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Dynamical Systems:

A Basic Primer

1 Basic bifurcations

1.1 Fixed points and linear stability

A dynamical system is defined simply by:

$$\dot{x} = f(x),\ x, f \text{ vectors in } \mathcal{R}^N \tag{1}$$

where x may be as simple as a scalar ($N = 1$) or as complicated as a velocity field ($N = \infty$) or its numerical discretization (N from 10 to 10^9). A fixed point $\bar{x}$ is a solution to:

$$0 = f(\bar{x}) \tag{2}$$

The *linear stability* of $\bar{x}$ is determined by the behavior of an infinitesimal perturbation $\epsilon(t)$. For $N = 1$:

$$\begin{aligned} \frac{d}{dt}(\bar{x} + \epsilon) &= f(\bar{x} + \epsilon) \\ \cancel{\dot{\bar{x}}} + \dot{\epsilon} &= \cancel{f(\bar{x})} + f'(\bar{x})\epsilon + \frac{1}{2} f''(\bar{x})\epsilon^2 + \cdots \approx f'(\bar{x})\epsilon \\ \epsilon(t) &= e^{tf'(\bar{x})}\epsilon(0) \end{aligned} \tag{3}$$

A perturbation ϵ will grow exponentially in time if $f'(\bar{x}) > 0$, i.e. if $\bar{x}$ is unstable. In contrast, if $f'(\bar{x}) < 0$, then ϵ decreases exponentially in time and $\bar{x}$ is stable,

The function f depends on a parameter μ, for example a Reynolds or Rayleigh number. A *steady bifurcation* is a change in the number of fixed points (roots of f); this is closely connected to stability.

1.2 Saddle-node bifurcations

The simplest function that can change the number of its roots as μ is varied is a quadratic polynomial, like that shown in figure 1. The *normal form* of the *saddle-node bifurcation* is

$$f(x, \mu) = \mu - x^2 \tag{4}$$

The fixed points of (4) are:

$$\bar{x}_\pm = \pm\sqrt{\mu} \tag{5}$$

which exist only for $\mu > 0$. Their stability is determined by

$$f'(\bar{x}_\pm) = -2\bar{x}_\pm = -2(\pm\sqrt{\mu}) = \mp 2\sqrt{\mu} \tag{6}$$

which shows that $\bar{x}_+ = \sqrt{\mu}$ is stable, whereas $\bar{x}_- = -\sqrt{\mu}$ is unstable. The *bifurcation diagram* of figure 2 shows the steady states $\bar{x}_\pm$ as a function of μ, along with their stability.

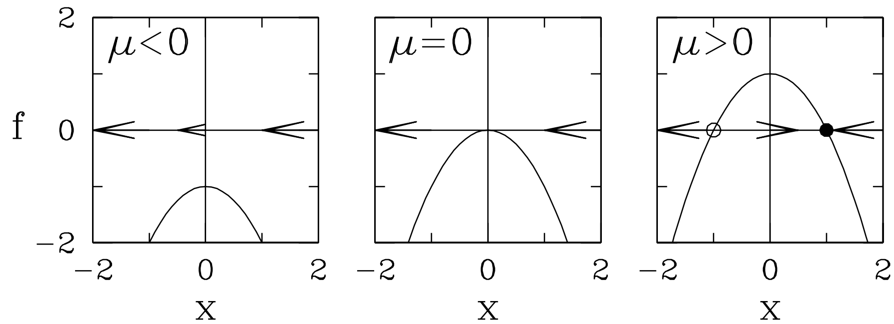
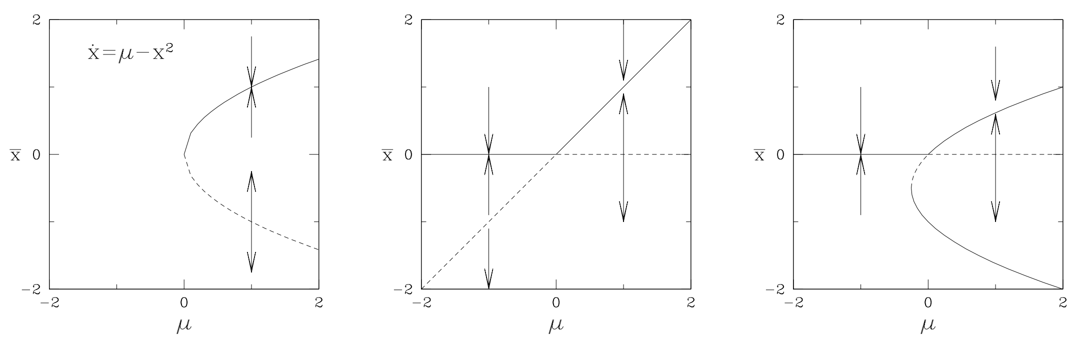


