Goal: Materialize topological SC.

1. Go beyond serendipitous discoveries & post-dictions
2. Get your approximations endorsed by experiments.

Strategy?

Recall : $H = H_{kin} + H_{int}$,

① Manipulate $H_{kin}$

- 1D wire (see von Oppen)
- TI | SC Heterostructure
- p-dope TMD
  arXiv: 1606.00857

② Manipulate $H_{int}$

- Metal | QSL Heterostructure
  arXiv: 1603.02692

-※ Poll: word cloud

* Mean-field description of TSC

① Quadratic gapped Hamiltonian cf. TI, QH
② particle - hole symmetry
③ time-reversal { symmetric cf. TI
   broken cf. QH
④ topologically non-trivial eigenstate spectrum (see Kane)
⑤ majorana bound-state at the vortex core and at an edge
⑥ odd-parity triplet.

-※ Poll: Hubbard model.

⑦ Insufficient for materialization.

* $\boxed{\text{Triplet} \Leftrightarrow \text{odd-parity}}$

Consider a Cooper pair wave function

$$\Psi_{c.p.}(\vec{r}_1, \vec{r}_2 ; \sigma_1, \sigma_2) = g(\vec{r}_1 - \vec{r}_2)\, \chi(\sigma_1, \sigma_2)$$

position & spin of each electron

in the absence of spin-orbit

Demand fermionic statistics

$$\Psi_{c.p.}(\vec{r}_1, \vec{r}_2 ; \sigma_1, \sigma_2) = -\Psi_{c.p.}(\vec{r}_2, \vec{r}_1 ; \sigma_2, \sigma_1)$$

1) singlet SC $\Leftrightarrow$ $\chi(\sigma_1, \sigma_2) = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

$\Rightarrow$ $g(\vec{r}_1 - \vec{r}_2)$ should be symmetric, i.e., even-parity

$l = 0$ (s-wave), $2$ (d-wave), ...

with the spin-channel fixed, the order parameter

$\Delta_{\vec{k}}$ is a complex function

2) triplet SC $\Leftrightarrow$ $\chi(\sigma_1, \sigma_2)$ symmetric

$\Rightarrow$ $g(\vec{r}_1 - \vec{r}_2)$ should be odd-parity

$\Leftrightarrow$ $g_{\vec{k}} = -g_{-\vec{k}}$

$l = 1$ (p-wave), ...

Generally

$$\Psi_{c.p.}(\vec{r}_1, \vec{r}_2 ; \sigma_1, \sigma_2) = g_{\uparrow\uparrow}(\vec{r}_1 - \vec{r}_2)|\uparrow\uparrow\rangle + g_{\uparrow\downarrow}(\vec{r}_1 - \vec{r}_2)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + g_{\downarrow\downarrow}(\vec{r}_1 - \vec{r}_2)|\downarrow\downarrow\rangle$$

the o.p. is a complex matrix in spin-space

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\uparrow\uparrow}(\vec{k}) & \Delta_{\uparrow\downarrow}(\vec{k}) \\ \Delta_{\uparrow\downarrow}(\vec{k}) & \Delta_{\downarrow\downarrow}(\vec{k}) \end{pmatrix}$$

* $\boxed{\text{"Spinless fermion" description of Equal spin pairs.}}$

1) ABM ($^3$He-A) vs BW ($^3$He-B) phase and T-rev.

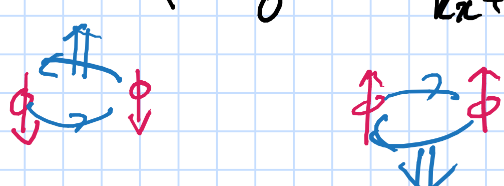
- T-rev
  - i) $\uparrow \to \downarrow$
  - ii) $\vec{k} \to -\vec{k}$
  - iii) complex-conjugate

