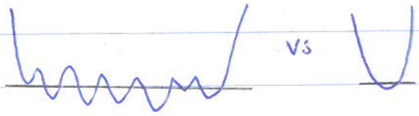
26/07/2017 GERARD BEN AROUS COMPLEXITY OF FUNCTIONS OF MANY VARIABLES

CONTEXT: Consider a fonction $f(x)$ with $x \in \mathbb{R}^N$ or $x \in M_N$ ← manifold → f smooth with values in $\mathbb{R}$

↳ functions are more or less complex



← level set → points for which $f \leq$ level, more or less complex.

A LITTLE TOPOLOGY:

One way to characterize complexity is the number of critical points.

$\mathrm{Crit}_k(f) = \#$ critical points of f of index $k = \begin{cases} \nabla f(x) = 0 \\ \nabla^2 f(x) \text{ has } k \text{ negative e.v.} \end{cases}$

↳ local minimum $\equiv k = 0$

more precise: $\mathrm{Crit}_k(f, u) = \# \ldots$ such that $f(x) \leq u$.

Consider the special case M_N compact; f Morse function (all critical points are non degenerate)

MORSE THEORY: Topological invariant: $b_k(M)$ Be numbers → counts the number of holes of dimension k in the manifold

weak MI: $\mathrm{Crit}_k(\cdot) \geq b_k(M)$

strong MI: $\mathrm{Crit}_0 - \mathrm{Crit}_1 + \mathrm{Crit}_2 - \ldots + (-1)^N \mathrm{Crit}_N = b_0 - b_1 + \ldots + (-1)^N b_N = \chi(M)$

↳ example: sphere S^{N-1} $b_0 = b_{N-1} = 1$, $b_k = 0$ $0 < k < N-1$

- $\mathrm{Crit}_k(f) \geq b_k$
- $\sum (-1)^k \mathrm{Crit}_k(f) = 1 + (-1)^{N-1} = \chi(S^{N-1}) = \begin{cases} 0 \\ 2 \end{cases}$.

A random smooth function can be way more complex that what Morse theory.

↳ Choose $f(x)$ Gaussian $\begin{cases} \mathbb{E} f(x) = m(x) \\ \mathrm{Cov}(f(x), f(y)) = C(x,y) \end{cases}$ → random realization.

↳ f: Polynomial, homogeneous of degree p.

On S^{N-1}: a) $f_1(x) = x_1^3$ not complex

b) $f_2(x) = \sum_{ijk=1}^{N} c_{ijk} x_i x_j x_k$ $\quad c_{ijk} \sim \mathcal{N}(0,1)$ iid → p-spin model on a sphere. $\left(H_3(x) = \sqrt{N} f\left(\frac{x}{\sqrt{N}}\right)\right)$

↳ expect to be complex.

c) $f_3(x) = \lambda f_1 + f_2$ → whether this function is complex depends on λ "signal to noise ratio".

- We want as well to understand the topology of sublevel sets. $A_u = \{x, f(x) \leq u\}$

$\chi(A_u) = \sum (-1)^k \mathrm{Crit}_k(f, u)$

KAC-RICE FORMULA: A way to compute the annealed complexity: $\mathbb{E}\left(\mathrm{Crit}_k^m(f, u)\right)$ ← moments of nb of critic. (Gaussian case).

(Azais - Wschebor)

(Adler - Taylor)

Kac-Rice formula gives a dictionary to transform these complexity questions into RMT. random matrix theory

$$\mathbb{E}(\mathrm{Crit}_k(f,\mu)) = \int_{M_N} dx \int_{-\infty}^{\mu} dv\, \mathbb{E}\Big(\underbrace{|\det \nabla^2 f(x)|}_{\text{hard part}} \mathbb{1}_{i(x)=k} \,\Big|\, \nabla f(x)=0,\ f(x)=v \Big)\, \varphi_x(v,0)$$

$\varphi_x(\cdot,\cdot)$ density of dist of $(f(x), \nabla f(x))$

Much simpler without the absolute value → "CO-AREA FORMULA":

$$\mathbb{E}(\chi(A_\mu)) = \int_{M_N} dx \int_{-\infty}^{\mu} dv\ \mathbb{E}\big(\det \nabla^2 f(x) \mid \nabla f(x)=0,\ f(x)=v \big)\, \varphi_x(v,0)$$

↳ random $N\times N$ matrix.

So what we need to understand is the distribution of $\nabla^2 f(x)$ conditioned by $\begin{cases} f(x)=v \\ \nabla f(x)=0 \end{cases}$

↳ $N\times N$ gaussian random matrix (symmetric).

Rk: There are no reasons for the matrix entries not to be correlated! Which is very hard to handle, kind of useless...

