Lecture I: MBL

Outline:

1. MBL transition - RG framework
   (A) Criterion MBL vs. thermal regions
   (B) Rare MBL regions in the thermal phase
       "Griffiths regime" and sub-diffusive transport
   (C) Rare thermal regions in the MBL phase
       → Avalanche instability
   (D) 2-parameter scaling theory

2. MBL meets open systems:
   Can we see sharp signatures of MBL in a
   system that is weakly coupled to a bath and drive?

Lenančič, Altman & Rosch PRL 121, 267603 (2018)
Lenančič, Alberton, Rosch, Altman PRL 125, 116601 (2020)
1. MBL transition

\[ H = \sum_{i=1}^{L} \frac{1}{2} \hbar \sigma_i^2 + \sum_{i,j} J \sigma_i^z \sigma_{j+1}^z \]

\[ \Rightarrow H_0 \]

(a) \[ J(\sigma_i^z, \sigma_{i+1}^z) = J \cdot (\sigma_i^z \sigma_{i+1}^z - \gamma (\sigma_i^x + \sigma_{i+1}^x)) \]

A. As a basic criterion for localization on a system of size \( L \) we may ask if an eigenstate of \( H_0 \) can decay/mix into the quasi-continuum made from other eigenstates of \( H_0 \) due to the coupling \( J \).

Fermi Golden rule (FGR)

Nearby eigenstates of \( H_0 \) typically differ by \( O(L) \) spins

\[ \Rightarrow \frac{1}{\varepsilon} \approx \Gamma \sim \frac{1}{L^{3/2}} \cdot p(E) \]

Matrix element for flipping \( O(L) \) spins

\[ z = \text{localization length} \]

\[ \text{DOS: } p(E) = \frac{1}{\Delta} \sim \frac{w}{2L} \]
Rough criterion for thermalization:

\[ \delta = \frac{E}{A} > 1 \quad \text{equivalently} \quad (\frac{E}{A})^2 e^{-\frac{1}{\delta}} > 1 \]

Implies a minimal localization length \( \frac{1}{\delta} < \frac{1}{\ln z} \) (in MBL phase)

Exercise 1: Derive a more precise and nicer version of this criterion using the 2-bit effective Hamiltonian from Huse’s lectures:

\[ H_{\text{eff}} = \sum_i \hat{H}_i + \frac{1}{2} \sum_{i<j} \hat{J}_{ij} \{ \hat{T}^z_i \hat{T}^z_j, \hat{T}^x_i \hat{T}^x_j, \hat{T}^y_i \hat{T}^y_j, \hat{T}^z_i \hat{T}^z_j \} \]

\[ \| \hat{J}_{ij} \| \sim J_0 e^{-1 + j^2 / L^2} \]

To test for delocalization consider the stability to perturbation by a local operator acting on one site, say \( i = 0 \).

Under what condition can the perturbation mix a macroscopic number of localized eigenstates?

[See Serbyn, Papic and Abanin, Phys. Rev. X 5, 041047 (2015)]

Does the jump in \( z_{\text{loc}} \) imply a 1st order transition and no diverging scale? No! We'll see that using this criterion in an RG scheme yields a critical fixed point.
Block RG scheme [Vosk, Huse, Altman PRX 5, 031032 (2015)]

The sample is assumed to be composed of regions (labeled by $i$) characterized by variable thermalization rates $\Gamma_i$.

A region is considered thermal or MBL depending on the local ratio $g_i = \Gamma_i/\Delta_i$ where $\Delta_i \sim w 2^{-li}$.

In the paper we derived an RG scheme to gradually merge regions that thermalize at the scale of the running cutoff.

Here instead I want to consider the consequences of the following:

(i) Rare MBL regions in the thermal phase
(ii) Rare thermal regions in the MBL phase
B. Rare MBL regions in the thermal phase - Griffiths regime

- Density of MBL inclusions of size $l > l_0$ near the critical point should depend only on a correlation length $\xi \gg l_0$.

\[ p(l) \sim \frac{1}{\xi} e^{l/\xi}, \quad p_L(l) = L \cdot p(l|_L) \sim 1 \]

- The largest MBL region creates a bottleneck for thermalization (in 1d) and thus sets the thermalization time $T_{th}$.

