Lecture I : MBL

Outline:

1. MBL transition - RG framework

   (A) Criterion MBL vs. thermal regions

   (B) Rare MBL regions in the thermal phase
   → "Griffiths regime" and sub-diffusive transport

   (C) Rare thermal regions in the MBL phase
   → Avalanche instability

   ⇒ (D) 2-parameter scaling theory

2. MBL meets open systems:

   Can we see sharp signatures of MBL in a system that is weakly coupled to a bath and drive?

   Lenarčič, Altman & Rosch PRL 121, 267603 (2018)
   Lenarčič, Alberton, Rosch, Altman PRL 125, 116601 (2020)

1. MBL transition

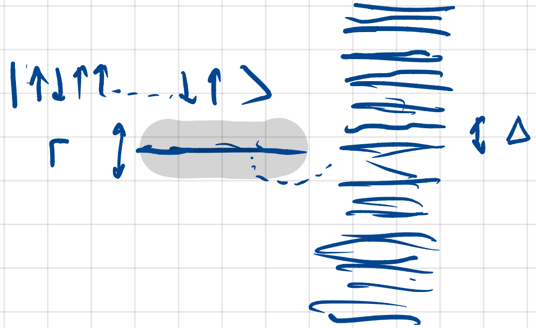
(1) $$H = \underbrace{\sum_{i=1}^{L} h_i \sigma_i^z}_{H_0} + \underbrace{\sum_{i,i+1} J[\vec{\sigma}_i, \vec{\sigma}_{i+1}]}_{\hat{V}}$$

(2) e.g $$J(\vec{\sigma}_i, \vec{\sigma}_{i+1}) = J\cdot\left(\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - \eta(\sigma_i^x + \sigma_{i+1}^x)\right)$$

A. As a basic criterion for localization on a system of size L we may ask if an eigenstate of H_0 can decay/mix into the quasi-continuum made from other eigenstates of H_0 due to the coupling J.

<u>Fermi Golden rule</u> (FGR)

Nearby eigenstates of H_0 typically differ by $O(L)$ spins



$$\Rightarrow \frac{1}{\tau} = \Gamma \sim \underbrace{J^2 e^{-L/\xi_{loc}}} \cdot \rho(E)$$

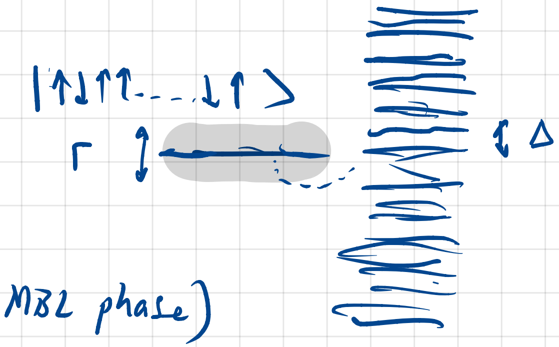
Matrix element for flipping $O(L)$ spins.

ξ = "localization length"

DOS: $\rho(\epsilon) = \frac{1}{\Delta} \sim \frac{W}{2^L}$

Rough criterion for thermalization:

$g \equiv \frac{\Gamma}{\Delta} > 1$ equivalently $\left(\frac{J}{W}\right)^2 e^{-L(\xi_{loc}^{-1} - \ln 2)} > 1$



Implies a minimal localization length $\xi_{loc} < \frac{1}{\ln 2}$ (in MBL phase)

Exercise 1: Derive a more precise and nicer version of this criterion using the l-bit effective Hamiltonian from Huse's lectures:

(3) $$H_{eff} = \sum \tilde{h}_i \tau_i^z + \sum_{i<j} \hat{J}_{ij}\left[\{\tau_i^z, \tau_{i+1}^z, \tau_{i+1}^0, \ldots, \tau_{j-1}^z, \tau_{j-1}^0, \tau_j^z\}\right]$$

$$\|\hat{J}_{ij}\| \sim J_0\, e^{-|i-j|/2\xi}$$

To test for delocalization consider the stability to perturbation by a local operator acting on one site, say $i=0$.
Under what condition can the perturbation mix a macroscopic number of localized eigenstates?

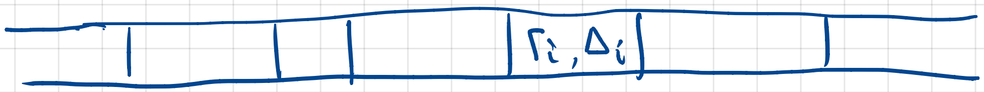
[See Serbyn, Papic and Abanin, Phys. Rev. X 5, 041047 (2015)]

Does the jump in ξ_{loc} imply a 1st order transition and no diverging scale?

No! We'll see that using this criterion in an RG scheme yields a critical fixed point.