Rk: Now this gives the annealed complexity → to compute quenched ⟨ replica method using annealed moments. / supersymmetry method.

SIMPLEST CASE: Isotropic processes: $C(x,y) = C(d(x,y))$ → only function of the distance

$\begin{cases} C(x) = f^2 = 1 \\ \nabla C(x) = 2f\nabla f = 0 \end{cases}$ ⟹ $f(x) \perp\!\!\!\perp \nabla f$; $\nabla f \perp\!\!\!\perp \nabla^2 f$.

⟹ In this case we can compute the distribution of $\nabla^2 f(x)$ conditionned on $f(x)=v$.

EXAMPLE: spherical p-spin: $f(x) = \sum C_{ijk} x_i x_j x_k$ $\quad m(x)=0$.

$C(x,y) = \langle x,y\rangle$ = function of distance on S^{N-1}.

$\nabla^2 f(x) \overset{(d)}{=} GOE_{N-1} - \underset{\text{constante}}{c}\, u\, I_d$

↳ need to compute: $\mathbb{E}(|\det(GOE_{N-1} - cuI)|\, \mathbb{1}_{i=k})$ → expectation of absolute value of characteristic polynomial → HARD.

$$\mathbb{E}(|\det(GOE_{N-1} - cuI|) = \mathbb{E}\Big(e^{N\left(\frac{1}{N}\sum_{i=1}^{N} \log|cu-\lambda_i|\right)}\Big)$$

$\mu_N = \frac{1}{N-1}\sum_i^{N-1} \delta_{\lambda_i}$

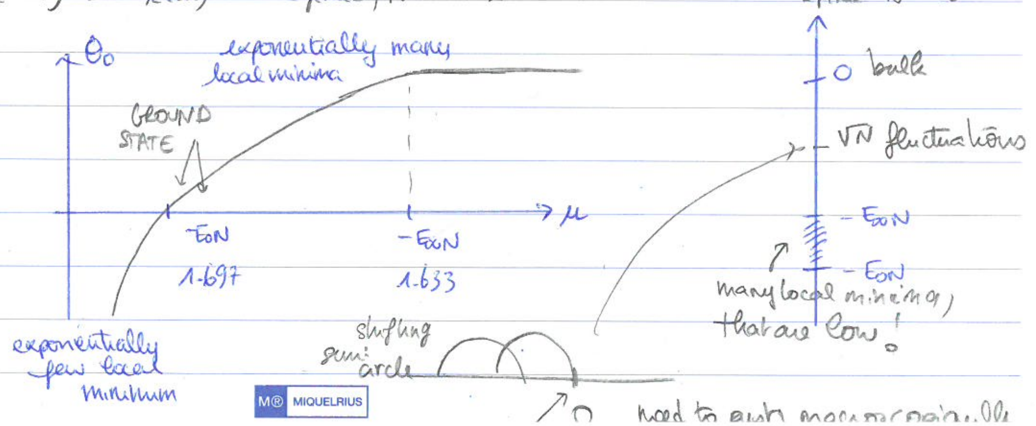
$\int \log|cu-y|\, d\mu_N(y) \overset{?}{\sim} e^{N\int \log|cu-y|\, d\sigma(y)}$

σ: ½ circle

+ BUNCH OF COMMENTS ON CONCENTRATION

$$\lim \frac{1}{N}\log \mathbb{E}(\mathrm{Crit}_k(u)) = \Theta_k(u) \quad . \ k \text{ fixed},\ N\to+\infty.$$

complexity of local minima:



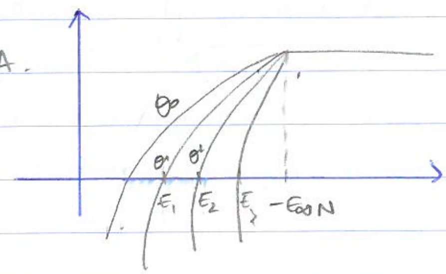
need to go to macroscopically

E. Subag → 2^{nd} moment method to prove concentration and the position of the ground state (conserve bound to the annealed computation) $u < -E_0$ $\frac{\mathrm{Crit}_0(u)}{\mathbb{E}(\ )} \to 1$

Subag-Zeitouni 16 - extreme values: Gumbel distribution: N independent centered gaussian variables
↳ distribution of extreme values -
↳ these points in large dimensional are basically orthogonal → independent
↳ if covariance = function of inner product

Now what about saddle points of higher energies:

Auffinger/Cerny BA.



Rk: We simplified → gaussian, isotropic..

Quadratic forms → critical points = $\vec{e.v.}$ → many of them but not exponentially

Gibbs measure at very low temperature JAGANNATH
↳ absence of temperature chaos:

Now if instead of looking at homogeneous, look at mixture polynomials.
↳ center of GOE is random