- The typical size $l_{th}$ of the largest MBL block:

\[ p(l_{th}) = \frac{1}{\xi} \exp\left[-l_{th}/\xi\right] = \frac{1}{L} \]

\[ \Rightarrow l_{th} \sim \frac{\xi}{\xi} \left[ \ln\left(\frac{L}{\xi}\right) + 1 \right] \]

\[ l_{th} \sim \frac{\xi}{\xi} \ln L \]

(5) thermalization time:

\[ T \sim T_0 e^{l_{th}/l_0} \sim T_0 L^{3/l_0} \]
Dynamical scaling:

The relation \( z \sim L^{\frac{3}{2}/b_0} \) between the system's length and thermalization time defines the dynamical exponent.

\[
(6) \quad z = \frac{3}{2}/b_0 \quad \Rightarrow \quad z \to \infty \text{ at the critical point together with } \xi \to \infty
\]

Transport time: Imbalances in hydrodynamic conserved quantities (e.g., particle number, energy) across the bottleneck will take longer to relax.

\[
(7) \quad \tau_{tr} = L^{1+\frac{3}{2}/b_0} \quad \Rightarrow \quad \tau_{tr} = L^{z+1}
\]

Note that such bottleneck effects are special to \( d=1 \).

If there was a localization transition in \( d>1 \), we would expect diffusive transport in the thermal phase with \( D \to 0 \) at the transition.
Exercise 2: Experiments in J. Bloch's group observed the decay of a density wave (DW) imposed on the initial state [see in particular Lüschen et al. PRL 119, 260401 (2017)]. Show that in the Griffiths phase the density wave decays as $t^{-1/2}$ at long times.

Hint: in what fraction of the system the DW is still 0 at time $t$?

Remark 1: Naively, in a diffusive system a DW at wave-vector $q$ will decay exponentially as $\exp[-t\cdot(Dq^2)]$ however because of interactions even non conserved quantities like the density wave develop a power law decay but it is much faster than $t^{-1/2}$.

Remark 2: The experiments mentioned above were done with a quasi-periodic lattice, so one would not expect to see such Griffiths phenomena as in a random system.
Rare thermal regions in the MBL phase can lead to an avalanche instability of MBL [de Roeck & Havener 2015; see also David's lecture]

 Briefly: a thermal region serves as a mini-bath that can thermalize its surrounding MBL l-bits.

If r-1 closest l-bits are thermalized what is the fate of the rth l-bit?

Coupling matrix-element: $V(r) \propto \sqrt{l/2l_{bath}} e^{-l/2l_{bath}}$

Level spacing of bath: $\Delta(r) \approx W/2(l+r)$

$$\Rightarrow g(r) = \frac{V(r)^2}{\Delta(r)^2} \sim \left(\frac{W}{2l}\right)^2 \frac{l}{2} e^{-\frac{l}{l_{bath}}}$$

$\Rightarrow$ MBL phase with $\frac{3}{l_{bath}} \frac{l}{l_{bath}}$ is unstable in $d=1$

$\Rightarrow$ Generalization to $d>1$ predicts no stable MBL phase for $d>1$
Contrast with the original perturbative criterion (BAA)


**BAA criterion** for 1d fermions with weak interaction $U$ (and $T \to 0$)

Matrix element for decay to particle-hole excitations: $\sum \epsilon_i c_i^\dagger c_i^\dagger c_i c_i$

$\mathbf{V} \sim \left( \frac{W}{\Delta^2} \right);\quad U = \frac{3 \hbar \omega_c}{\omega_0}$

Number of decay channels

Resonance condition:

\[
\frac{\mathbf{V}}{\Delta^2} \approx \frac{3^2 U}{\hbar \nu_c \omega_0} > 1 \Rightarrow U_c \approx \frac{\hbar \nu_c \omega_0}{3^2 \omega_0}
\]

