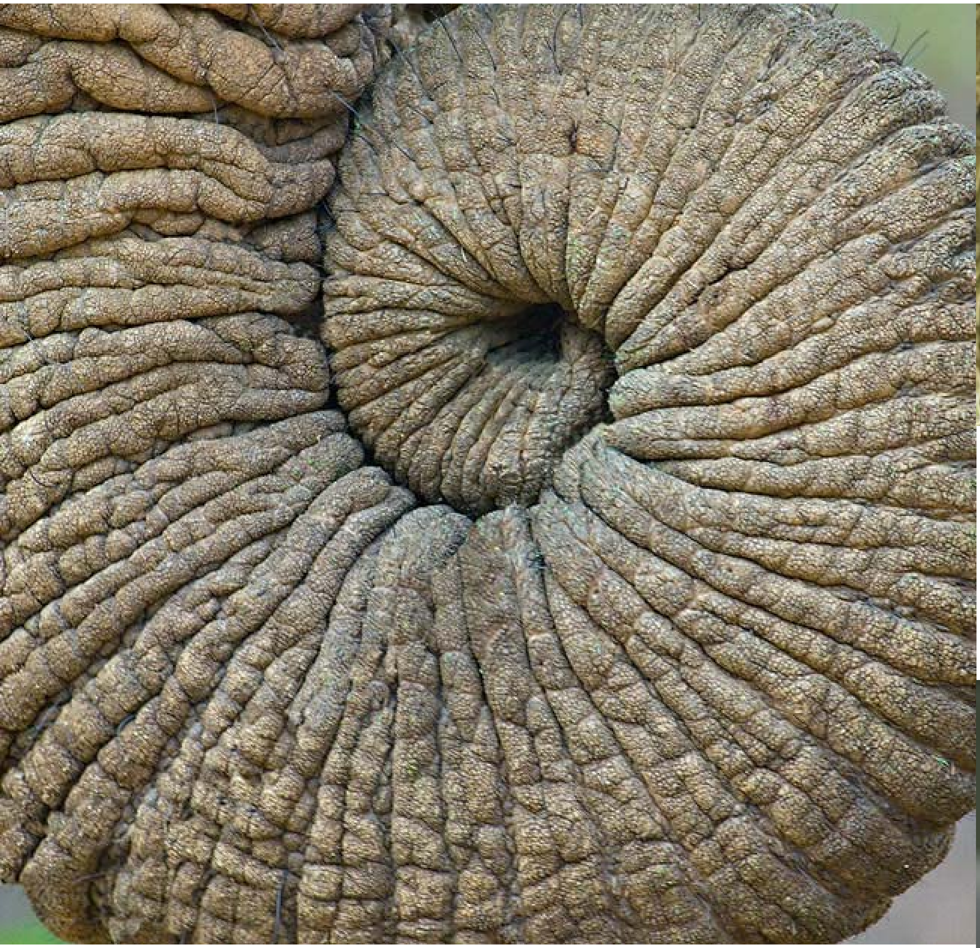
Objects that are flexible purely for geometric reasons (sheets, filaments and ribbons) make an overwhelming variety of patterns in nature and our technological world.

Can we organize this profusion of shape and form by identifying building blocks? Are there elementary excitations of elastic materials that we can study?



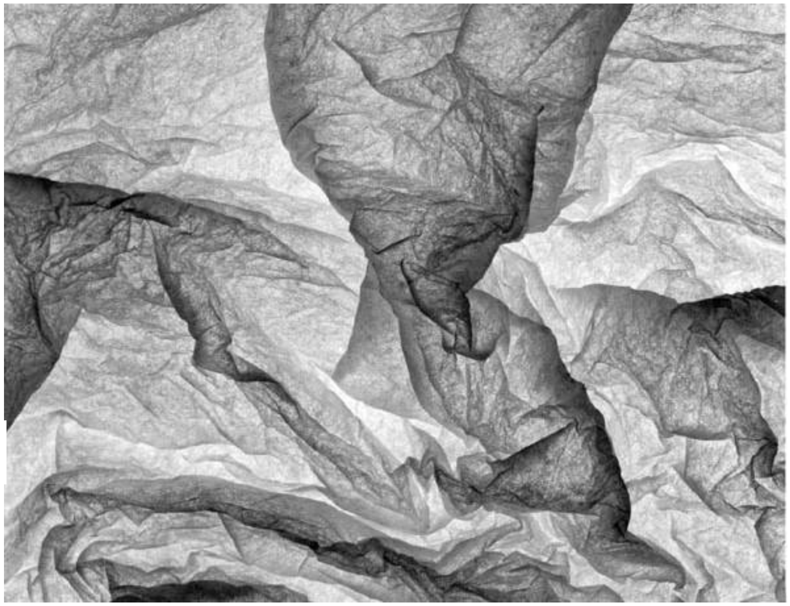
Sea urchin

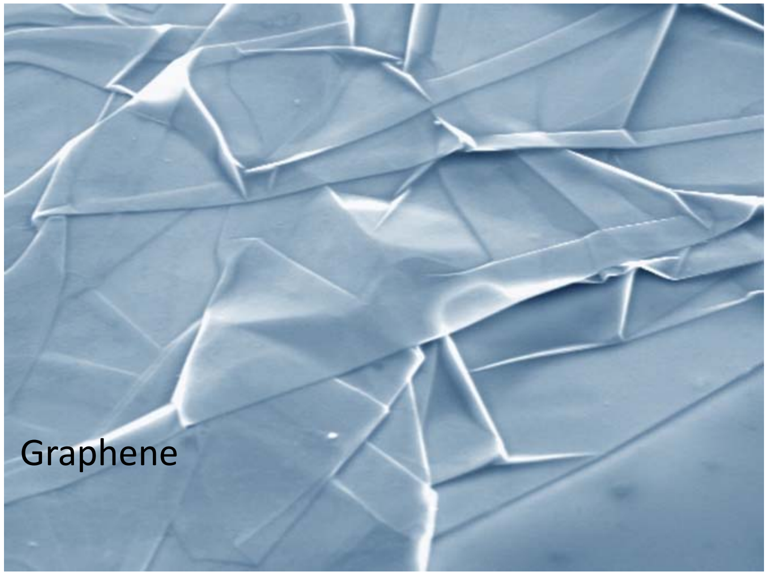
Sharon, Swinney, Marder

Yva Momatiuk and John Eastcott/PhotoResearchers, Inc.

that a leaf or flower—just like a torn sheet of plastic—can use an enhanced, uniform growth at its margins to generate such complex patterns. Examples of wavy edges in nature include, from left to right, some lichens (shown, *Sticta limbata*), orchids (shown, *Schomborgkia beysiana*), sea slugs (represented by *Glossodoris hikuerensis*) and ornamental cabbage. (Lichen photograph courtesy of Stephen Sharnoff; sea slug photograph courtesy of Jeff Jeffords.)