Figure 1: The function $f = \mu - x^2$ has 0, 1, or 2 roots, if $\mu < 0$, $\mu = 0$, or $\mu > 0$.



a) saddle-node b) transcritical c) stabilized transcritical

Figure 2: Bifurcation diagrams for (a) a saddle node bifurcation, (b) a transcritical bifurcation, and (c), a transcritical bifurcation with an additional stabilizing saddle-node bifurcation. Stable branches are shown as solid curves, unstable branches as dashed curves.

1.3 Transcritical bifurcations

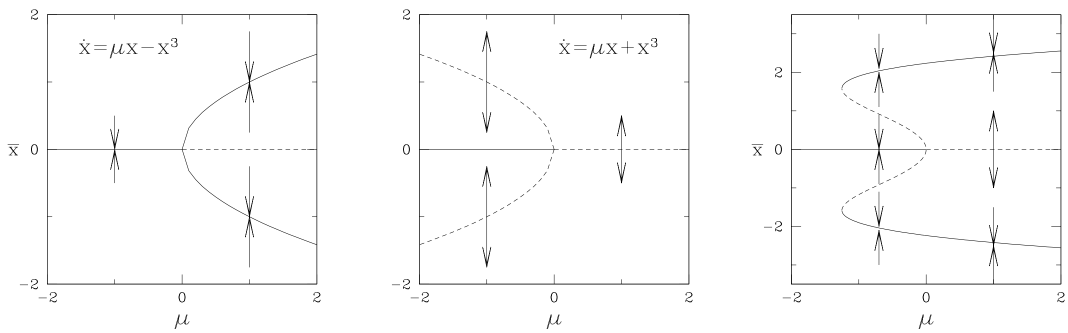
If the constraints on the problem are such as to forbid a constant term in the Taylor expansion of f, then the truncated Taylor expansion leads to the normal form of a *transcritical bifurcation*:

$$\dot{x} = \mu x - x^2 \tag{7}$$

The search for fixed points and their linear stability leads to:

$$0 = \bar{x}(\mu - \bar{x}) \Longrightarrow \begin{cases} \bar{x} = 0 \\ \bar{x} = \mu \end{cases} \qquad f'(\bar{x}) = \mu - 2\bar{x} = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ -\mu & \text{for } \bar{x} = \mu \end{cases} \tag{8}$$

Thus $\bar{x} = 0$ is stable for $\mu < 0$, unstable pour $\mu > 0$, whereas $\bar{x} = \mu$ does the opposite: these fixed points merely exchange their stability. Since trajectories of (7) go to infinity, a higher-order term is sometimes



a) supercritical b) subcritical c) stabilized subcritical

Figure 3: Pitchfork bifurcation diagrams A pitchfork bifurcation is (a) supercritical if the new branches are stable and (b) subcritical if they are unstable. (c) Additional saddle-node bifurcations prevent trajectories from diverging to infinity.

added to the normal form as shown in figure 2 in order to stabilize the trajectories, e.g.

