Lecture 2

- How do granular materials respond to stress? Why are stresses localized?

1. How do we think about this question for elastic materials.?

$$\nabla \cdot \sigma = \vec{f}_{ext}$$

2D

$$\partial_y \sigma_{yy} + \partial_x \sigma_{xy} = f_y$$
$$\partial_y \sigma_{yx} + \partial_x \sigma_{xx} = f_x$$

AND $\sigma_{xy} = \sigma_{yx}$ (torque balance).

2 eqns for 3 unknowns.

$$\sigma_{ij} = C_{ijkl} \underbrace{\epsilon_{kl}}_{\text{strain tensor}}$$

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right]$$

where $\vec{u}$ is the displacement field measured from a reference state (zero stress)

Compatibility Condition

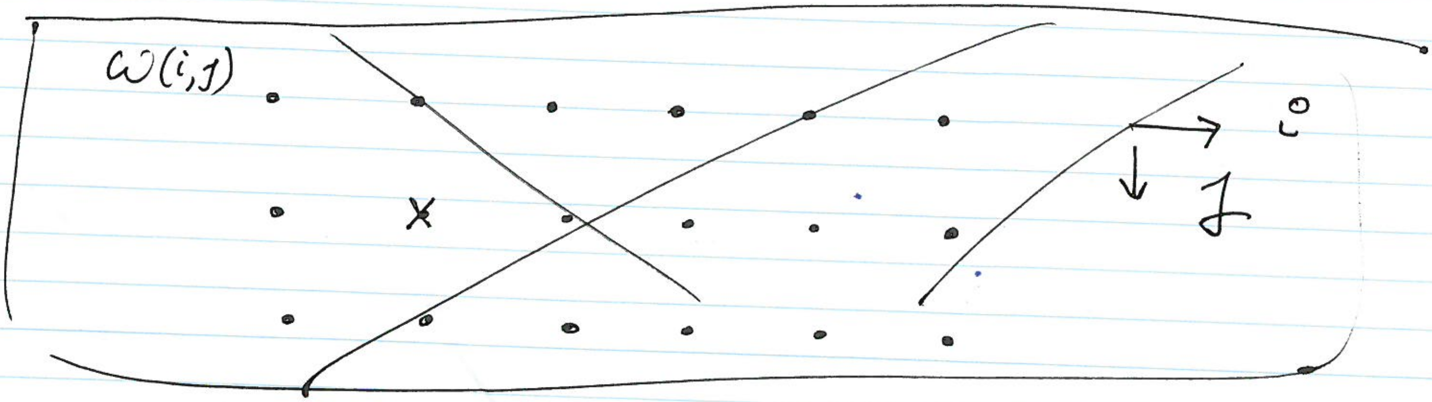
$$\partial_y^2 u_{xx} + \partial_x^2 u_{yy} - 2\partial_x \partial_y u_{xy} = 0$$

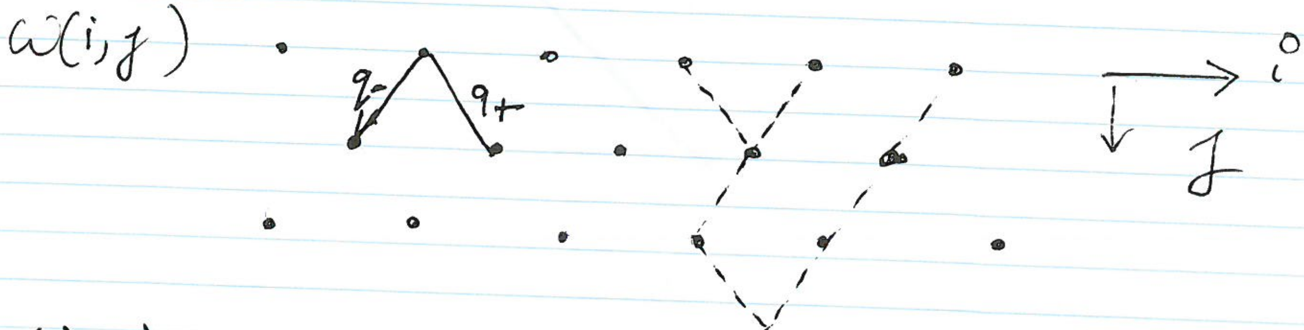
$$\left[\partial_y^4 + t\,\partial_x^4 + 2r\,\partial_x^2 \partial_y^2\right] \sigma_{ij} = 0$$

The lack of a reference state $\Rightarrow$ these closure relations are not available to us.