- $$\hat{\Delta}^{2D}_{ABM} = \underbrace{\Delta_0 (k_x + i k_y)}_{l_z = 1 \text{ is a good quantum \#.}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T \Delta_{\uparrow\uparrow}(\vec{k}) = \Delta^*_{\downarrow\downarrow}(-\vec{k}) = -\Delta_0 (k_x - i k_y) \neq \Delta_{\uparrow\uparrow}(\vec{k})$$

$\Rightarrow$ T- breaking

- $$\hat{\Delta}^{2D}_{BW} = \Delta_0 \begin{pmatrix} -k_x + i k_y & 0 \\ 0 & k_x + i k_y \end{pmatrix}$$



$$T \Delta_{\uparrow\uparrow}(\vec{k}) = \Delta^*_{\downarrow\downarrow}(-\vec{k}) = \Delta_{\uparrow\uparrow}(\vec{k})$$

$$T \Delta_{\downarrow\downarrow}(\vec{k}) = \Delta_{\downarrow\downarrow}(\vec{k})$$

$\Rightarrow$ T - invariant, pairs are spin-orbit locked.

2) Equal spin pairs $\Leftrightarrow$ odd parity spinless fermion pairs.

$$H_{BdG} = \sum_k \left[ \xi_k c_k^\dagger c_k + \frac{1}{2} ( \Delta_k^* c_{-k} c_k + \Delta_k c_k^\dagger c_{-k}^\dagger ) \right]$$

$$\xi_k = \frac{1}{2m} |\vec{k}|^2 - \mu$$

$$\Delta_k = \Delta_0 (k_x + i k_y)$$ (Read & Green PRB 2000. see Read)

※ Poll: Interacting ?

Diagonalize (Bogoliubov transform)

$$\gamma_k = u_k c_k - v_k c_{-k}^\dagger$$

$$\gamma_k^\dagger = u_k^* c_k^\dagger - v_k^* c_{-k}$$

with $\frac{v_k}{u_k} \Delta_k^* = -(E_k - \xi_k)$

$$\Rightarrow H_{BdG} = \sum_k E_k \gamma_k^\dagger \gamma_k + \text{const.}$$

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$$

Topological Quantum Phase transition

$\mu > 0$ : weak pairing, topological

$\mu < 0$ : strong pairing, trivial

3) N-electron wavefunction & Pfaffian

$$|\Psi_{BCS}\rangle = \prod_k (u_k + v_k c_k^\dagger c_{-k}^\dagger)|0\rangle$$

$$= \prod_k u_k (1 + g_k c_k^\dagger c_{-k}^\dagger)|0\rangle, \quad g_k \equiv \frac{v_k}{u_k}$$

project $|\Psi_{BCS}\rangle$ to N-particle state:

$$\langle \vec{r}_1, \vec{r}_2, \cdots \vec{r}_N | \Psi_{BCS} \rangle = \mathcal{A}\, g(\vec{r}_1 - \vec{r}_2)\, g(\vec{r}_3 - \vec{r}_4) \cdots g(\vec{r}_{N-1} - \vec{r}_N)$$

$$g(\vec{r}) = \frac{1}{L^2} \sum_k e^{i\vec{k}\cdot\vec{r}} g_{\vec{k}}$$

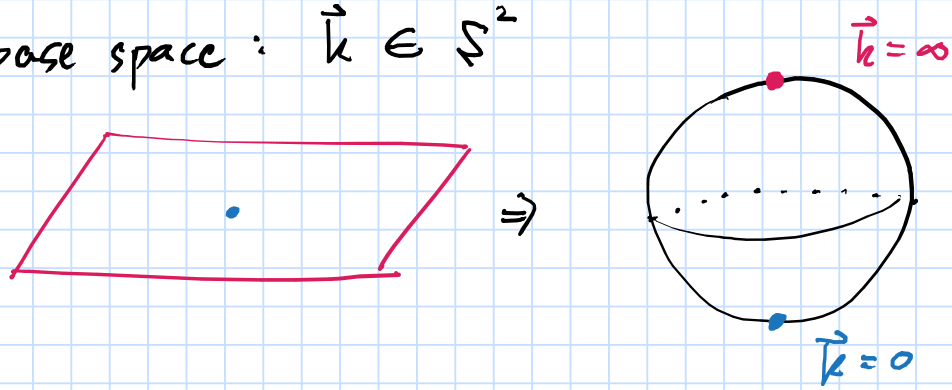
$\mathcal{A}$ : anti-symmetrization op.