$$\dot{x} = \mu x - x^2 - \alpha x^3 \tag{9}$$

1.4 Pitchfork bifurcations

If symmetry requires $f(x)$ to be odd in x, the resulting cubic polynomial takes one of two forms:

$$f(x,\mu) = \mu x - x^3 \tag{10}$$
$$f(x,\mu) = \mu x + x^3 \tag{11}$$

The corresponding bifurcation diagrams are given in figure 3. Equation (10) is the normal form of a *supercritical pitchfork bifurcation*. The fixed points and stability are:

$$0 = \bar{x}(\mu - \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \text{for all } \mu & f'(0) = \mu \\ \bar{x} = \pm\sqrt{\mu} & \text{for } \mu > 0 & f'(\pm\sqrt{\mu}) = -2\mu < 0 \text{ for } \mu > 0 \end{cases} \tag{12}$$

The fixed point $\bar{x} = 0$ is stable for $\mu < 0$ and becomes unstable at $\mu = 0$, where new branches of stable fixed points $\bar{x} = \pm\sqrt{\mu}$ are created. For the subcritical case, we have

$$0 = \bar{x}(\mu + \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \text{for all } \mu & f'(0) = \mu \\ \bar{x} = \pm\sqrt{-\mu} & \text{for } \mu < 0 & f'(\pm\sqrt{-\mu}) = -2\mu > 0 \text{ for } \mu < 0 \end{cases} \tag{13}$$

Contrary to the supercritical case, the fixed points $\pm\sqrt{\mu}$ exist in the région $\mu < 0$ where $\bar{x} = 0$ is stable and they are unstable. A fifth-order term is sometimes added to (11) to prevent trajectories from diverging to infinity.

2 Systems with two or more dimensions

2.1 Linear stability analysis in two or more dimensions

We now consider $\dot{x} = f(x)$, with $x, f \in \mathcal{R}^N$, $N > 1$. To study the linear stability of a fixed point $\bar{x}$ satisfying $0 = f(\bar{x})$, we perturb it by an infinitesimal $\epsilon(t) \in \mathcal{R}^N$.

$$\begin{aligned} \frac{d}{dt}(\bar{x} + \epsilon) &= f(\bar{x} + \epsilon) \\ \cancel{\dot{\bar{x}}} + \dot{\epsilon} &= \cancel{f(\bar{x})} + Df(\bar{x})\,\epsilon + \epsilon\, D^2 f(\bar{x})\,\epsilon + \dots \qquad (14) \end{aligned}$$

$$\dot{\epsilon} = Df(\bar{x})\,\epsilon \qquad (15)$$

In (15), $Df(\bar{x})$ is the *Jacobian* of f, i.e. the matrix of partial derivatives, evaluated at the fixed point $\bar{x}$. To clarify its meaning we rewrite these equations explicitly for each component:

$$\dot{\epsilon}_i = \frac{\partial f_i}{\partial x_j}(\bar{x})\,\epsilon_j \qquad (16)$$

whose solution is

$$\epsilon(t) = e^{Df(\bar{x})t}\epsilon(0) \qquad (17)$$

The behavior of (17) depends on its spectrum, i.e. its set of eigenvalues. It can be shown that (in most cases) the eigenvalues of the exponential of a matrix A are the exponential of the eigenvalues. Let $A = V\Lambda V^{-1}$. Then

$$\begin{aligned} e^{At} &= I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \dots \\ &= VV^{-1} + tV\Lambda V^{-1} + \frac{t^2}{2}V\Lambda V^{-1}\,V\Lambda V^{-1} + \frac{t^3}{6}V\Lambda V^{-1}\,V\Lambda V^{-1}\,V\Lambda V^{-1} + \dots \\ &= V\left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \frac{t^3}{6}\Lambda^3 + \dots\right]V^{-1} \\ &= Ve^{\Lambda t}V^{-1} \qquad (18) \end{aligned}$$

The same reasoning can be applied to any analytic function f of a matrix, using the Taylor series of $f(A)$. Thus, we only need to know how to take exponentials of the matrix of eigenvalues.