Stress-only approach

q - model





$$\omega(i,j) = \omega_0 + q_+(i-1,j-1)\,\omega(i-1,j-1) + q_-(i+1,j-1)\,\omega(i+1,j-1)$$

Ordered $\quad q_+ = q_- = \frac{1}{2}$

$$\omega(i,j) = \omega_0 + \frac{1}{2}\left[\omega(i-1,j-1) + \omega(i+1,j-1)\right]$$

Master eqn for random walks in space "i" and time "j"

$$\frac{dP(n,t)}{dt} = -\sum r_{n,m} P(n,t) + \sum r_{m,n} P(m,t)$$

q_+ or $q_- = 1$ the other is zero.

coalescing random walks

- Imposes scalar force balance
- Disorder is coded in the q's [contact loading and ∂th.]
- Uniform distribution of q s
  (sort of an Edwards assumption)
- Advantage $\Rightarrow$ Solve for $P_J(\omega)$

for a uniform distbn of q's

$J \to \infty$ $\quad P(\omega) = \frac{\omega}{\langle \omega^2 \rangle} e^{-\frac{\omega}{\langle \omega \rangle}}$

Exponential tail [from unifm distbn]

- Exponential tail is robust as long as we dont take the $q = 0, 1$ case.

[Bouchaud lectures: Liu et al Science, PRE ~ 2002]

The scalar model → continuum limit.

$$\partial_t w = \xi + D_0 \partial^2_{xx} w \qquad \text{if } q_\pm = \frac{1}{2}$$

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- Like diffusion of a δ-fn

→ Broadens 

no "chains"

- Missing vectorial force-balance.

(Continuing in this vein of thinking)

some simple models

$F_V^{(i,j)}, F_H^{(i,j)}$ two components transmitted

- assume force gets transmitted only along dirn of leg (frictionless)

2. a fraction p of F_V gets transmitted along middle leg

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F_H was absent in q model.

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- biases propagation of F_V along F_H

$$\partial_y F_V + \partial_x F_H = \rho + a\frac{(1-p)}{2} \partial^2_{yy} F_V$$

$$\partial_y F_H + \partial_x \left[c_0^2 F_V\right] = \frac{a}{2} \partial^2_{xx} F_H$$

$c_0^2 = (1-p)\tan^2\psi$

- eliminate $F_H \rightarrow \left[\partial_y^2 - c_0^2 \partial_x^2\right] F_V = 0$

Wave equation!

- characteristics

$$x - c_0 y \quad , \quad x + c_0 y$$

along which the wave propagates
$\Rightarrow$ force chains as soon as you ~~inco~~ include vector force bal.

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- Connect to

$$\partial_y \sigma_{yy} + \partial_x \sigma_{xy} = \rho$$
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- if. ignore ∂_{xx}^2, ∂_{yy}^2 in F_H, F_V eqn.

- $\sigma_{xy} = F_H$, $\boxed{\sigma_{xx} = c_0^2 \sigma_{yy}}$

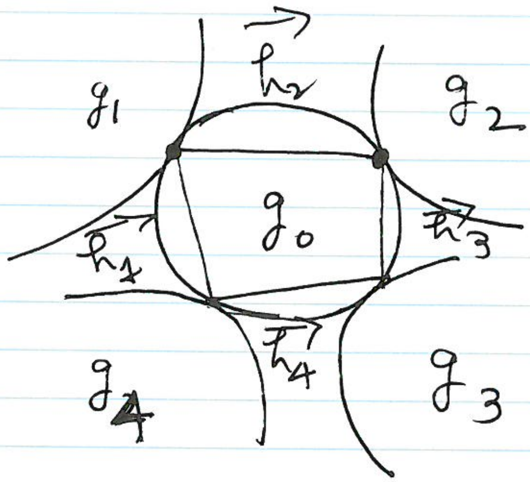
- Any closure of the form

$$\sigma_{xx} = c_0^2 \sigma_{yy}\, g\left(\frac{\sigma_{xy}^2}{\sigma_{yy}^2}\right)$$

leads to wave-like propagation.