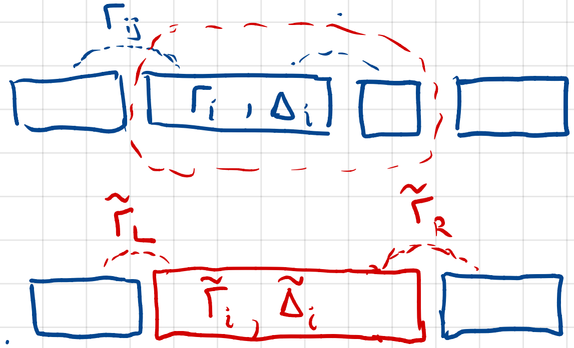
Block RG scheme [Vosk, Huse, Altman PRX 5, 031032 (2015)]

The sample is assumed to be composed of regions (labeled by i) characterized by variable thermalization rates Γ_i



A region is considered thermal or MBL depending on the local ratio $g_i = \Gamma_i / \Delta_i$ where $\Delta_i \sim W 2^{-l_i}$

In the paper we derived an RG scheme to gradually merge regions that thermalize at the scale of the running cutoff.

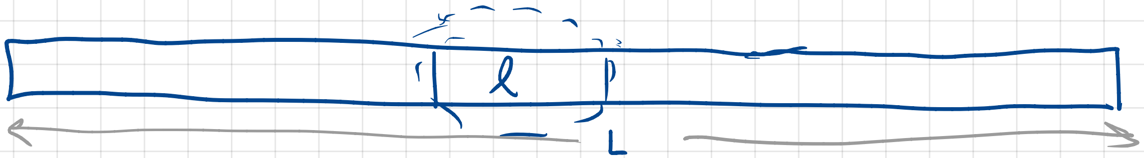


Here instead I want to consider the consequences of the following

(i) Rare MBL regions in the thermal phase

(ii) Rare thermal regions in the MBL phase.

B. Rare MBL regions in the thermal phase - Griffiths regime



- Density of MBL inclusions of size $> \ell$ near the critical point should depend only on a correlation length $\xi \gg \ell_0$

(4) $$\rho(\ell) \sim \frac{1}{\xi} e^{-\ell/\xi} \qquad P_L(\ell) = L \cdot \rho(\ell_{th}) \sim 1$$

- The longest MBL region creates a bottleneck for thermalization (in 1d) and thus sets the thermalization time τ_{th}

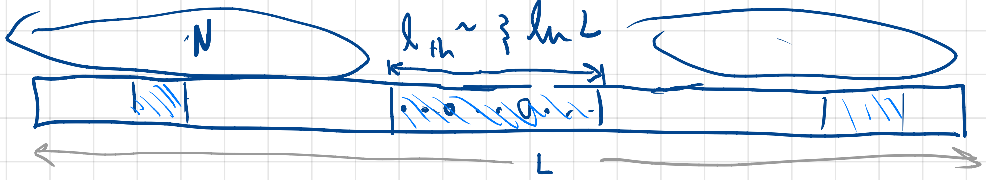
- The typical size ℓ_{th} of the longest MBL block:

$$\rho(\ell_{th}) = \frac{1}{\xi} \exp[-\ell_{th}/\xi] = \frac{1}{L}$$

$$\Rightarrow \ell_{th} \sim \xi \left[\ln\left(\frac{L}{\xi}\right) + 1 \right] \qquad \ell_{th} \sim \xi \ln L$$

(5) Thermalization time: $\tau \sim \tau_0 \, e^{\ell_{th}/\ell_0} \sim \tau_0 L^{\xi/\ell_0}$

Dynamical scaling:



The relation $\tau \sim L^{\xi/l_0}$ between the system's length and thermalization time defines the dynamical exponent

(6) $\boxed{z = \xi/l_0}$ $\Rightarrow$ $z \to \infty$ at the critical point together with $\xi \to \infty$

→ Transport time: Imbalances in hydrodynamic conserved quantities (e.g. particle number, energy) across the bottleneck will take longer to relax

$\tau_{tr} = \tau \cdot L$ because we need to transfer $O(L)$ particles across

(7) $\Rightarrow$ $\tau_{tr} = L^{1+\xi/l_0}$ $\qquad z_{tr} = z + 1$

Note that such bottleneck effects are special to $d=1$
If there was a localization transition in $d>1$ we would expect diffusive transport in the thermal phase with $D \to 0$ at the transition.

Exercise 2: Experiments in I. Bloch's group observed the decay of a density wave (DW) imposed on the initial state [see in particular Lüschen et. al PRL 119, 260401 (2017)]. Show that in the Griffiths phase the density wave decays as $t^{-1/2}$ at long times

Hint: in what fraction of the system the DW is still $O(1)$ at time t?

Remark 1: Naively, in a diffusive system a DW at wave-vector q will decay exponentially as $\exp[-t\cdot(Dq^2)]$ however because of interactions even non conserved quantities like the density wave develop a power law decay but it is much faster than $t^{-1/2}$.

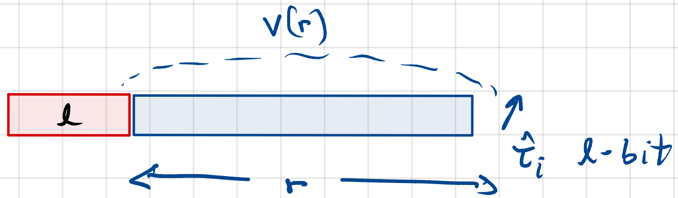
Remark 2: The experiments mentioned above were done with a quasi-periodic lattice, so one would not expect to see such Griffiths phenomena as in a random system.