For a 2×2 matrix with real eigenvalues, we have:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Longrightarrow e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \qquad (19)$$

For a 2×2 matrix with complex eigenvalues $\mu \pm i\omega$, we have:

$$\Lambda = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \Longrightarrow e^{t\Lambda} = \begin{pmatrix} e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) \\ e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) \end{pmatrix} \qquad (20)$$

For a mixture of real and complex eigenvalues, as in the example depicted in figure 4, we have:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \mu & -\omega & 0 \\ 0 & \omega & \mu & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \Longrightarrow \exp(t\Lambda) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) & 0 \\ 0 & e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} \quad (21)$$

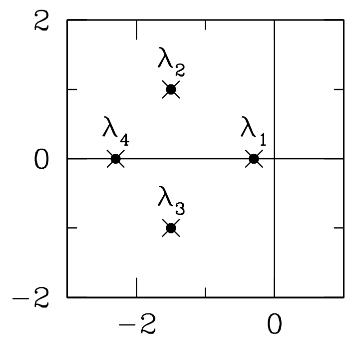


Figure 4: Eigenvalues of a Jacobian having two real eigenvalues and a pair of complex conjugate eigenvalues, with $0 > Re(\lambda_1) > Re(\lambda_2) = Re(\lambda_3) > Re(\lambda_4)$.

The question "stable or unstable?" becomes "stable or unstable *in which directions*?" The fixed point $\bar{x}$ is considered to be linearly stable if the real parts of *all* of the eigenvalues of $Df(\bar{x})$ are negative, and unstable if even *one* of the eigenvalues has a postive real part. The reasoning behind this is that initial random perturbations will contain components in all directions. If there is instability in one direction, then this component will grow and we will diverge away from $\bar{x}$, initially in the unstable direction.

2.2 Circle pitchfork bifurcation

We consider the evolution equation

$$\frac{dz}{dt} = (\mu - \alpha|z|^2)z \quad (22)$$

The steady states of (22) are $z = 0$ and $|z| = \sqrt{\mu/\alpha}$, which exists only for $\mu/\alpha > 0$, as illustrated in figure 5. We consider μ to be the bifurcation parameter, such as a relative Reynolds number $(Re - Re_c)/Re_c$ or Rayleigh number $(Ra - Ra_c)/Ra_c$. The transition occurring at $\mu = 0$ is called a circle pitchfork, because a "circle" of steady states, $z = \sqrt{\mu/\alpha}\, e^{i\theta}$, is created as μ crosses zero. We may write (22) in Cartesian coordinates $z = x + iy$:

$$\frac{dx}{dt} = (\mu - \alpha(x^2 + y^2))x \quad (23a)$$

$$\frac{dy}{dt} = (\mu - \alpha(x^2 + y^2))y \quad (23b)$$

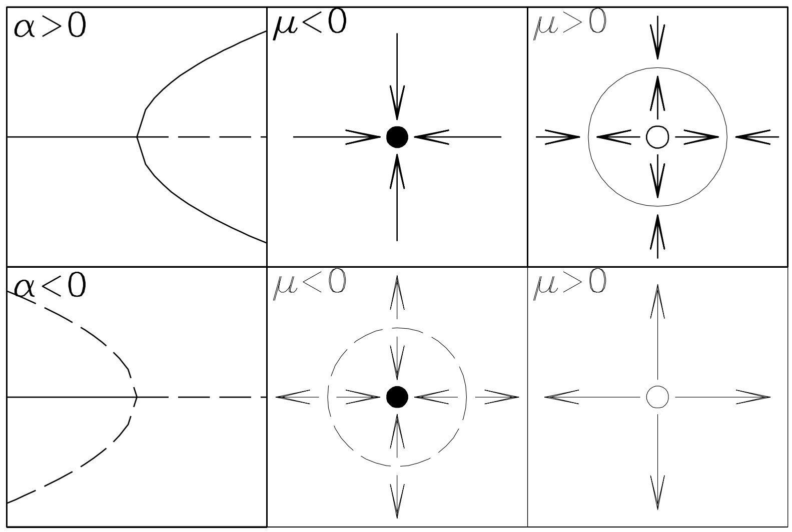


Figure 5: Circle pitchfork bifurcation. The upper diagrams correspond to the supercritical case, the lower diagrams to the subcritical case. Leftmost are the bifurcation diagrams in the (μ, r) plane, middle and rightmost are phase portraits in the (x, y) plane. Stable steady states are designated by solid dots or the solid curve. Unstable steady states are shown as hollow dots or the dashed curve.