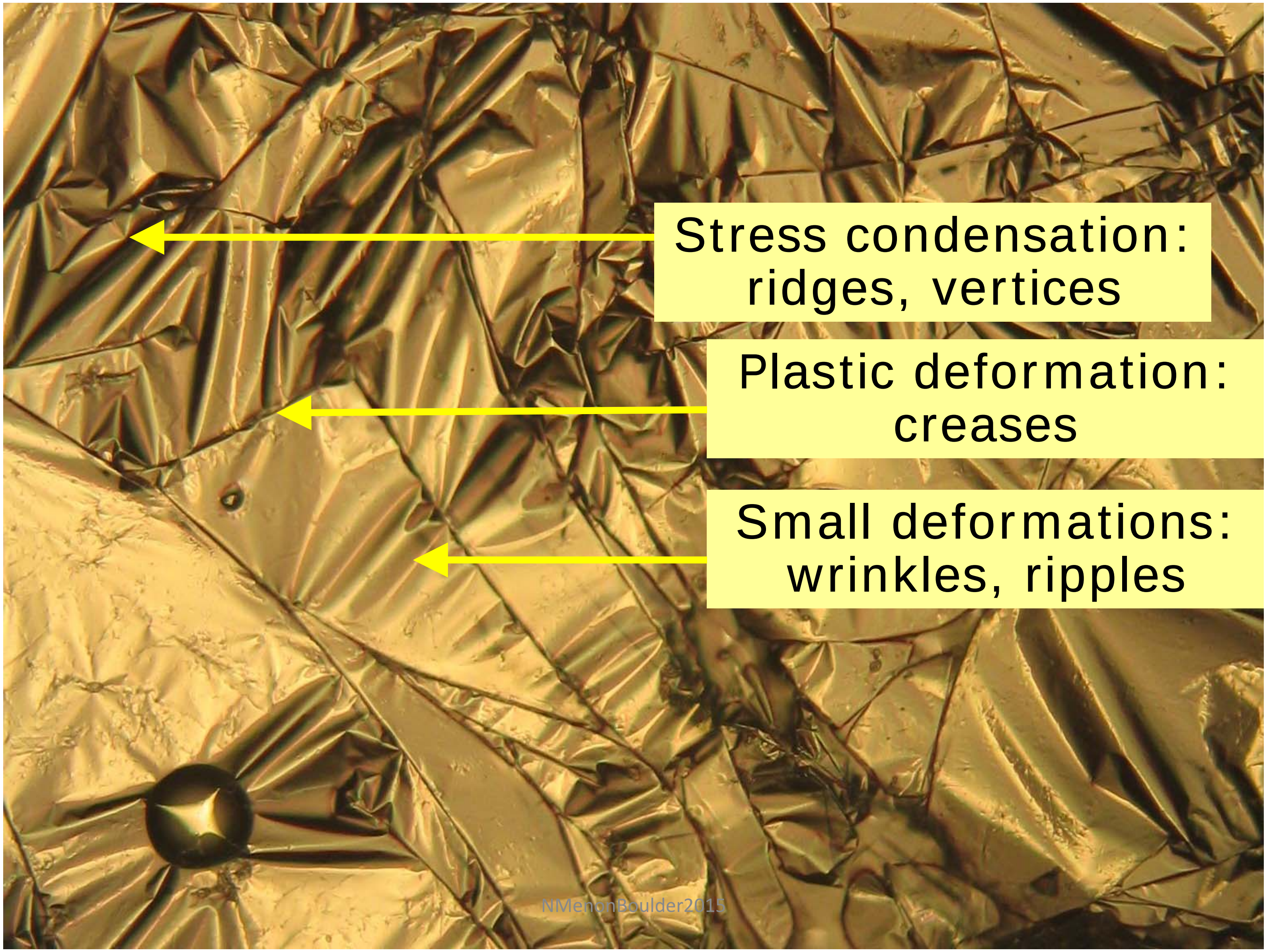
Fabric





Graphene

Earth’s skin



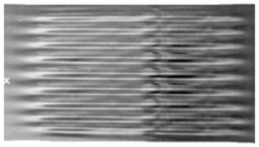
Stress condensation: ridges, vertices

Plastic deformation: creases

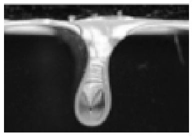
Small deformations: wrinkles, ripples

Overall goals of our discussion

Wrinkles

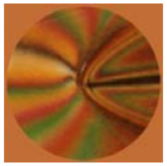


Folds

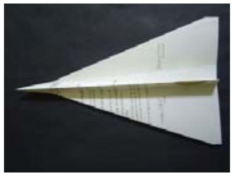


Ridges

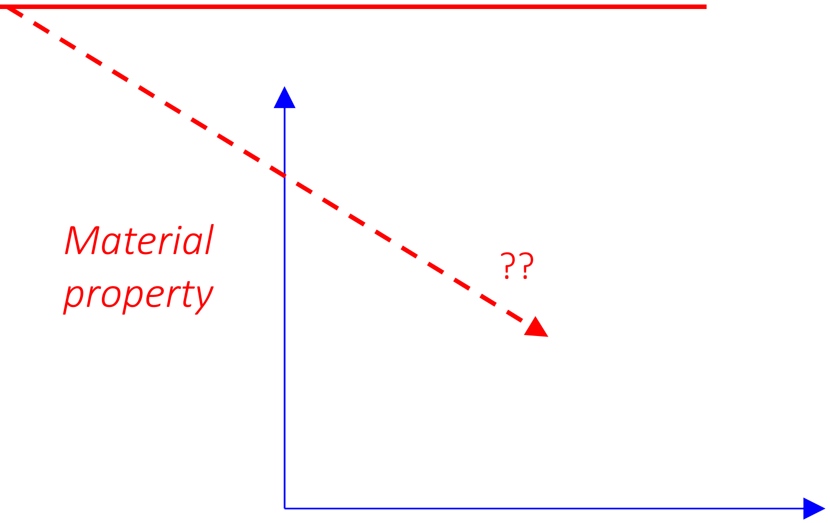
d-cones



Creases (plasticity)