(C) Rare thermal regions in the MBL phase

can lead to an avalanche instability of MBL
[de-Roeck & Huveneers 2015; see also David's lecture]

Briefly: a thermal region serves as a mini-bath that can thermalize its surrounding MBL l-bits.

If $r-1$ closest l-bits are thermalized what is the fate of the r-th l-bit?



coupling matrix-element: $V(r) \approx \frac{J}{\sqrt{2^{\ell+r}}} e^{-\ell/2\xi_{loc}}$

Level spacing of bath: $\Delta(r) \approx W/2^{(\ell+r)}$

(8) $\Rightarrow$ $g(r) = \frac{V(r)^2}{\Delta(r)^2} \sim \left(\frac{J}{W}\right)^2 2^{\ell} \underbrace{e^{-(\xi_{loc}^{-1} - \ln 2) r}}$

$\Rightarrow$ MBL phase with $\xi_{loc} > \frac{1}{\ln 2}$ is unstable in $d=1$

$\Rightarrow$ Generalization to $d>1$ predicts no stable MBL phase for $d>1$

contrast with the original perturbative criterion (BAA)

[Basko, Aleiner & Altshuler Ann. Phys. 2006]

BAA criterion for 1d fermions with weak interaction U (and $T \to \infty$)

Matrix element for decay to particle-hole excitations: $\sum_{\alpha\beta\gamma\delta} U c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta$

$$V \sim \left(\frac{W}{\Delta_\xi}\right) \cdot U = \frac{\xi_{loc}}{l_0} U,$$

# of decay channels

$W \sim \hbar v_F / l_0$ = bandwidth of p-h excitations

$\Delta_\xi \sim \hbar v_F / \xi_{loc}$ = level spacing in a localization volume

Resonance condition: $\dfrac{V}{\Delta_\xi} \sim \dfrac{\xi^2 U}{\hbar v_F l_0} > 1 \Rightarrow \boxed{U_c \approx \dfrac{\hbar v_F l_0}{\xi^2_{loc}}}$

$\Rightarrow$ Anderson localization in 1d is stable to weak interaction

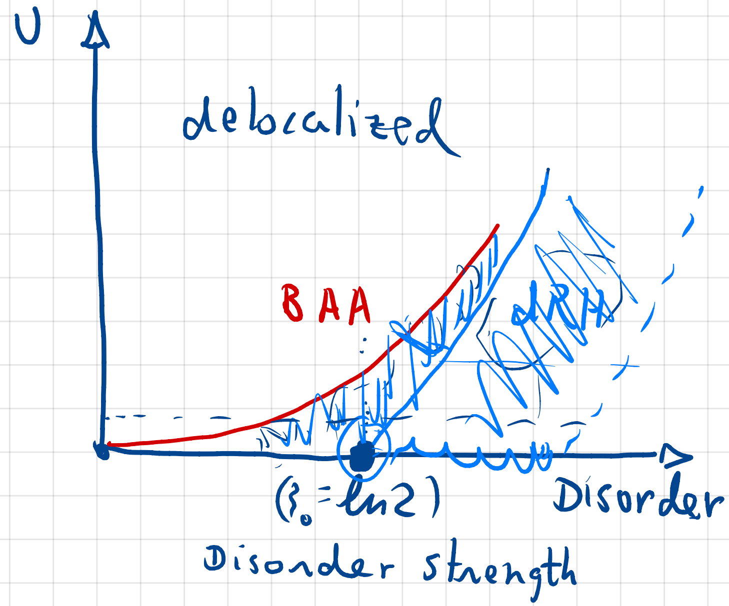
contrast with:

dRH criterion [de Roeck & Huveneers 2015]

Anderson localization in 1d is unstable to weak interaction if the non-interacting localization length is $\xi_{loc} > 1/\ln 2$

$$H_0 = -\sum_i (c^\dagger_i c_{i+1} + H.c) + \sum_i h_i c^\dagger_i c_i$$

$$H_{int} = \sum_{ij} U_{ij} c^\dagger_i c_i c^\dagger_j c_j$$



(D) 2-parameter scaling theory of the MBL transition

→ Dumitrescu et. al Phys. Rev. B 99, 094205 (2019)

If the MBL transition is driven by the avalanche instability then a minimal model for the transition should include at least the following two variables

→ ξ_{loc}^{-1} (average) inverse localization length

→ ρ density of thermal inclusions (assumed to be small)

Initially ρ and ξ^{-1} are determined by the microscopic parameters. Now we coarse grain we expect them to flow

→ i. According to the avalanche criterion, if $\xi_{loc}^{-1} < \xi_c^{-1} = \ln 2$ $(\ln q)$ then ρ becomes relevant (renormalized upward)