$$\Psi_{BCS}(\vec{r}_1, \cdots \vec{r}_N) = \mathrm{Pf}[\hat{g}] \equiv \frac{1}{2^{N/2}\left(\frac{N}{2}\right)!} \sum_P \mathrm{sgn}(P) \prod_{i=1}^{N/2} g(\vec{r}_{P(2i-1)} - \vec{r}_{P(2i)})$$

Poll :

* $\boxed{\text{Topological index.}}$

base space : $\vec{k} \in S^2$




target space : $\hat{n}_{\vec{k}} = \frac{1}{E_{\vec{k}}} (\mathrm{Re}\Delta_{\vec{k}}, -\mathrm{Im}\Delta_{\vec{k}}, \xi_{\vec{k}}) \in S^2$

$\hat{n}_{\vec{k}}$ is a normal vector on a unit sphere

$\because \quad E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + |\Delta_{\vec{k}}|^2}$

Now $\vec{k} \to \hat{n}_{\vec{k}}$ for given $\mu$ and $\Delta_{\vec{k}}$ is a map $S^2 \to S^2$

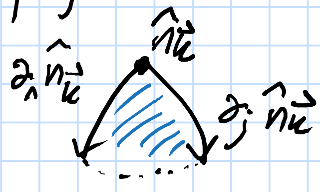
Q. classes of topologically distinct maps ?

Homotopy class $\underset{\substack{\uparrow \\ \text{base space} \\ S^2}}{\pi_2} (S^2) = \mathbb{Z}$ $\leftarrow$ target space

1) $\hat{n}_{\vec{k}}$ wraps $S^2$ integer $\mathcal{M}[\hat{n}_{\vec{k}}]$ times as $\vec{k}$ varies

$$\mathcal{M}[\hat{n}_{\vec{k}}] \equiv \frac{1}{4\pi} \frac{1}{2} \int d\vec{k} \; \varepsilon_{ij} \; \hat{n}_{\vec{k}} \cdot (\partial_i \hat{n}_{\vec{k}} \times \partial_j \hat{n}_{\vec{k}})$$

: Pontryagin index or Chern number




area of the oriented triangle

2) $\mathcal{M}[\hat{n}_k]$ depends on $\Delta_{\vec{k}}$ (and the sign of $\mu$)

3) $\mathcal{M}[\hat{n}_k] = 0$ : topologically trivial.

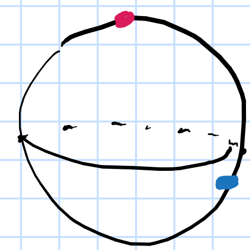
Examples for $\xi_{\vec{k}} = \frac{k^2}{2m} - \mu$, $\lim_{\vec{k}\to\infty} \xi_k = \lim_{\vec{k}\to\infty} E_{\vec{k}} = \frac{k^2}{2m}$

1) singlet s-wave : $\Delta_{\vec{k}} = \Delta_0$ constant.

$$\hat{n}_{\vec{k}} = \frac{1}{E_{\vec{k}}} (\Delta_0, 0, \xi_{\vec{k}}),$$

$$\vec{k} = \infty : \hat{n}_{N.P.} = (0, 0, 1) = N.P.$$

$$\vec{k} = 0 : \hat{n}_{S.P.} = \frac{1}{\sqrt{\mu^2 + \Delta_0^2}} (\Delta_0, 0, -\mu)$$



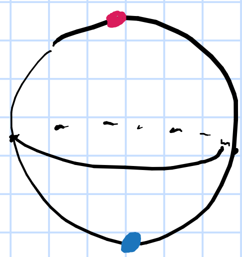
$\Rightarrow \mathcal{M} = 0$ : topologically trivial