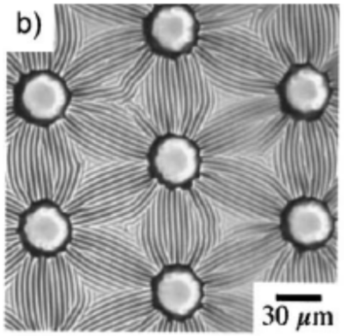
- These structures are generated by elastic instabilities
- What are the energetics and stability of these constructs?
- Where do all these structures belong?
- How to specify these axes?



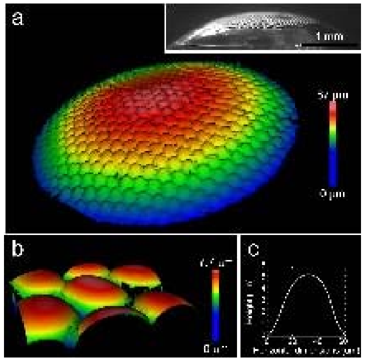
Instability not as “failure” but technological tool

Patterning (actuatable ones at that), metrology, coatings, surface control

Nanoscale elastic patterning

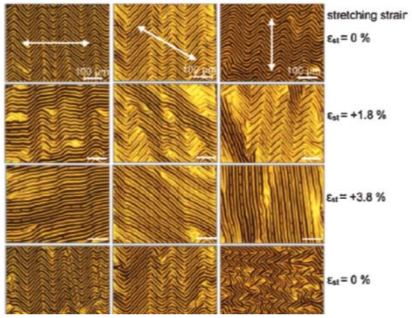


Bowden et al, 1999

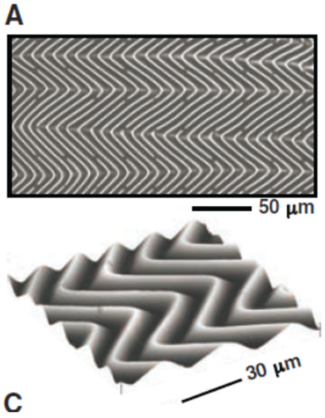


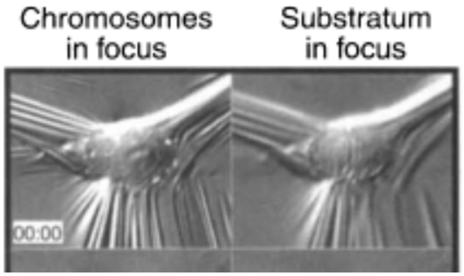
Crosby, Breid 2010

Stretchable electronics



Rogers 2011





Burton, Nature 97

Plan

Overall theme	Pattern formation via elastic instabilities
Elasticity	Stress, large deformation strain, Hooke’s Law; moduli for plates
1D Euler buckling	two approaches
1D wrinkling	Scaling analysis, generality of “substrate”; going beyond single mode (briefly)
1D Folds	mechanical stability, exact solution, system size dependence
2D Wrinkling	Lamé problem as archetype, two limits of FvK, increased dimensionality of phase space; (briefly) other geometries
Crumples	Ridges, d-cones and e-cones;
Wrapping	Idea of asymptotic isometry; Folds in 2-D

Things I will not do

Mainly mechanics, will not work at thermal scales

Focus on sheets, not on filaments, ribbons (but others have more than compensated for that)

No free surface instabilities

Focus on statics, not on dynamics (lots of open problems and opportunities here)

- Strain describes change in distance between material points

Material point moves from $\vec{x} \to \vec{x}'$ Get a displacement field

$$u_i = x_i' - x_i'$$

Change in distance between material points $d\vec{x}$ apart

$dl^2 = dx_i\, dx_i$

$$dl'^2 = dx_i'\, dx_i' = (dx_i + du_i)(dx_i + du_i)$$
$$= (dx_i + \partial_j u_i\, dx_j)(dx_i + \partial_j u_i\, dx_j)$$
$$= (dl^2 + 2\, \partial_j u_i\, dx\, dx_j + \partial_j u_i\, \partial_k u_i\, dx_j\, dx_k)$$
$$= dl^2 + 2\, \epsilon_{ij}\, dx_i\, dx_j$$

Strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \underbrace{\partial_i u_k\, \partial_j u_k}_{\text{Nonlinear in deformation}}\right)$$

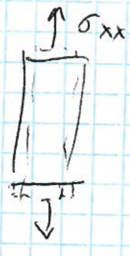
Sometimes called

Green's strain tensor (as opposed to linear Cauchy)

$\boxed{\text{Stress- Strain Relation}}$ – Hooke's Law

Young's modulus $Y = \dfrac{\sigma_{xx}}{\epsilon_{xx}}$

Poisson ratio $\Lambda = -\dfrac{\epsilon_{yy}}{\epsilon_{xx}}$



Typical moduli:

10^{11}	Pa	metals
10^{9}	Pa	glassy polymers
10^{5}-10^{6}	Pa	Elastomers

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\, \delta_{ij}\, \epsilon_{kk}$$