ii. Having a finite ρ renormalizes ξ_{loc}^{-1} downward

At the fixed points we expect

MBL: $\rho_* = 0$, $\xi_{loc,*}^{-1} > \xi_c^{-1}$

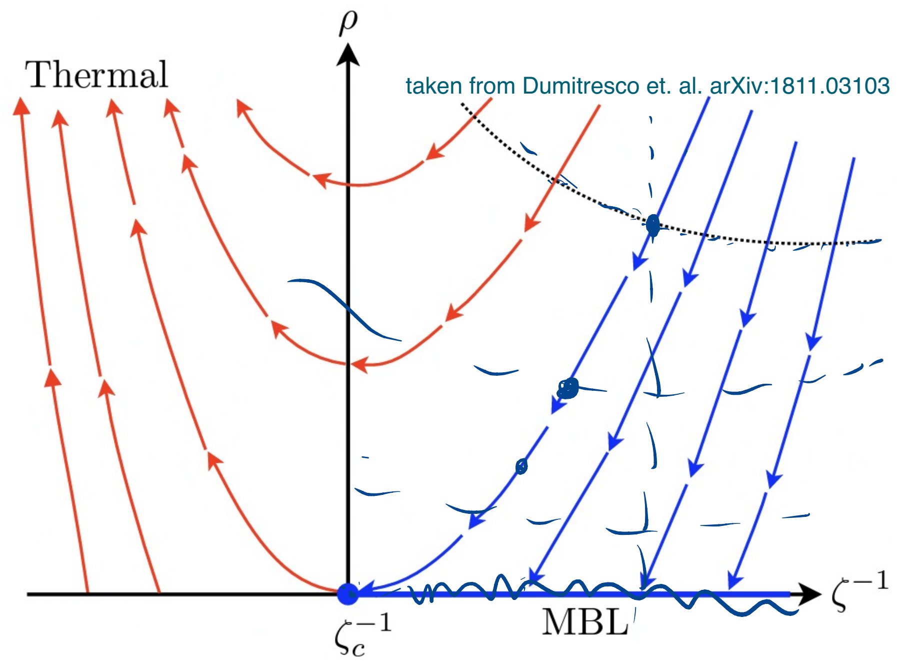
Thermal $\rho_* = 1$, $\xi_{loc,*}^{-1} = 0$

scaling equations

$\ln b = \ln \frac{a}{a_0}$

The simplest scaling equations (lowest order in ρ) that capture the physics of (1) and (2) above:

$$(9)\quad \begin{cases} \dfrac{d\rho}{d\ln b} = -c_1\left(\xi_{loc}^{-1} - \xi_c^{-1}\right)\rho + \ldots \\[2ex] \dfrac{d\xi_{loc}^{-1}}{d\ln b} = -c_2\,\rho\,\xi_{loc}^{-1} + \ldots \end{cases}$$



Equivalent to Kosterlitz-Thouless flow eqns through a change of variables $y = \sqrt{\rho}$

correspondence:

$\boxed{\text{EA, Refael, Polkovnikov and Kafri 2004 PRL}}$

KT transition		MBL transition
Low T critical phase	$\longleftrightarrow$	MBL phase
Vortex fugacity	$\longleftrightarrow$	Density of thermal inclusions
stiffness (K/T)	$\longleftrightarrow$	ξ_{loc}^{-1}

consequences

i. Average size of thermal inclusions

According to (8) the effective coupling to a seed thermal region decays as $e^{-(\xi_{loc}^{-1} - \xi_c^{-1})r}$ thus we can relate:

(10) $$\rightarrow \langle l_T \rangle \approx \boxed{\frac{1}{\xi_{loc}^{-1} - \xi_c^{-1}}} \leftarrow$$

ii. Distribution of lengths of thermal inclusions:

Concrete RG schemes predict a power-law distribution at the critical point $p(l_T) \sim l^{-\alpha}$

(11) $$\Rightarrow \langle l_T \rangle = \int dl \, l^{1-\alpha} = \frac{1}{\alpha - 2} \qquad \alpha > 2$$

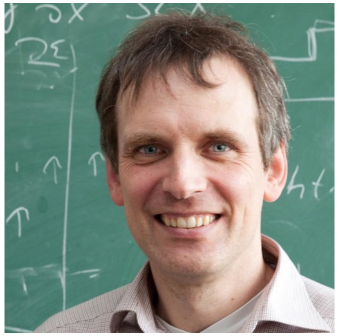
comparing the divergence in (10) and (11) we conclude that at criticality $\alpha \to 2 \Rightarrow p_c(l_T) = l_T^{-2}$

$$\Rightarrow \langle l_T \rangle_c \sim \ln L$$

(typical l_T is finite)

Signatures of MBL criticality in a driven open system

Lenarcic, EA, Rosch, PRL 267603 (2018)
Lenarcic, Alberton, Rosch, EA, PRL 116601 (2020)



The essence of Many Body localization (closed system)

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + h_i^x S_i^x + h_i^z S_i^z \qquad h_i^\alpha \in [-h, h]$$

(Griffith phase)