- Breaks rotational symmetry.

We have the tools to do better!



$$\vec{f}_{g_0,g_1}, \quad \vec{f}_{g_0,g_2}, \quad \vec{f}_{g_0,g_3}, \quad \vec{f}_{g_0,g_4}$$

$$\vec{f}_{g_0,g_1} = \vec{h}_1 - \vec{h}_2 + \vec{\Phi}_{g_1} - \vec{\Phi}_{g_0}$$

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$$= -\vec{f}^{\,body}_{g_0} = (\quad) + ?$$

Fails if we have body forces such as gravity

$$\sum \vec{f}_{g_0,g_i} = -\vec{f}^{\,body}_{g_0} \neq 0$$

- Each contact has two loops and two grains involved. Can we introduce auxiliary fields on the grains?

$$-\vec{f}^{\,body}_{g_0} = \vec{\Phi}_{g_1} + \vec{\Phi}_{g_2} + \vec{\Phi}_{g_3} + \vec{\Phi}_{g_4} - 4\vec{\Phi}_{g_0}$$

This is the discrete Laplacian

$$-\vec{f}^{\,body}_{g_0} = \square^2 \vec{\Phi}_{g_0}$$

Reminder q model $\partial_t \omega = \partial^2_{\cancel{xx}} \omega$

Define a ket $|\Phi\rangle = \{\vec{\Phi}_1 \cdots \vec{\Phi}_N\}$ for N-grain

and $|\vec{f}^{\,body}\rangle = \{\vec{f}^{\,body}_1 \cdots f^{\,body}_N\}$

$$\boxed{\square^2 |\Phi\rangle = -|f^{\,body}\rangle}$$

$$\square^2 = \begin{bmatrix} -z_1 & 0 & 1 & \cdots & 1 & \cdots \\ 0 & -z_2 & \cdot & \cdots & \cdot & \cdot \\ & & & & & \\ & & & & & \end{bmatrix}$$

z_1 entries are $= 1$
z_2 " " $= 1$

Graph Laplacian [Graph with loops]
Real-space network captured by $\square^2$

- Response to
  depends on
- How localiz
  then is de
  nature of
  $\square^2$
- $\square^2 = \sum_{i=1}^{N} \lambda$
- One zero e
  with $|\lambda_1\rangle =$
- All other λ
- Eliminate
  $|\tilde{f}\rangle = |f$
- Can invert
  aft
- $|\phi\rangle = -$
  $(\square^2)^{-1}\,\square^2 =$

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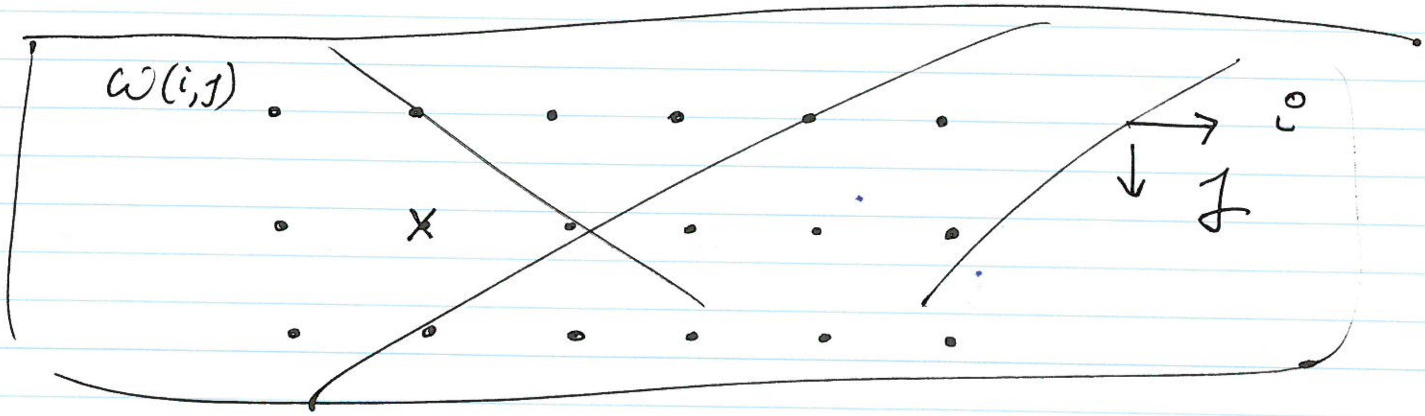