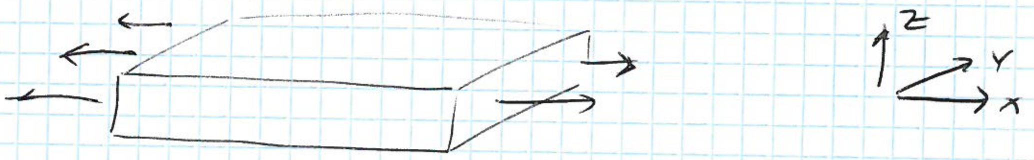
(Cauchy 1822, Lamé 1852)

OR $\varepsilon_{ij} = \dfrac{(1+\Lambda)}{E}\sigma_{ij} - \dfrac{\Lambda}{E}\sigma_{kk}\,\delta_{ij}$

Explicitly $\epsilon_{xx} = \sigma_{xx}/E - A/E\,(\sigma_{yy} + \sigma_{zz})$; $\varepsilon_{xy} = \dfrac{1+\Lambda}{E}\sigma_{xy}$

Thin plates : Not interested in variations across thickness. Integrate out z-dependence

Stretching



$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{F}{Et}$$

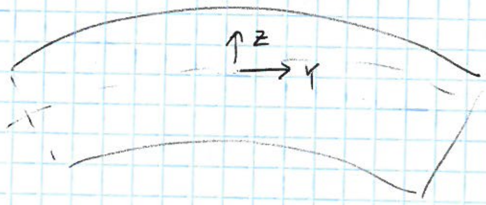
Et ← Y – stretching modulus.

Stretching energy / area $\mathcal{E} = \frac{1}{2} Y \epsilon_{xx}^2$

$$\boxed{Y = Et}$$

Bending:

[Also look at Alex Levine notes]



Deflection $w = w(y)$

$$\epsilon_{yy} = \frac{\partial^2 w}{\partial y^2}$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \Lambda \frac{\sigma_{xx}}{E}$$

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - A \frac{\sigma_{yy}}{E} = 0$$

$$\Rightarrow \sigma_{xx} = \Lambda \sigma_{yy}$$

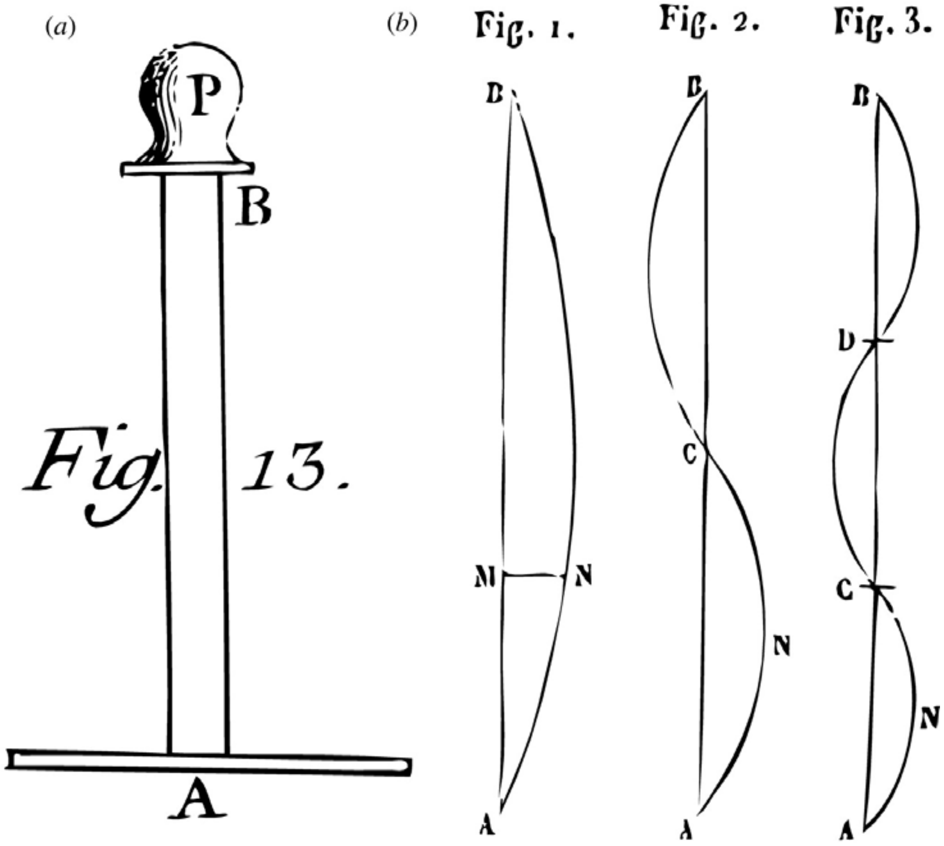
$$\epsilon_{yy} = \frac{1-\Lambda^2}{E} \cdot \sigma_{yy} \Rightarrow \sigma_{yy} = \frac{E \epsilon_{xx}}{1 \cdot \Lambda^2} = -\frac{Ez}{1-\Lambda^2} \frac{\partial^2 w}{\partial y^2}$$

Bending moment

$$M_{,} = \int_{-t/2}^{t/2} \sigma_{yy} z \, dz = - \int_{-t/2}^{t/2} \frac{Ez^2}{1-\Lambda^2} \frac{\partial^2 w}{\partial y^2} dz$$

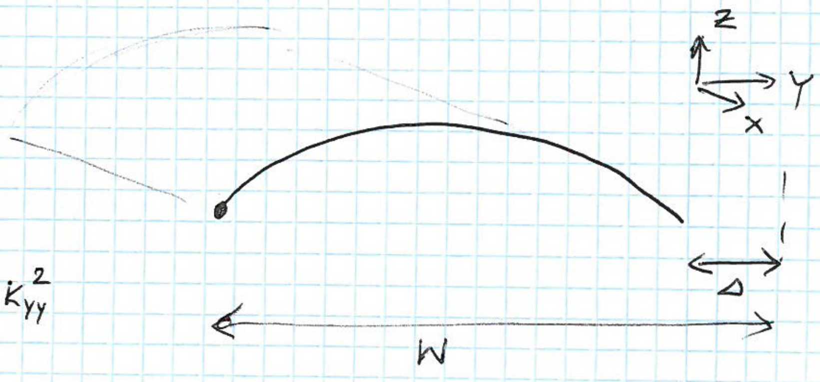
$$= B \frac{\partial^2 w}{\partial y^2} \rightarrow B = \frac{Et^3}{12(1-\Lambda^2)}$$

Euler buckling (a) illustrations from Euler (1744) (b) illustrations from Lagrange 1770.