$\Rightarrow$ Anderson localization in 1d is stable to weak interaction contrast with:

**dRH criterion** [de Rock Ahtuneeers 2015]

Anderson localization in 1d is unstable to weak interaction if the non-interacting localization length is $\xi > 1/\ln 2$

\[
\mathbf{H}_0 = -\sum \epsilon_i c_i^\dagger c_i + \mathbf{H}_c + \sum \mathbf{h}_j c_j^\dagger c_j
\]

\[
\mathbf{H}_{\text{int}} = \sum_{ij} V_{ij} c_i^\dagger c_i^\dagger c_j c_j
\]

If the MBL transition is driven by the avalanche instability, then a minimal model for the transition should include at least the following two variables:

- $\tilde{\xi}_{\text{loc}}^{-1}$ (average inverse localization length)
- $g$ density of thermal inclusions (assumed to be small)

Initially $g$ and $\tilde{\xi}_{\text{loc}}^{-1}$ are determined by the microscopic parameters. Now we coarse-grain we expect them to flow according to the avalanche criterion, if $\tilde{\xi}_{\text{loc}}^{-1} < \tilde{\xi}_{\text{c}}^{-1} = \ln z$ then $g$ becomes relevant (renormalized upward).

Having a finite $g$ renormalizes $\tilde{\xi}_{\text{loc}}^{-1}$ downward.

At the fixed points we expect

- MBL: $\tilde{\xi} = 0$, $\tilde{\xi}_{\text{loc},x}^{-1} = \tilde{\xi}_{\text{loc}}^{-1}$
- Thermal: $\tilde{\xi} = 1$, $\tilde{\xi}_{\text{loc},x}^{-1} = 0$
The simplest scaling equations (lowest order in $g$) that capture the physics of (1) and (2) above:

\[
\begin{align*}
\frac{d s}{d \ln b} &= -c_1 \left( s_{loc}^{-1} - s_{c}^{-1} \right) \rho + \cdots \\
\frac{d s_{loc}^{-1}}{d \ln b} &= -c_2 \left( s_{c}^{-1} \right) \rho_{loc}^{-1} + \cdots
\end{align*}
\]

Equivalent to Kostelec-Thouless flow eqns through a change of variables $y = \sqrt{\rho}$

**correspondence:**
EA, Retzki, Polkovnikov and Kafri 2004
PRL

\[
\ln b = \ln \frac{a}{a_c}
\]

**KT transition**

- Low $T$ critical phase $\leftrightarrow$ MBL phase
- Vortex fugacity $\leftrightarrow$ Density of thermal inclusions
- Stiffness ($K/T$) $\leftrightarrow$ $s_{loc}^{-1}$
i. Average size of thermal inclusions

According to (8) the effective coupling to a seed thermal region decays as \( e^{- \left( \frac{t}{\delta_{\text{loc}} - \delta_c} \right)^r} \), thus we can relate:

\[
\langle l_T \rangle \approx \frac{1}{\delta_{\text{loc}}^{-1} - \delta_c^{-1}} e^{-}\left( \frac{t}{\delta_{\text{loc}} - \delta_c} \right)^r
\]  

\( (10) \)

ii. Distribution of lengths of thermal inclusions:

Concrete RG schemes predict a power-law distribution at the critical point \( p(l_T) \sim l^{-\alpha} \)

\[
\Rightarrow \langle l_T \rangle = \int dl_T l^{-\alpha} = \frac{1}{\alpha - 2} \quad \alpha > 2
\]  

\( (11) \)

Comparing the divergence in (10) and (11) we conclude that at criticality \( \alpha \to 2 \Rightarrow p_c(l_T) \sim l^{-2} \)

\[
\Rightarrow \langle l_T \rangle_c \sim \ln L
\]

(typical \( l_T \) is finite)
Signatures of MBL criticality in a driven open system

Lenarcic, EA, Rosch, PRL 267603 (2018)
Lenarcic, Alberton, Rosch, EA, PRL 116601 (2020)
The essence of Many Body localization (closed system)

\[ H = \sum_i S_i^z S_{i+1}^z + h_i^x S_i^x + h_i^y S_i^y \]

\[ h_i^z \in [-h, h] \]

\[ \tau \sim l^2 \]

