi c e n}} + \frac{\hat{\phi}}{\sqrt{2 \pi e n}}

to get \quad H = \hbar^2 (a^2 + q^2)