Alain Goriely et al. Proc. R. Soc. A 2008;464:3003-3019

Euler Buckling of a plate



Elastic energy density

$$e = \frac{1}{2} Y \epsilon_{YY}^2 + \frac{1}{2} B \kappa_{YY}^2$$

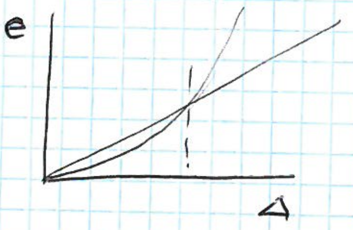
We analyze this situation in two ways

① Elastica

Compare pure compression against pure bending.

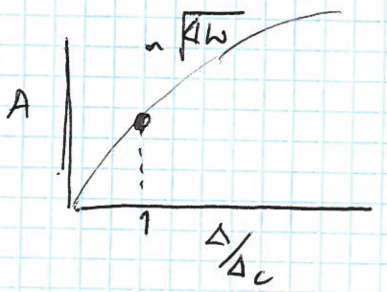
- Compression $\quad \epsilon_{YY} = \Delta/W \quad \Rightarrow \quad e_s = \frac{1}{2} Y \left(\frac{\Delta}{W}\right)^2$
- Bending - assume inextensibility constraint

$$\int_0^{W-\Delta} dy \sqrt{1+\xi'^2} - W = 0$$



Small slope $|\xi'| \ll 1 \rightarrow \int_0^{W-\Delta} (1+\frac{1}{2}\xi'^2)dy - W = 0$

$$e = \frac{1}{2} B \xi''^2 + \sigma \left(\frac{1}{2}\xi'^2 - \Delta\right) = 0$$



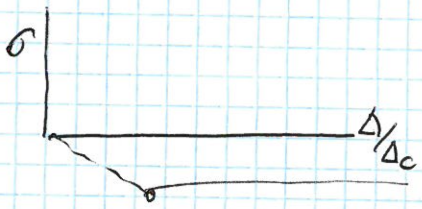
Euler Lagrange : $\quad B\xi'''' - \sigma\xi''$

With torque $= 0$ & $\xi(0) = \xi(W) = 0$

$\xi(y) = A \sin\left(\sqrt{\frac{\sigma}{B}}\, y\right) \qquad \sigma = \frac{\pi^2 B}{W^2}$

$A = \frac{2}{\pi}\sqrt{\Delta W}$

$e = \frac{1}{2} B \frac{\Delta}{W^3}$



Transition at $\Delta/\Delta_c = 1$ where $\quad \frac{B\Delta_c}{W^3} = \frac{Y\Delta_c^2}{W^2}$

$$\Rightarrow \Delta_c = \frac{B}{YW} \Rightarrow \frac{\Delta_c}{W} = \frac{B}{YW^2} \sim \left(\frac{t}{W}\right)^2$$

$\sim$ [illegible] VK^{-1}

② Full treatment of Euler buckling in small slope

$\vec{u}(y) = u(y)\hat{y} + \xi(y)\hat{z}$

$\epsilon_{YY} = u' + \frac{1}{2}\xi'^2$ (omitting a $\frac{1}{2}u'^2$ term)

$\sigma_{YY} = Y\epsilon_{YY}$

Energy density, $e = \frac{1}{2} Y \epsilon_{YY}^2 + \frac{1}{2} B (\xi'')^2$

The corresponding Euler-Lagrange equations are

① $B\xi'''' - \sigma_{YY}\xi'' = 0$ - Normal force balance, from variation w.r.t. $\xi(y)$

② $\sigma_{YY}' = 0 \Rightarrow \sigma_{YY} = \text{const.}$ — In-plane force balance from variation w.r.t. $u(y)$

This pair of Föppl - von Karman equations may be solved by a series solution.

Try $\xi(y) \simeq A(\Delta)\sin\left(\frac{\pi y}{W}\right)$ + other terms

$u(y) \simeq -\frac{\Delta}{W}y$ + other terms

Plug $\xi(y)$ into the first FvK equation.

$$B\left(\frac{\pi}{W}\right)^4 A - \sigma_{YY}\left(\frac{\pi}{W}\right)^2 A = 0 \Rightarrow \sigma_{YY} = \frac{B\pi^2}{W^2} \quad —③$$

But $\sigma_{YY} = Y\epsilon_{YY} = Y\left[-\frac{\Delta}{W} + \frac{1}{2}\frac{A^2\pi^2}{W^2}\sin^2\left(\frac{\pi Y}{W}\right)\right]$

$= Y\left[-\frac{\Delta}{W} + \frac{\pi^2}{W^2}\frac{A^2}{4}\right.$ + oscillating term which will have to be fixed by other terms in series $\left.\right]$

−④

Put ③ & ④ together, to get

$$A^2 = \frac{2^2}{\pi^2} W \left(\Delta - \Delta_c \right) \quad \text{where} \quad \frac{\Delta_c}{W} = \frac{\pi^2}{W^2} \frac{B}{Y}$$

The first term in this attempt at a series solution (we wont talk about convergence) already yields the same threshold as in the elastica ie. $\Delta_c/_W$.

Further $A \sim \sqrt{W}\sqrt{\Delta - \Delta_c}$ goes to the elastica solution as $\Delta \gg \Delta_c$

We were lucky to have a solution in this simple problem, but both in this problem, and in other instabilities to come, can do linear stability analysis

for small $\frac{\Delta}{\Delta_c} - 1$: called post-buckling.

Here for large Δ

$\frac{\Delta}{\Delta_c} \gg 1$ get back elastica.