Thermal (Griffith phase)

\[ h_c \]

MBL

Local integrals of motion

\[ \tilde{H} = \sum_i \tilde{V}_i \tau_i^z + \sum_{ij} \tilde{V}_{ij} \tau_i^x \tau_j^x + \ldots \]

Huse and Oganesyan 2013,
Serbyn and Abanin 2013
Vosk and EA 2013
MBL in an open system?

\[ H = \sum_i \hat{v}_i \tau_i^z + \sum_{ij} \hat{v}_{ij} \tau_i^z \tau_j^z + \ldots \]

The system thermalizes with the bath

\[ \langle \tau_i^z \rangle \sim e^{-\varepsilon t} \to 0 \]

Thermal (Griffith phase)
MBL in an open system?

- Couple the system also to a hot bath or drive

- Weak coupling limit: $\varepsilon \to 0 \quad \theta = \text{const}

- Infinite time: $\varepsilon t \to \infty$

Distinct steady states?"
Thermalizing system

conserved total energy:

\[ \dot{\mathcal{H}} = \mathcal{A} - \mathcal{E}(T - T_b) \]

steady state \[ \rho = e^{-\beta \mathcal{H}} \]

- unique temperature determined by \( \theta \) alone. Non-linear response.

MBL system

conserved local energies:

\[ \dot{\mathcal{G}}_i = \mathcal{G} \mathcal{G}_i \mathcal{G}_i - \mathcal{G}_i \mathcal{G}_i \mathcal{G}_i (T_i - T_b) + \mathcal{G}_i \mathcal{G}_i \mathcal{G}_i (T_i - T_i) + \mathcal{G}_i \mathcal{G}_i \mathcal{G}_i (T_i - T_{i-1}) \]

\[ \rho = e^{-\sum \beta_i \mathcal{G}_i} \quad T_i \sim T_b + \theta \frac{\partial \mathcal{G}_i}{\partial \mathcal{G}_i} \]

- Many local temperatures!
Digression: Quantum formulation [Zenarcic, Lange, Rosch arxiv:1706.05700]

\[ \dot{\rho} = L \rho = -i [H_0, \rho] + \varepsilon \hat{D} \rho \]

where \( L \) could be Lindblad form

we want the steady state:

\[ \lim_{t \to \infty} \rho(t) = \rho_\infty = \rho_0 + \delta \rho \]

\[ \rho_0 = \lim_{\varepsilon \to 0} \rho_\infty = ? \]

Suppose \( N \) commuting IOM:

\[ [H_i, H_j] = 0 \quad i,j = 0, \ldots, N \]

GGF ansatz \( \rho_0 = \frac{e^{-i \sum \lambda_i H_i}}{\text{Tr}(e^{-i \sum \lambda_i H_i})} \) satisfies \( \mathcal{L}_0 \rho_0 \) for all \( \{ \lambda_i \} \)

\[ \Rightarrow \{ \lambda_i \} \text{ determined by } \mathcal{L}_1 \]

Exercise 3: Show that the steady state conditions \( \langle \dot{H}_i \rangle = 0 \) to lowest order in \( \varepsilon \) are given by \( \text{Tr}(H_i \hat{D} \rho_0) = 0 \). These give \( N \) equations for the \( N \) \( \lambda_i \).
Numerical results

Exact diagonalization
strict limit $\varepsilon \to 0$ $\Delta t \to \infty$

1) observe singular onset of the temperature variance at the MBL transition.