or in polar coordinates $z = re^{i\theta}$:

$$\frac{d\left(re^{i\theta}\right)}{dt} = \left(\frac{dr}{dt} + ri\frac{d\theta}{dt}\right) e^{i\theta} = \left(\mu - \alpha r^2\right) r\ e^{i\theta} \tag{24}$$

$$\frac{dr}{dt} = \left(\mu - \alpha r^2\right) r \tag{25a}$$

$$\frac{d\theta}{dt} = 0 \tag{25b}$$

Equation (25a) shows that the amplitude r undergoes an ordinary pitchfork bifurcation and (25b) shows that the phase θ shows no tendency to move. The stability of the trivial state and the bifurcating circle of states can be calculated from either (23) or (25) via the Jacobian matrix:

$$J(x, y) = \begin{pmatrix} \mu - \alpha(3x^2 + y^2) & -2\alpha xy \\ -2\alpha xy & \mu - \alpha(x^2 + 3y^2) \end{pmatrix} \tag{26}$$

$$J(r, \theta) = \begin{pmatrix} \mu - \alpha 3r^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{27}$$

To calculate the stability of the trivial state, we use

$$J(x=0,y=0)=\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \tag{28}$$

which has μ as a double eigenvalue. The trivial state (0,0) is stable for $\mu < 0$ and has two unstable directions (corresponding to the two directions of the plane) for $\mu > 0$. Note that the polar form $J(r,\theta)$ of equation (27) would seem to indicate that the two eigenvalues of (0,0) are μ and 0. This contradictory result arises from the fact that θ is not well defined at $r = 0$, so that $J(r,\theta)$ is also not well defined at $r = 0$.

For the bifurcating circle of states, we may write

$$J(r=\sqrt{\mu/\alpha},\theta)=\begin{pmatrix} \mu-3\mu & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} -2\mu & 0 \\ 0 & 0 \end{pmatrix} \tag{29}$$

whose eigenvalues are -2μ and 0. The circle pitchfork can be supercritical or subcritical, according to the sign of α. If $\alpha > 0$, then the circle of bifurcating states exists for $\mu > 0$ and the eigenvalue -2μ corresponds to contraction onto the circle $\sqrt{\mu/\alpha}$, i.e. stability. If $\alpha < 0$, then the circle of bifurcating states exists for $\mu < 0$ and the eigenvalue -2μ corresponds to expansion away from the circle, i.e. instability. The eigenvalue 0 corresponds to the phase invariance, i.e. the fact that the system shows no tendency to move in the direction θ. The eigenvalues can be confirmed by evaluating $J(x,y)$ of equation (26) for $x^2+y^2=r^2=\mu\alpha$:

$$J(x,y)=\begin{pmatrix} -2\alpha x^2 & -2\alpha xy \\ -2\alpha xy & -2\alpha y^2 \end{pmatrix} \tag{30}$$

whose eigenvalues are:

$$-\alpha(x^2+y^2)\pm\sqrt{(\alpha(x^2-y^2))^2+4\alpha x^2y^2}=\alpha(x^2+y^2)\pm\sqrt{\alpha(x^2+y^2))^2}=-\mu\pm\mu \tag{31}$$

The eigenvector corresponding to eigenvalue $\mu = 0$ resulting from phase invariance, called the *marginal* direction, points in the θ direction, i.e. it is $(0,1)$ in the polar representation and $(-y,x)$ in the Cartesian representation.