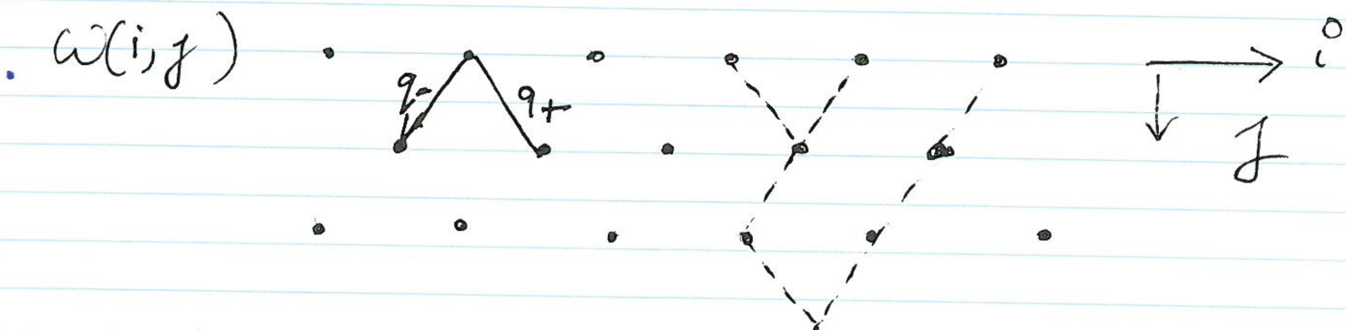

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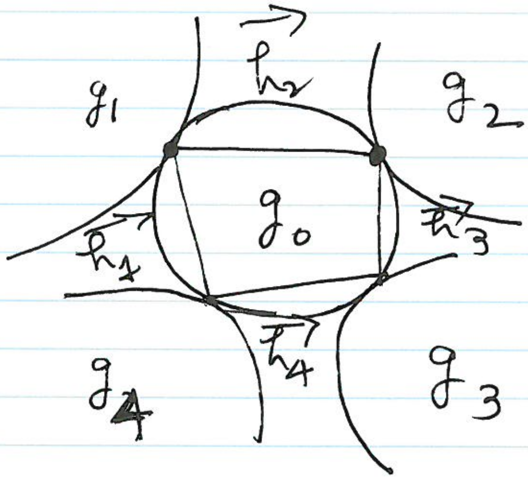

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Graph Laplacian [Graph with loops]

Real-space network captured by $\square^2$

- Response to a given $|f^{body}\rangle$ depends on the spectrum of $\Box^2$
- How localized the response is then is determined by the nature of the eigenfunc of $\Box^2$
- $$\Box^2 = \sum_{i=1}^{N} \lambda_i \, |\lambda_i\rangle\langle\lambda_i|$$
- One zero eigenvalue. with $|\lambda_1\rangle = \{1, \cdots 1\}$ because of row sum $= 0$
- All other λ are $-$ve.
- Eliminate $|\lambda_1\rangle$ by
  $$|\tilde{f}\rangle = |f^{bo}\rangle - \frac{1}{N}\sum_{i=1}^{N} |f^{body}\rangle$$
- Can invert ~~by~~ after projecting out zero m.
- $$|\phi\rangle = -[\Box^2]^{-1}\,|\tilde{f}^{body}\rangle$$
  $$(\Box^2)^{-1}\,\Box^2 = 1 - |\lambda_1\rangle\langle\lambda_1|$$

$$|\phi\rangle = \sum_{i\neq 1} \frac{1}{\lambda_i} \langle \lambda_i | \tilde{f}^{\,body} | \lambda_i \rangle$$

Can construct "new" $\vec{f}_{g_0, g_1}$

from $|\vec{\phi}\rangle$

<u>NOTE</u> : Whenever I mentioned the "Coulomb Condition" it referred to the constraint $f_t \leq \mu f_N$: nothing to do with E&M ! Sorry for not explaining this granular terminology. Mohr-Coulomb failure in granular solids is the original model of failure where the condition above is violated.