Wrinkles in 1D

Cerda and Mahadevan PRL 2003

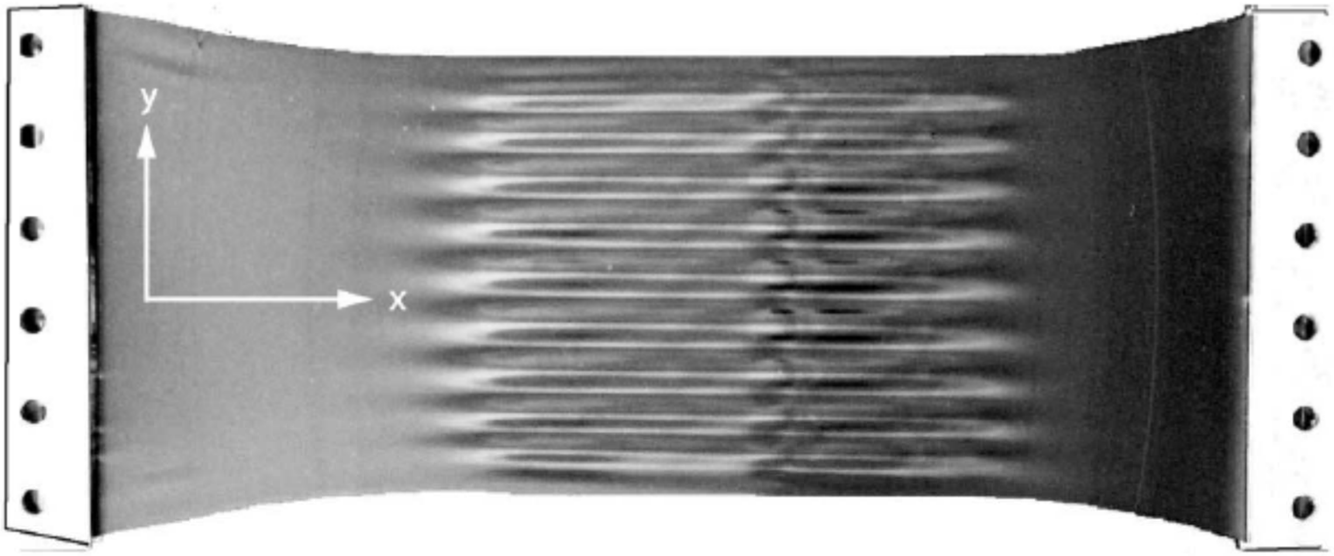
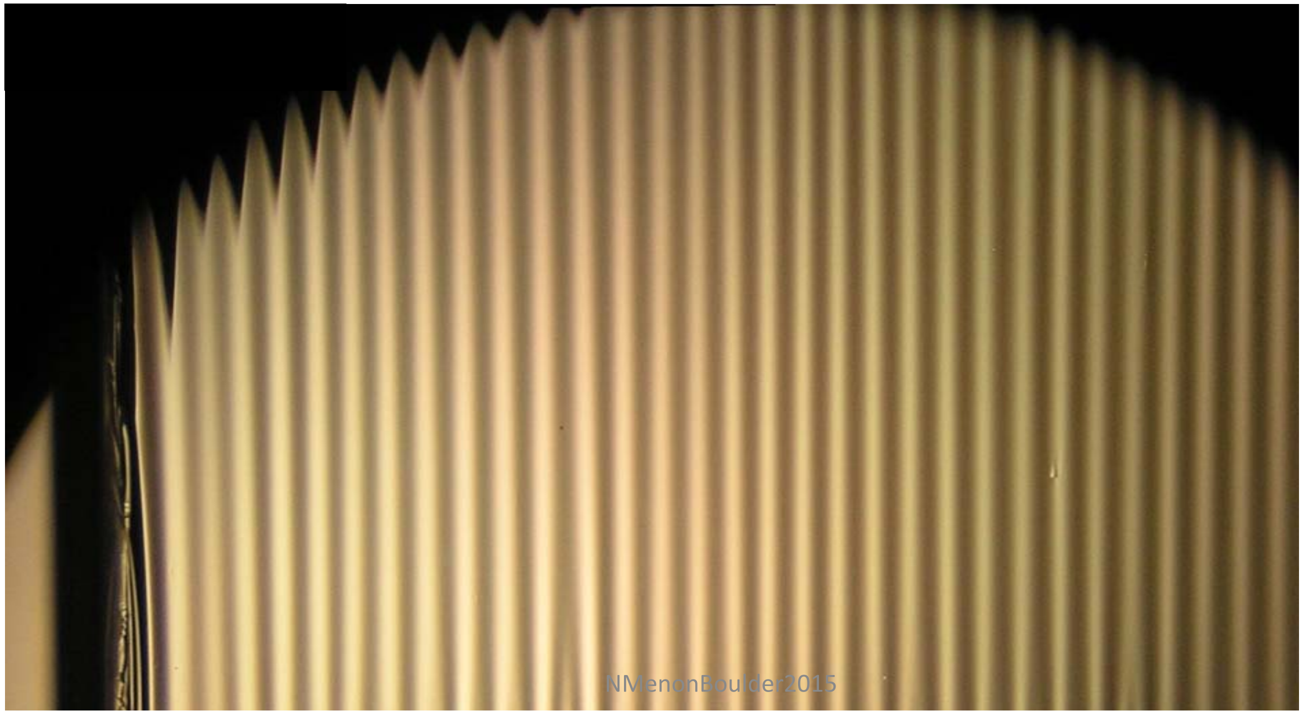


FIG. 1. Wrinkles in a polyethylene sheet of length $L \approx$ 25 cm, width $W \approx 10$ cm, and thickness $t \approx 0.01$ cm under a uniaxial tensile strain $\gamma \approx 0.10$. (Figure courtesy of K. Ravi-Chandar)

Wrinkles in 1D

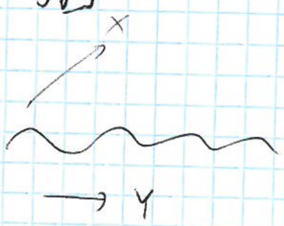


Huang PRL 2010

11) Wrinkling a la Cerda-Mahadevan

Start with the case of a fluid foundation.