2.3 Hopf Bifurcation

If λ_1, λ_2 are a complex conjugate pair whose real part μ changes sign, then a *Hopf bifurcation* takes place. Its normal form, i.e. the simplest nonlinear equation displaying this behavior, can be written:

$$\dot{z}=(\mu+i\omega)z-|z|^2z \tag{32}$$

Writing $z = x+iy$, (32) becomes

$$\dot{x}+i\dot{y} = (\mu+i\omega)(x+iy)-(x^2+y^2)(x+iy) \tag{33}$$

$$\dot{x} = \mu x-\omega y-(x^2+y^2)x \tag{34}$$

$$\dot{y} = \omega x+\mu y-(x^2+y^2)y \tag{35}$$

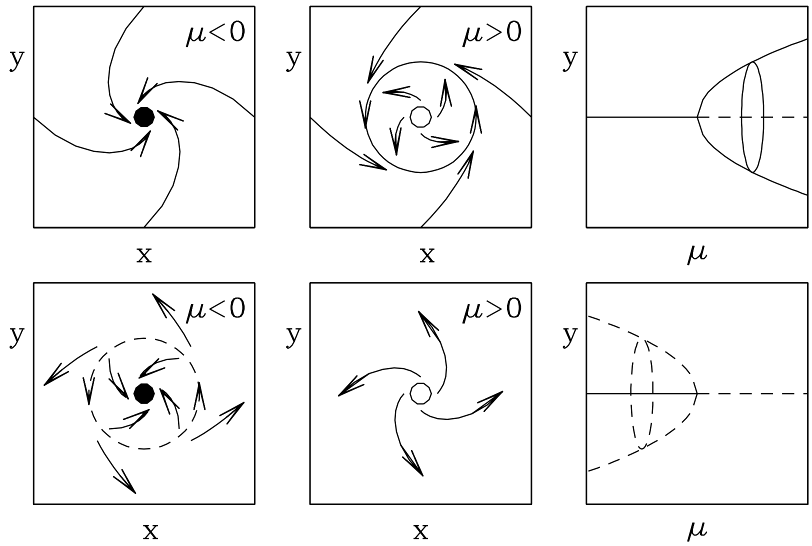


Figure 6: Hopf bifurcations. Top: supercritical case, $\dot{z} = \mu z - |z|^2 z$. Bottom: subcritical case, $\dot{z} = \mu z + |z|^2 z$. Left and middle: behavior of trajectories for $\mu < 0$ and for $\mu > 0$. The solid points and circles correspond to stable fixed points and limit cycles. The hollow points and dashed circles correspond to unstable fixed points and limit cycles. Right: bifurcation diagrams showing the solutions as a function of μ. The solid curves correspond to stable solution branches, the dashed curves to unstable solution branches. The ellipses represent one member of the branch of limit cycles.

We can also use a polar representation $z = re^{i\theta}$. The normal form (32) then becomes:

$$(\dot{r} + ir\dot{\theta})e^{i\theta} = (\mu + i\omega)re^{i\theta} - r^2 re^{i\theta} \quad (36)$$

$$\dot{r} = \mu r - r^3 \quad (37)$$

$$\dot{\theta} = \omega \quad (38)$$

Equation (37) describes a pitchfork in the radial direction, and (38) describes rotation. The fixed points of (37) are $r = 0$ and $r = \sqrt{\mu}$ (where we retain only $r > 0$). For the normal form, we can calculate the *limit cycle*, that is, the periodic solution of (32) approached by all trajectories, regardless of initial condition.

$$z(t) = \sqrt{\mu}e^{i\omega(t-t_0)} \quad (39)$$

As for the pitchfork, there also exists a *subcritical* version of the Hopf bifurcation, with normal form:

$$\dot{z} = (\mu + i\omega)z + |z|^2 z \quad (40)$$

The behavior of systems (32) and (40) in the neighborhood of a Hopf bifurcation is shown in figure 6.