2) spinless $p-ip$ : $\Delta_{\vec{k}} = k_x - i k_y$

$$\hat{n}_{\vec{k}} = \frac{1}{E_{\vec{k}}} (k_x, k_y, \xi_{\vec{k}})$$

$$\vec{k} = \infty : \hat{n}_{N.P.} = (0, 0, 1) = N.P.$$

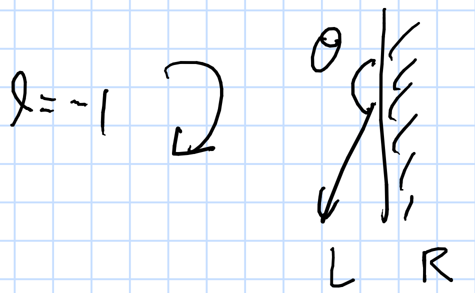
$$\vec{k} = 0 : \hat{n}_{S.P.} = \frac{1}{\mu} (0, 0, -\mu) = (0, 0, -1) = S.P.$$



$\Rightarrow \mathcal{M} = \pm 1$ : topologically non-trivial

## * Sketchy derivation of the Edge State

Consider an edge




$$k_x = -k\sin\theta = k\cos(\theta+\pi/2)$$
$$k_y = k\cos\theta = k\sin(\theta+\pi/2)$$

$$\Delta_k \propto (k_x - i k_y) \quad \text{with } l=-1$$
$$= \Delta_0 \frac{k}{k_f} e^{i\theta}$$

Finding edge state $\Leftrightarrow$ solving boundary QM.

Focus on the 1D problem at each point

$$H^{L/R}_{Bdg} = \begin{pmatrix} -i\partial_x & \Delta e^{i\phi_{L/R}(y)} \\ \bar{\Delta} e^{-i\phi_{L/R}(y)} & i\partial_x \end{pmatrix}$$

Seek a bound-state solution $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} e^{-\kappa|x|}$

with eigenvalue $E_0 = \sqrt{|\Delta|^2-\kappa^2}$

<u>$x>0$</u> $\quad i\kappa u_0 + \Delta_R v_0 = E_0 u_0$

<u>$x<0$</u> $\quad -i\kappa u_0 + \Delta_L v_0 = E_0 u_0$

Match the boundary:

$$\frac{(E_0 - i\kappa)}{(E_0 + i\kappa)} = \frac{\Delta_R}{\Delta_L} = e^{-i\Phi}$$

$$\Phi \equiv \phi_L - \phi_R$$

$$E_0 = \Delta\cos\Phi/2 \qquad \kappa = \Delta\sin\Phi/2$$

Back to 2D. $\Phi = \theta$, $E_0 = 0$ bound state for $\theta = \pi$.

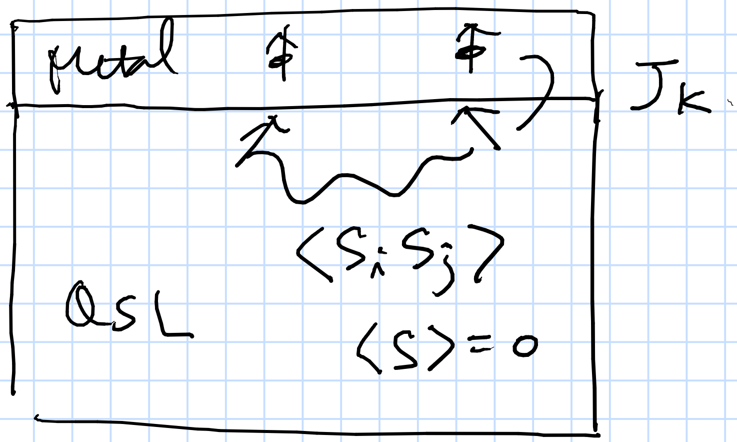
☐ Metal / QSL Heterostructure.