$$\tau \sim \ell^z$$

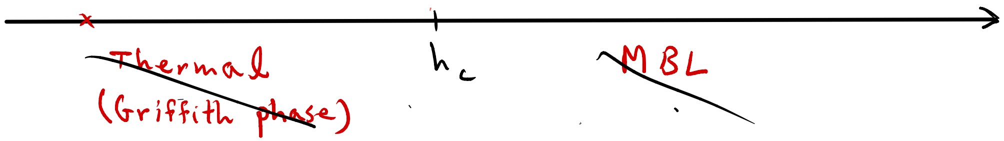
Local integrals of motion

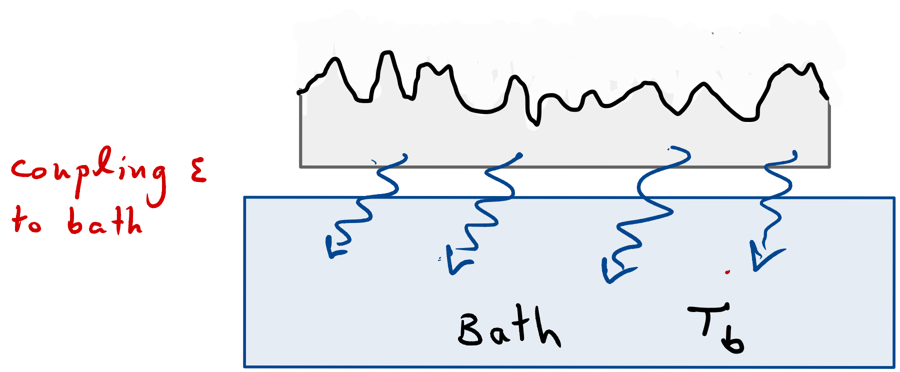
$$\tilde{H} = \sum_i \tilde{V}_i \tau_i^z + \sum_{ij} V_{ij} \tau_i^z \tau_j^z + \dots$$

Huse and Oganesyan 2013,
Serbyn and Abanin 2013
Vosk and EA 2013

MBL in an open system ?

$$H = \sum_i \tilde{V}_i \tau_i^z + \sum_{ij} V_{ij} \tau_i^z \tau_j^z + \dots$$



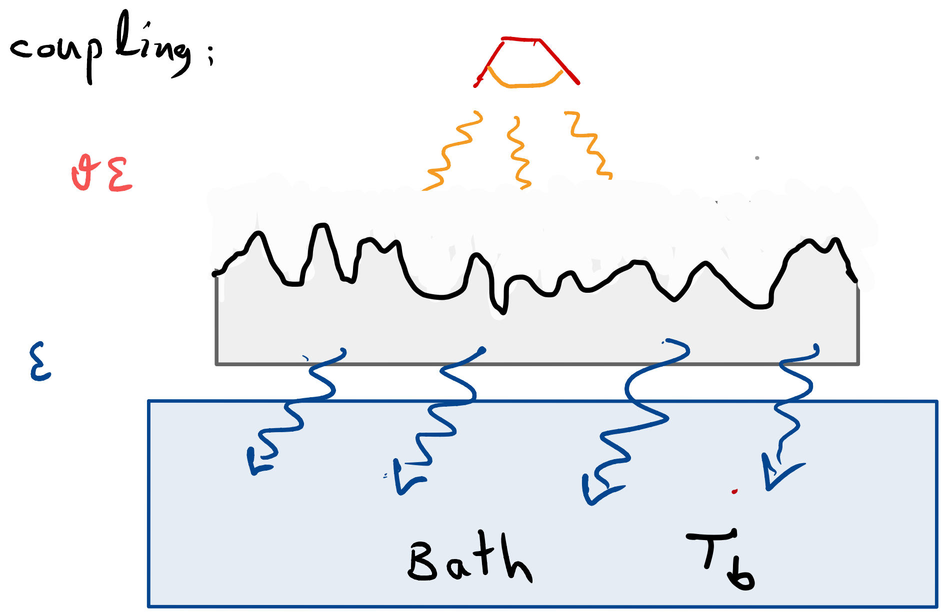


The system thermalizes with the bath

$$\langle \tau_i^z \rangle \sim e^{-\varepsilon t} \longrightarrow 0$$

MBL in an open system ?

- couple the system also to a hot bath or drive

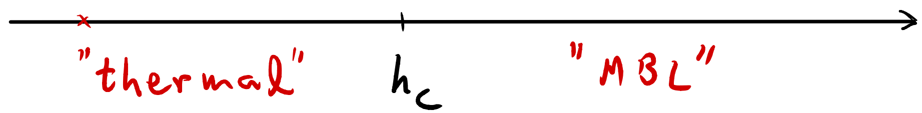


- weak coupling limit: $\varepsilon \to 0$

  $\vartheta = \text{const}$

- Infinite time: $\varepsilon t \longrightarrow \infty$

Distinct steady states ?