How is the singularity broadened by a finite $\varepsilon$ in a realistic system? Universality? Finite $\varepsilon$ scaling?
Hydrodynamic description

\[ \frac{\partial e}{\partial t} - \nabla [\kappa(r) \nabla T(r)] = \varepsilon \theta g_{1}(r) - \varepsilon g_{2}(r)(T(r) - T_b) \]

\[ g_{1,2} = 1 + \xi_{1,2}(r) \quad T = \bar{T} + \delta T \]

\[ -\nabla [\kappa(r) \nabla \delta T] = \varepsilon \theta (1 + \xi_{1}(r)) - \varepsilon (1 + \xi_{2}(r))(\bar{T} - T_b + \delta T) \]

Expand in powers of: \( \xi_{1,2}(r) \) (\( \delta T \sim \xi_{1}, \xi_{2} \))

Zeroth order: \( \bar{T} = T_b + \theta \)

Linear order:

\[ (-\kappa \nabla^2 + \varepsilon) \delta T = \varepsilon \theta \xi_{1}(r) \quad <\xi(r)\xi(r')> = \mu \delta(r-r') \]

Let's solve this in the two phases
MBL phase: $\bar{E} \sim x E \quad \Rightarrow \quad (-x \nabla^2 + 1) \delta T = \delta \xi(r)$

$$\delta T = \theta \int dr' G(r'-r) \xi(r') \delta G(r'-r')$$

$$\langle \delta T^2 \rangle = \theta^2 \int dr' dr'' G(r') G(r'') \langle \xi(r') \xi(r'') \rangle$$

$$= \theta \int \frac{d^2 q}{2\pi} \left( \frac{1}{\chi q^2 + \epsilon} \right)^2 \sim \frac{2\pi \theta}{\alpha} \theta^2$$

$\Rightarrow$ spatial temperature fluctuations are finite ($o(\theta)$) at zeroth order in

Thermal phase: $\bar{E}(q) \sim q^{2-2} \quad \tilde{G}(q) = \left( \chi q^2 + \epsilon \right)^{-1}$

$$\delta T \equiv \sqrt{\langle \delta T^2 \rangle} \sim \Theta W \xi^{\frac{1}{2\epsilon}}$$

Exercise 4: Derive the last relation
Numerical approach

This limit of an open system of taking $t \to \infty$ first (steady state) and $\varepsilon$ small, motivates a numerical scheme; use TEBD to compute the steady state $\hat{\rho}_\infty$ of the Lindblad evolution:

$$\dot{\rho} = i[H, \rho] + \varepsilon \left( D_{\text{cold}}[\rho] + g \cdot D_{\text{hot}}[\rho] \right)$$

Pros:
- $\varepsilon$ limits the growth of the bond dimension => can handle much larger systems than ED.
- The level broadening $\varepsilon$ possibly regulates small size transients related to level commensurabilities.

Cons:
- Larger physical dimension than ED of $H$
Our calculation: Model

\[ \dot{\rho} = i[H, \rho] + \varepsilon \left( D_{\text{cold}}[\rho] + \theta D_{\text{hot}}[\rho] \right) \]

\[ H = \sum_i S_i^z \cdot \bar{S}_i^z + h^x_i S_i^x + h^z_i S_i^z \]

\[ h^x_i \in [-h, h] \]

\[ D_{\text{hot}}[\rho] = \sum_i \left[ 4 S_i^z \rho S_i^z - \rho \right] \]

\[ D_{\text{cold}}[\rho] = \sum_{i,\alpha} L_{i\alpha} \rho L_{i\alpha}^\dagger - \frac{i}{\hbar} \{ L_{i\alpha} L_{i\alpha}^\dagger, \rho \} \]

Operational definition of local temperature:

\[ F(T_i) = \min_T \left[ \text{tr} \left( (\rho_{i,i+1}^{(i)} - \rho_{\text{th}}^{(i)}(T))^2 \right) \right] \]

\[ \Rightarrow T_i \]

\[ L_{i1} = S_i^z \rho_{i,i+1}^{(i)} \quad L_{i2} = \rho_{i1} S_i^z \]

\[ L_{i3} = S_i^+ \rho_{i,i+1}^{(i)} \quad L_{i4} = \rho_{i1} S_i^- \]
Numerical results (TEBD)

- Signature of critical dynamics in \( \varepsilon \) dependence of \( \delta \tau \sim \varepsilon^{1/2} \)

- Obtain divergence of \( z \)