* Non-phononic mechanism?

Poll RVB

① Dope RVB spin-liquid. ☹

② Borrow spin-correlation;



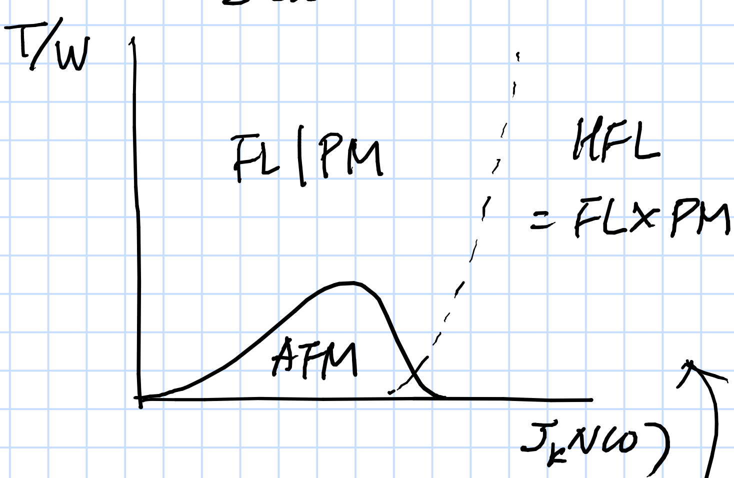

Simple isotropic metal
+
Gapped QSL.

i.e. $\langle S_i \rangle = 0$,
$\langle S_i S_j \rangle \neq 0$

spin-fermion model

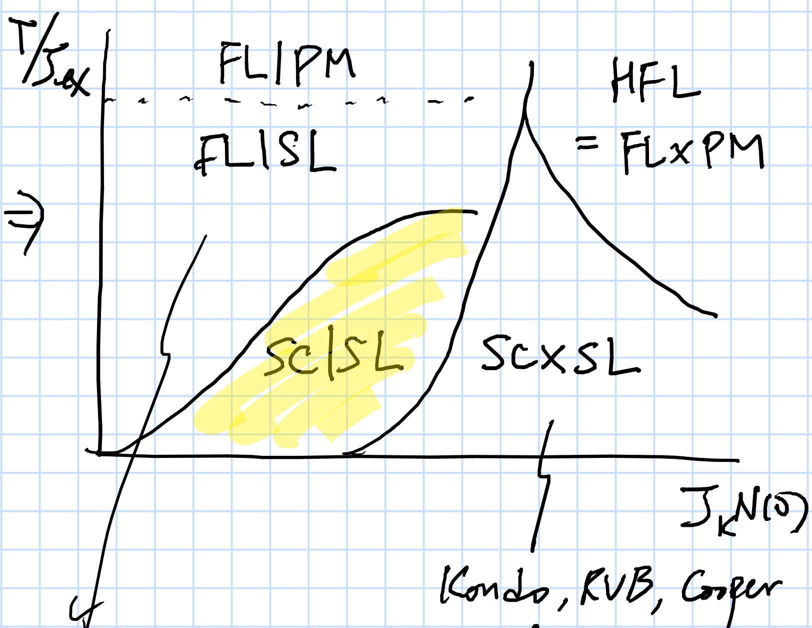
Energy scales: $E_F$, $J_{ex}$, $J_K$.

For $J_{ex} = 0$.




$\Rightarrow$

$J_{ex} \neq 0$ + Frustration




Doniach.

Kondo singlet

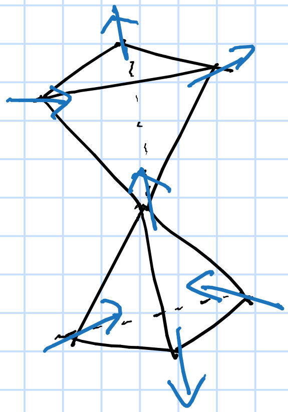
Senthil
Vojta
Sachdev
(2003)

Kondo, RVB, Cooper singlets

Coleman & Andrei
(1989)

Needed: A gapped, well-understood QSL.