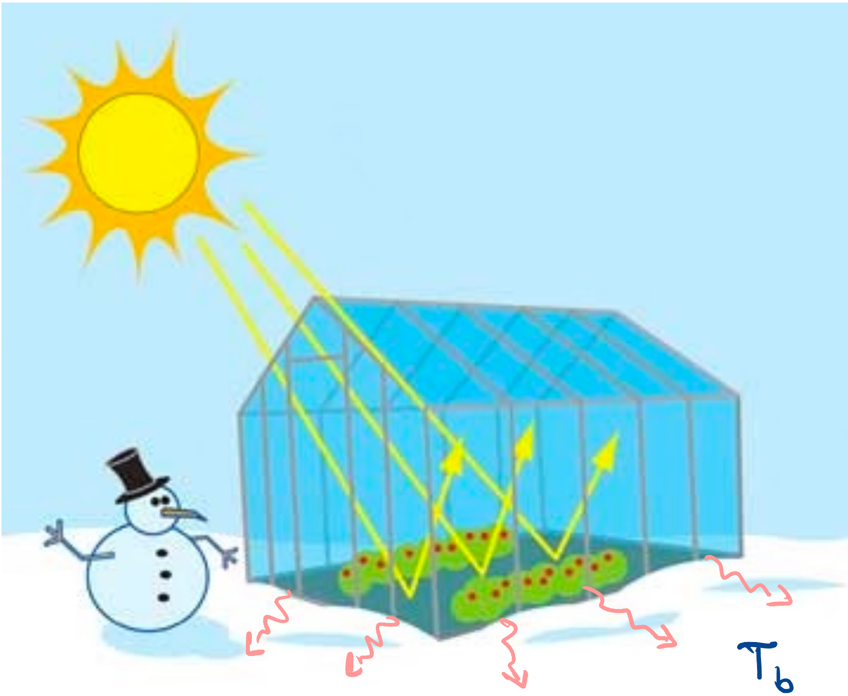
Thermalizing system

conserved total energy:

$$\partial_t \langle H \rangle = \vartheta \varepsilon - \varepsilon (T - T_b)$$

steady state $\Rightarrow$ $\rho = e^{-\beta H}$

$T = T_b + \vartheta$



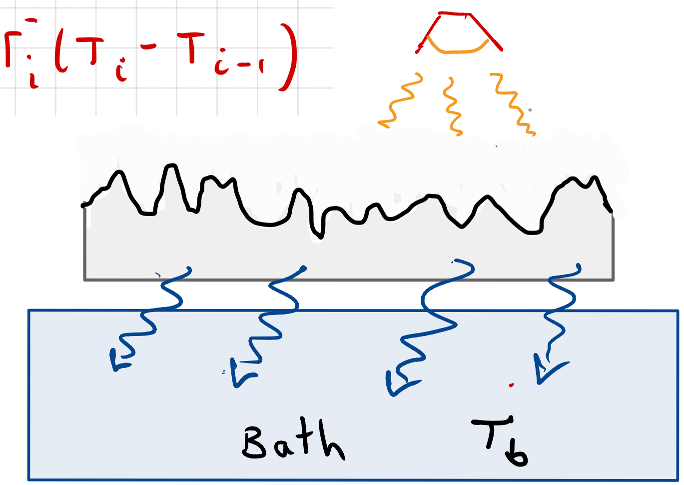
- unique temperature determined by ϑ alone. Non-linear response.

MBL system

conserved local energies:

$$\partial_t \langle \hat{H}_i \rangle = \vartheta \varepsilon g_{1i} - \varepsilon g_{2i} (T_i - T_b) + \varepsilon \Gamma_i^+ (T_i - T_{i+1}) + \varepsilon \Gamma_i^- (T_i - T_{i-1})$$

$\Rightarrow$ $\rho = e^{-\sum_i \beta_i H_i}$, $T_i \approx T_b + \vartheta \frac{g_{1i}}{g_{2i}}$



- Many local temperatures!

Digression: Quantum formulation [Lenarcic, Lange, Rosch arXiv: 1706.05700]

$$\dot{\rho} = \mathcal{L}\rho = \underbrace{-i[H_0,\rho]}_{=\mathcal{L}_0\rho} + \underbrace{\varepsilon \hat{D}\rho}_{\mathcal{L}_1\rho}$$

$\hat{D}$ could be Lindblad form

we want the steady state:

$$\lim_{t\to\infty} \rho(t) = \rho_\infty = \rho_0 + \delta\rho$$

$\mathcal{L}\hat{\rho}_\infty = 0$, $\mathcal{L}_0\hat{\rho}_0 = 0$

$$\rho_0 = \lim_{\varepsilon\to 0} \rho_\infty = ?$$

Suppose N commuting IOM: $[H_i, H_j] = 0 \quad i,j = 0,\dots,N$

GGE ansatz $\rho_0 = \dfrac{e^{-\sum_i \lambda_i H_i}}{\mathrm{Tr}\left(e^{-\sum_i \lambda_i H_i}\right)}$ satisfies $\mathcal{L}_0\rho_0$ for all $\{\lambda_i\}$

$\Rightarrow$ $\{\lambda_i\}$ determined by $\mathcal{L}_1$

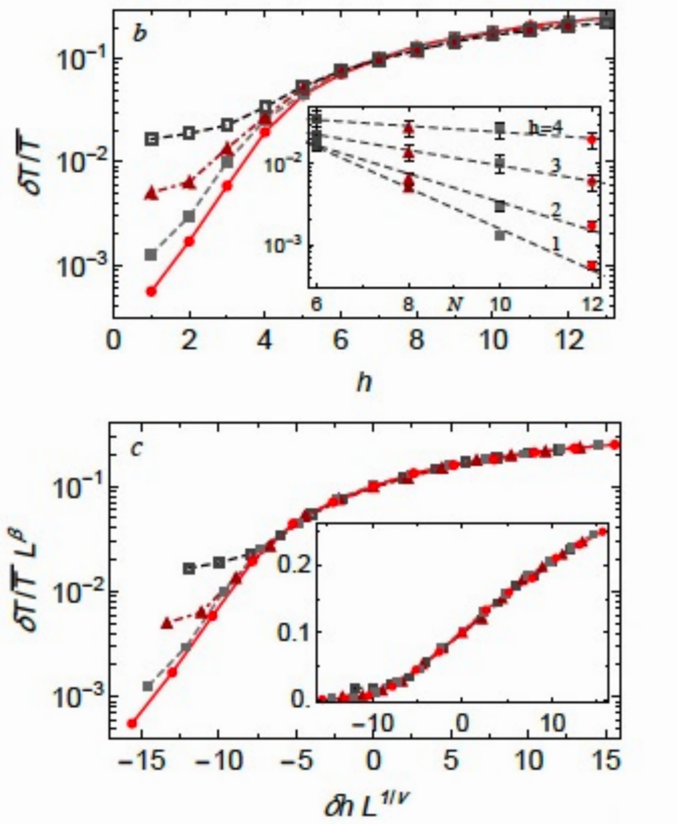
<u>Exercise 3</u>: show that the steady state conditions $\langle \dot{H}_i \rangle = 0$ to lowest order in ε are given by $\mathrm{Tr}(H_i \hat{D} \rho_0) = 0$
These give N equations for the N λ_i