Energy density $e = \frac{1}{2} B(\partial_{yy}\xi)^2 + \frac{1}{2} K \xi^2$ $[K = \rho g]$



Inextensible

$$\frac{\Delta l}{l} = -\frac{\Delta}{W} = \frac{\lambda}{l} - 1$$

$$l = \int_0^\lambda \sqrt{1+\xi'^2}\, dy \qquad \xi = A\sin(qy)$$

$$\simeq \int_0^\lambda \left(1+\frac{1}{2}\xi'^2\right) dy = \lambda + \lambda \frac{q^2A^2}{4}$$

$$-\frac{q^2A^2}{4} = \frac{-\Delta}{W} \quad \Rightarrow \quad qA = 2\sqrt{\Delta/W}.$$

[Please find threshold]

Put this in the energy density.

$$e = \frac{1}{2}\left[B\,(q^2A)^2 + K A^2\right] = \frac{1}{2}\left[B q^4 A^2 + KA^2\right]$$

$$= \frac{1}{2}\left[Bq^2\left(\frac{4\Delta}{W}\right) + K\left(\frac{4\Delta}{W}\right)\frac{1}{q^2}\right]$$

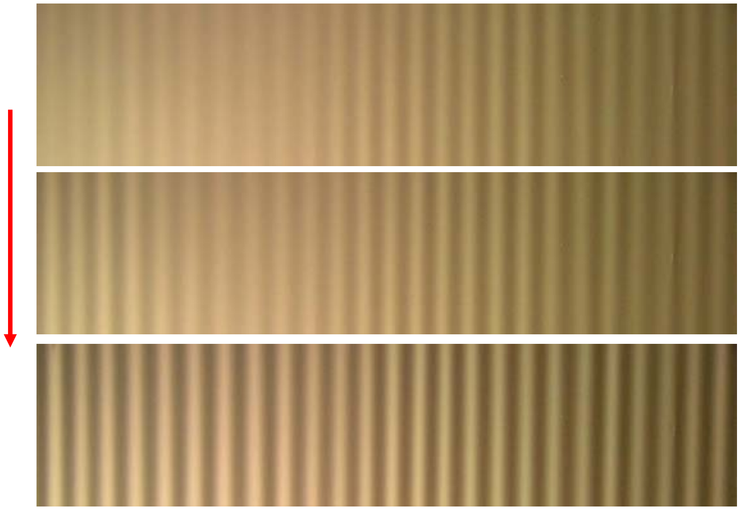
Minimize to get

$$q \sim \left(\frac{K}{B}\right)^{1/4}$$

Tensional case $\frac{T}{2}(\partial_x\xi)^2$

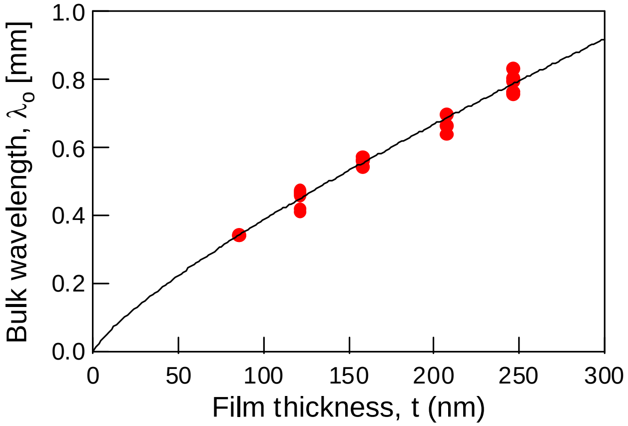
Here $K = T/L^2$; $qA = 2\sqrt{\gamma\Lambda}$