$\Rightarrow$ QSI.



2 in-2out ice-rule

$$\Leftrightarrow \quad \vec{\nabla}\cdot\vec{S}(\vec{r}) = 0$$

$$\vec{S} = \vec{\nabla}\times\vec{A}$$

$\rightsquigarrow$ emergent gauge field.

$$\langle A_a(\vec{q})\, A_b(-\vec{q})\rangle \sim \frac{1}{q^2}\left(\delta_{ab} - 2\hat{q}_a\hat{q}_b\right)$$

$$\Rightarrow \langle S_a(\vec{q})\, S_b(-\vec{q})\rangle \sim \delta_{ab} - \hat{q}_a\hat{q}_b$$

Experiments: $Pr_2Zr_2O_7$.
- Quantum Fluct ✓
- Gapped.
- "understanding"

* $H = H_c + H_K + H_{spin}$

$$H_K = J_K \sum_{\alpha\beta}\int d\vec{r}\; \psi^\dagger_\alpha(\vec{r})\,\vec{\sigma}_{\alpha\beta}\,\psi_\beta(\vec{r})\cdot\vec{S}(\vec{r}, z=0)$$

$$= -J_K \sum_{\alpha\beta}\int d\vec{r}\left(\vec{\nabla}\times\psi^\dagger_\alpha(\vec{r})\,\sigma_{\alpha\beta}\,\psi_\beta(\vec{r})\right)\cdot\vec{A}(\vec{r}, z=0)$$

Note: unusual gauge-matter coupling!!

integrate out spin. safest with gap.

$$\mathcal{L}_{int}(t) = -\frac{J_K^2}{2}\int dt'\int d\vec{r}\, d\vec{r}'\; \underbrace{\vec{s}(\vec{r},t)\cdot\langle\vec{S}(\vec{r},t)\,\vec{S}(\vec{r}',t')\rangle\cdot\vec{s}(\vec{r}',t')}$$

$$\left[(\vec{\sigma}_{\alpha\beta}\times\vec{\nabla})_a\,\psi^\dagger_\alpha\psi_\beta\right] D_{ab} \left[(\vec{\sigma}_{\alpha'\beta'}\times\vec{\nabla})_b\,\psi^\dagger_{\alpha'}\psi_{\beta'}\right]$$

$D_{ab}$: gauge propagator

Observations.

① unusual gauge-matter coupling

Minimal Coupling

$$\vec{j}\cdot\vec{A}$$

$$= e\sum_{k,\alpha} \vec{A}(\vec{q})\cdot\frac{\vec{k}}{m}\psi^\dagger_{\vec{k}+\vec{q}/2,\alpha}\psi_{\vec{k}-\vec{q}/2,\alpha}$$

$\Downarrow$

Repulsion against pairing

$$-\sum_{p_1 p_2 q} D(q)\frac{(\vec{p}_1\times\hat{q})\cdot(\vec{p}_2\times\hat{q})}{m^2}\psi^\dagger_{p_1+q}\psi_{p_1}\psi^\dagger_{p_2-q}\psi_{p_2}$$

$> 0$ for $\vec{p}_1 = -\vec{p}_2$

Spin-ice / electron

$$J_K\sum \psi^\dagger_\alpha\vec{\sigma}_{\alpha\beta}\psi_\beta\cdot(\vec{\nabla}\times\vec{A})$$

QSI induced interaction

$$-J_K^2 D(\vec{q})(\vec{\sigma}_{\alpha\beta}\times\hat{q})\cdot(\vec{\sigma}_{\alpha'\beta'}\times\hat{q}) < 0 \quad \text{if } \vec{\sigma}_{\alpha\beta}\parallel\vec{\sigma}_{\alpha'\beta'}$$

$\Rightarrow$ possibly attractive but anisotropic & SO coupled effective interaction.

② selection rule in mean-field theory

$\Rightarrow$ pairing only in $S=1$, odd-parity channel.