Numerical results

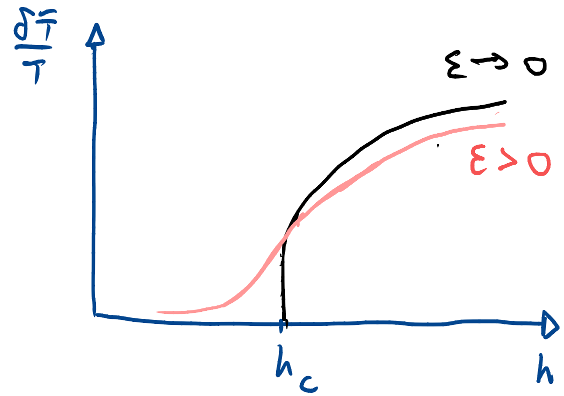
Exact diagonalization

strict limit $\varepsilon \to 0$ $\quad \varepsilon t \to \infty$

1) observe singular onset of the temperature variance at the MBL transition.



How is the singularity broadened by a finite ε in a realistic system? Universality? Finite ε scaling?



Hydrodynamic description

$$\cancel{\partial_t e} - \nabla[k(r)\nabla T(r)] = \varepsilon\vartheta g_1(r) - \varepsilon g_2(r)(T(r) - T_b)$$

$$g_{1,2} = 1 + \xi_{1,2}(r) \qquad T = \bar{T} + \delta T$$

$$-\nabla[k(r)\nabla\delta T] = \varepsilon\vartheta(1+\xi_1(r)) - \varepsilon(1+\xi_2(r))(\bar{T} - T_b + \delta T)$$

Expand in powers of: $\xi_{1,2}(r) \quad (\delta T \sim \xi_1, \xi_2)$

zeroth order: $\bar{T} = T_b + \vartheta$

Linear order:

$\xi_1 - \xi_2$

$$(-\bar{k}\nabla^2 + \varepsilon)\delta T = \varepsilon\vartheta\,\xi(r) \qquad \langle \xi(r)\xi(r')\rangle = w\,\delta(r-r')$$

Let's solve this in the two phases

MBL phase: $\bar{\kappa} \sim \chi \varepsilon \quad \Rightarrow \quad (-\chi \nabla^2 + 1)\delta T = \vartheta \xi(r)$

$$\delta T = \vartheta \int dr' \, G(r'-r) \, \xi(r')$$

$$\langle \delta T^2 \rangle = \vartheta^2 \int dr' \, dr'' \, G(r') G(r'') \overbrace{\langle \xi(r') \xi(r'') \rangle}^{w\,\delta(r'-r'')}$$

$$= w \vartheta^2 \int \frac{dq}{2\pi} \left(\frac{1}{\chi q^2 + 1} \right)^2 \approx \frac{2\pi \tilde{w}}{a} \vartheta^2$$

$\Rightarrow$ spatial temperature fluctuations are finite $(O(1))$ at zeroth order in

Thermal phase: $\bar{\kappa}(q) \sim \gamma q^{z-2} \qquad \tilde{G}(q) = (\gamma q^z + \varepsilon)^{-1}$

$$\overline{\delta T} \equiv \sqrt{\langle \delta T^2 \rangle} \sim \vartheta W \varepsilon^{\frac{1}{2z}}$$

$\boxed{\text{Exercise 4: Derive the last relation}}$

Numerical approach

This limit of an open system of taking $t \to \infty$ first (steady state) and ε small, motivates a numerical scheme:

use TEBD to compute the steady state $\hat{\rho}_\infty$ of the Lindblad evolution:

$$\dot{\rho} = i[H, \rho] + \varepsilon\left(D_{cold}[\rho] + g \cdot D_{hot}[\rho]\right)$$

Pros:

- ε limits the growth of the bond dimension
  $\Rightarrow$ can handle much larger systems than ED.
- The level broadening ε possibly regulates small size transients related to level commensurabilities.