Wrinkles in 1D – fluid substrate



t=246 nm, increasing compression

Wavelength independent of amplitude



Film thickness

$$q_o^{-1} = \lambda_o = \left(\frac{B}{\rho g}\right)^{1/4}$$

Tuning wavelength through B

Thickness

Young’s modulus

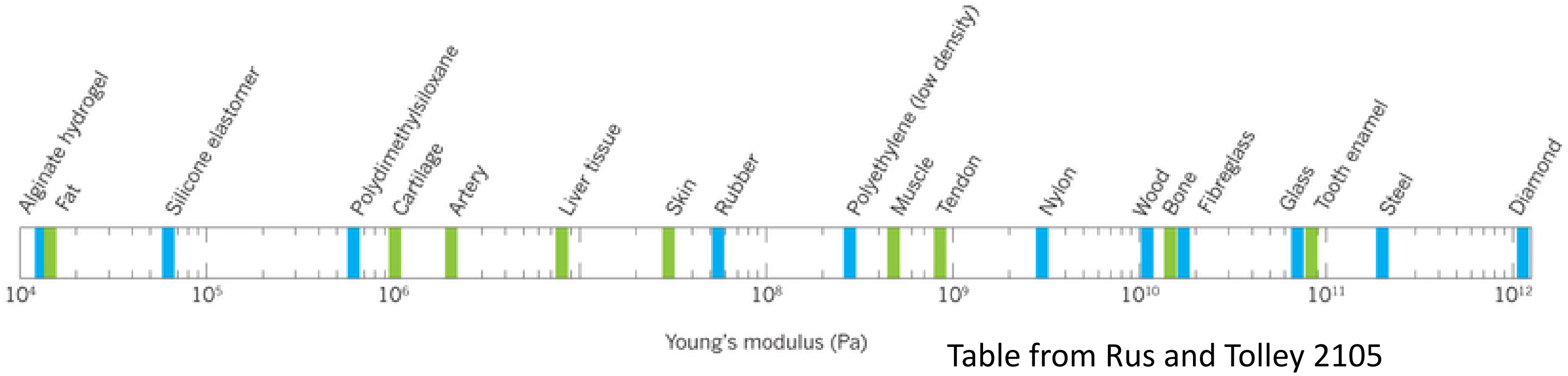


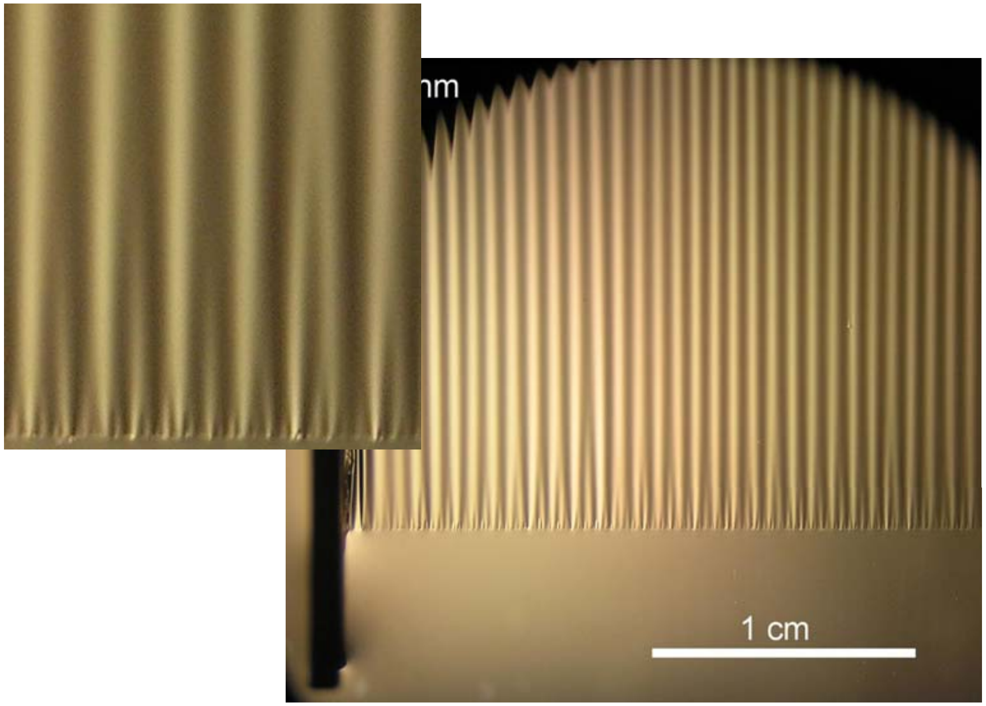
Table from Rus and Tolley 2105

Finger rafting

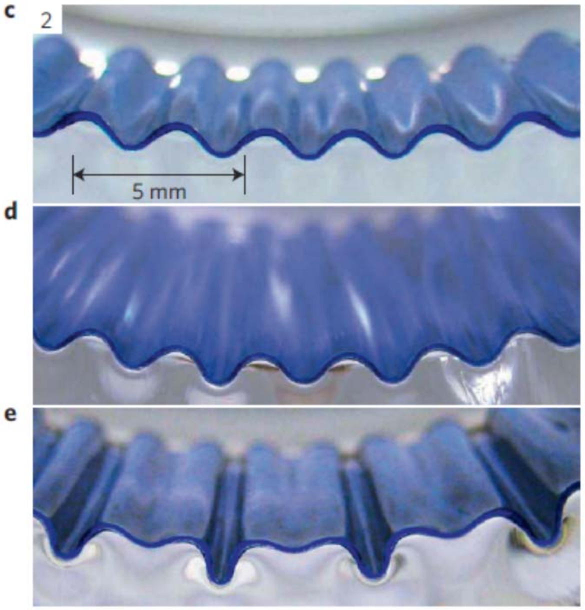
Vella and Wettlaufer, PRL 2004

Finger rafting is the block zippered pattern that forms when thin ice sheets floating on water collide creating "fingers" that push over and under each other alternately. This photo was taken off the Antarctic coast. (Credit: W.F. Weeks)

Wrinkles in 1D – beyond single mode



Cascade between two wavelengths, Huang 2010



Period doubling phenomena Brau et al 2010

Lecture 1:

Main source for wrinkling calculation -
Cerda, E., & Mahadevan, L. (2003). Geometry and physics of wrinkling. *Physical review letters*, *90*(7), 074302.

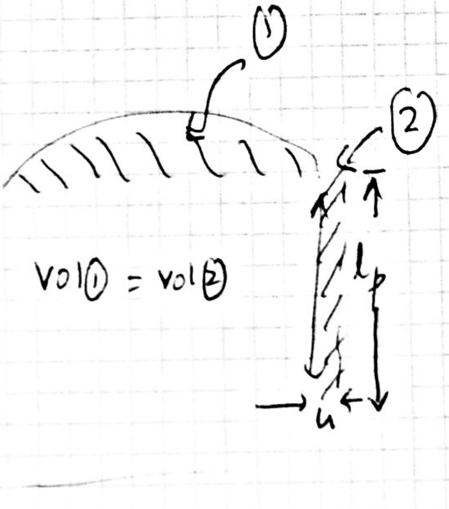
Discussion of Euler buckling regimes follows a pedagogical review in preparation by Benny Davidovitch and myself. Get in touch with me if you want a draft when it is ready (end of summer 2015?)

I have cited data and images where I showed them.

Useful (to me) books on elasticity:
Physics of Continuum Matter by B. Lautrup -- *nice exposition at an introductory level*
Elasticity by Landau and Lifshitz – *no comments needed*
Theory of Elasticity by Timoshenko and Goodier; Plates and Shells by Timoshenko and Woinowsky-Krieger – *both books are detailed expositions by major figure in engineering mechanics, good place to look up solutions for specific geometries*

Soft substrate

Incompressible substrate



$\text{Vol}① = \text{vol}②$

Typical displacement of u, over a penetration depth l_p $\Rightarrow A\lambda \sim u l_p$

Typical strain $\epsilon \sim \frac{u}{l_p} \sim \frac{A\lambda}{l_p^2}$

$$\text{Energy/area} \sim \frac{E_s \epsilon^2 (l_p \lambda)}{\lambda} \sim E_s \left(\frac{A\lambda}{l_p^2}\right)^2 l_p$$

$$\sim \left[\frac{E_s \lambda^2}{l_p^3}\right] A^2$$

For deep substrate $l_p \sim \lambda$ $\quad K \sim E_s/\lambda$

For shallow substrate $l_p \sim H$ $\quad K \sim \frac{E_s \lambda^2}{H^3}$

Where H is the depth of the subphase.