Cons:

- Larger physical dimension than ED of H

Our calculation: Model

$$\dot{\rho} = i[H,\rho] + \varepsilon\left(D_{cold}[\rho] + \vartheta \cdot D_{hot}[\rho]\right)$$

$L = 60$ sites

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + h_i^x S_i^x + h_i^z S_i^z$$

$h_i^\alpha \in [-h, h]$

$$D_{hot}[\rho] = \sum_i \left[4 S_i^z \rho S_i^z - \rho\right]$$

$$D_{cold}[\rho] = \sum_{i,\alpha} L_{i\alpha} \rho L_{i\alpha}^\dagger - \frac{1}{2}\{L_{i\alpha}^\dagger L_{i\alpha}, \rho\}$$

Non unital

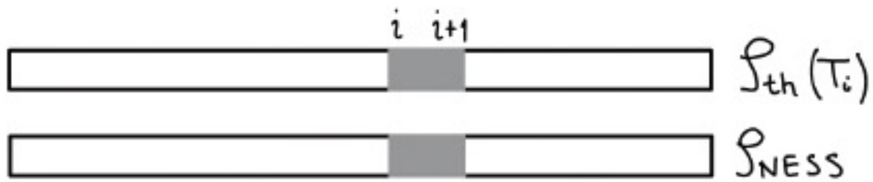
$L_{i1} = S_i^+ P_{\downarrow\, i+1}$, $L_{i2} = P_{\downarrow\, i} S_{i+1}^+$

$L_{i3} = S_i^- P_{\uparrow\, i+1}$, L_{i4} $P_{\uparrow\, i} S_{i+1}^-$

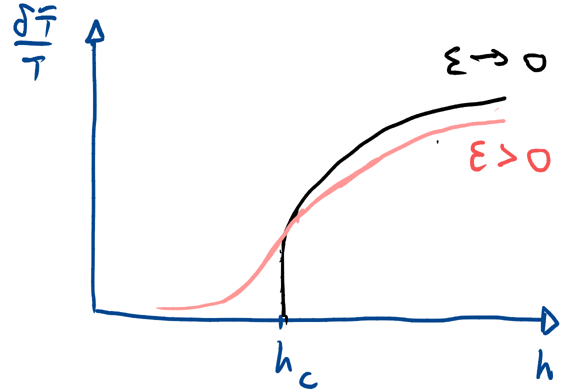
Operational definition of local temperature:

$$F(T_i) = \min_T \mathrm{tr}\left[\left(\rho^{(i,i+1)} - \rho_{th}^{(i,i+1)}(T)\right)^2\right]$$

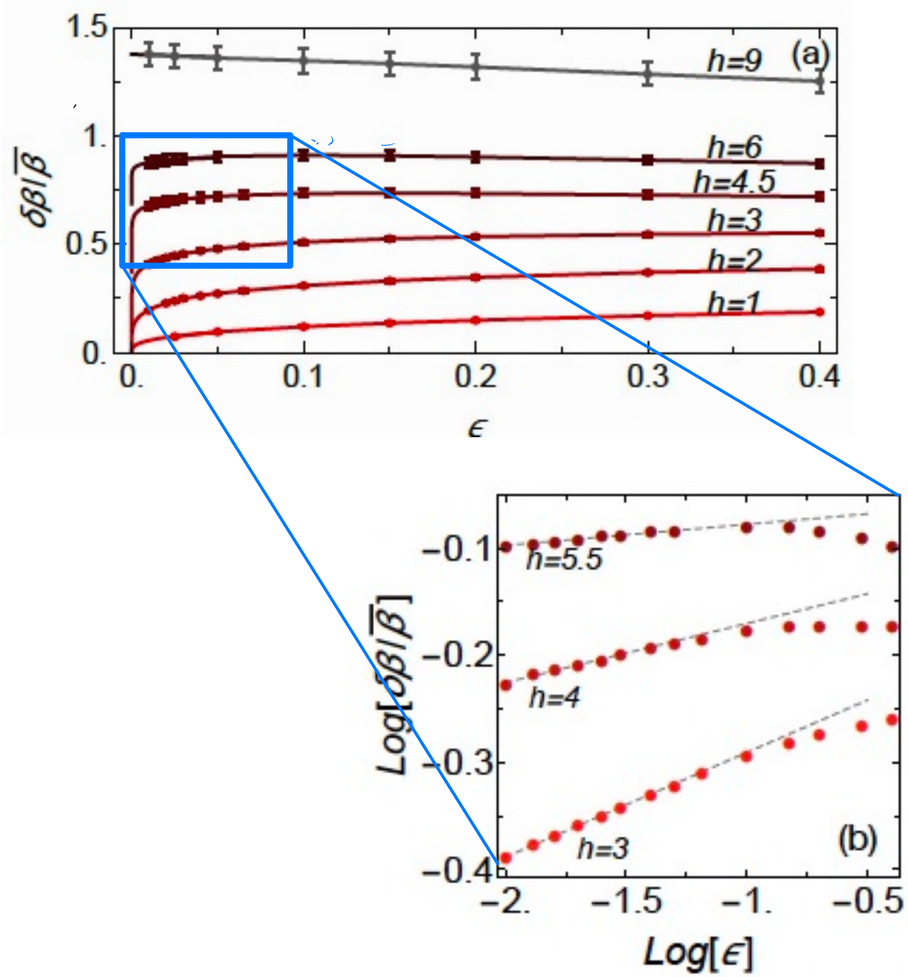
$\longrightarrow T_i$



Numerical results (TEBD)



- signature of critical dynamics in ε dependence of $\delta \overline{T}$ $\sim \varepsilon^{1/2z}$



- obtain divergence of z