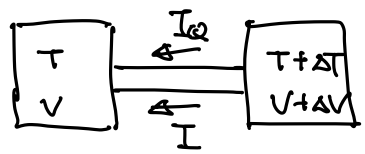
General Transport : Electric, Thermal & Thermoelectric

Consider two thermal baths connected by a transport channel,



In a linear responce regime, following four quantities are defined

$$I = G\,\Delta V \Big|_{I_Q=0} \quad , \qquad S = \frac{\Delta V}{\Delta T}\Big|_{I=0}$$

charge current — Thermo power

$$I_Q = -\kappa\,\Delta T \Big|_{I=0} \quad , \qquad I_Q = \pi I \quad (\Delta T = 0)$$

heat current — Peltier effect — Peltier coefficient.

Considering the linearity of the system all these four eqns can be summarized by

$$\begin{pmatrix} I \\ I_Q \end{pmatrix} = \begin{pmatrix} G & GS \\ +G\pi & -\kappa \end{pmatrix} \begin{pmatrix} \Delta V \\ \Delta T \end{pmatrix} \quad \text{—— (1)}$$

We will find how to describe these transport coefficients in general.

Boltzmann Transport Equation : Semiclassical approach

Let us first start with the semiclassical Boltzmann equation.

$$[\partial_t + \vec{v}\cdot\vec{\nabla}_r - e(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{\nabla}_k]\, f(\vec{r},\vec{k},t) = I_{coll}[f]$$

$f(\vec{r},\vec{k},t)$ → distribution function

$I_{coll}[f]$ → Collision Kernal.

eg.) $I_{coll} = -\frac{f-f_0}{\tau_0}$ [relaxation time approx]

Boltzman transport eqns can be rewritten in different forms, depending on the relative size of following characteristic length

- L : size of system (boundary)
- l_{el} : elastic scattering length (disorders)
- $l_{el\text{-}ph}$: electrons and other bosonic degree of freedoms
- l_{ee} : e-e interaction electron-thermalization.

Regime	Conserved quantities	Eqns
(Quasi) Ballistic $L \sim l_{el} \ll l_{ee}$	current for each momentum	$(\partial_t + \vec{v}\cdot\vec{\nabla}) f(\vec{r},\vec{k},E) = I_{coll}[f]$; $E = \epsilon(\vec{k}) + \mu(\vec{r})$; residual collision
Diffusive $l_{el} \ll L, l_{ee}$	Current for each energy	$D = \frac{v^2\tau}{d}$; $(\partial_t - D\vec{\nabla}^2) f(\vec{r},E,t) = I_{inel}[f]$
(Quasi) Equilibrium $l_{ee} \ll L, l_{el}$	Charge & Energy current "hydrodynamic regime"	Local equilibrium of electron gas is established. $f(r,E,t) = \frac{1}{e^{(E-\mu(\vec{r}))/k_B T(\vec{r})} + 1}$; $\left[\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{convection}}\right] n = -\underbrace{\vec{\nabla} p}_{\text{electron gas pressure}} - \underbrace{n\vec{\nabla}\phi}_{\text{electrostatic drive}} + \overbrace{\eta}^{\text{viscosity}} \vec{\nabla}^2 \vec{v} - \overbrace{\gamma}^{\frac{ne^2}{m\sigma_0}} \vec{v}$

Diffusive Transport

With the channel length L

$$\Delta V \rightarrow L\,\nabla\mu/e$$

$$\Delta T \rightarrow L\,\nabla T$$

$$I = A(-e)\frac{1}{4\pi^3}\int d^3k\, \vec{v}(\vec{k}) f(\vec{r},\vec{k}) = -eA\int dE\, \overbrace{N(E)}^{\text{Density of state}} \int \frac{d\hat{k}}{4\pi}\, \vec{v}(\hat{k}) f(\vec{r},\hat{k},E)$$

$$I_Q = A\frac{1}{4\pi^3}\int d^3k\,(E-\mu)\vec{v}(\vec{k}) f(\vec{r},\vec{k}) = A\int dE\, N(E)(E-\mu)\int \frac{d\hat{k}}{4\pi}\, \vec{v}(\hat{k}) f(\vec{r},\hat{k},E)$$

Assuming $e\Delta V,\ k_B\Delta T \ll k_B T \ll E_F \simeq \mu$; $f_0(E,\mu,T) = \frac{1}{e^{(E-\mu)/k_BT}+1}$

$f(r,E) \simeq f^0(E, \mu(r), T(r))$

$\nabla f(r,E) = (\partial_\mu f^0)\nabla\mu + (\partial_T f^0)\nabla T$

$$G = \frac{A}{L} e^2 \int dE\, N(E) D(E)\, \partial_\mu f^0 \ ; \quad GS = \frac{A}{L} e \int dE\, N(E) D(E)\, \partial_T f^0$$

$$G\Pi = \frac{A}{L} e \int dE\,(E-\mu) N(E) D(E)\, \partial_\mu f^0 \ ; \quad K = \frac{A}{L}\int dE\,(E-\mu) N(E) D(E)\, \partial_T f^0$$

(2)

Further assuming $N(E) \simeq N_F + c_F(E-\mu)$; $c_F = \left(\frac{\partial N}{\partial E}\right)_{E_F}$ $c_F > 0$ (electrons), < 0 (holes)

$D(E) \simeq D_F + c_D(E-\mu)$; $c_D = \left(\frac{\partial D}{\partial E}\right)_{E_F}$

(3)

Using ② & ③, one can show

- $$\frac{K}{G} = \underbrace{\frac{\pi^2 k_B^2}{3e^2}}_{= L_0 \text{ : Lorentz \#}} T \quad ; \quad \text{Wiedemann Frantz law}$$

(Slide 2-5)

- $$S = e L_0 \frac{1}{G}\frac{dG}{dE} T = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)\frac{k_BT}{G}\frac{dG}{dE}\bigg|_{E_F} \quad ; \quad \text{Mott Formula}$$

(slide 6-7)

Under magnetic fields

$$\overleftrightarrow{S} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right) k_BT \left(\overleftrightarrow{\sigma}^{-1}\right)\left(\frac{d\overleftrightarrow{\sigma}}{dE}\right)_{E_F}$$

- $\Pi = ST$: Onsager reprocity relation

Here

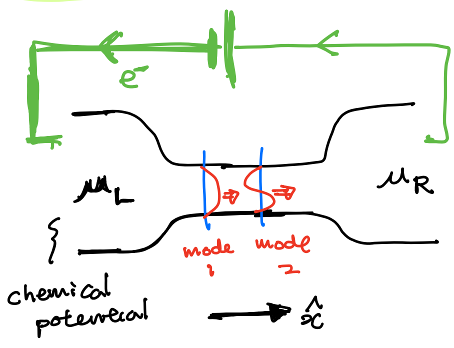
$$\Pi = \frac{I_Q}{I} = \frac{n\Delta Q v}{nev} = \frac{\Delta Q}{e} \quad : \text{Heat transfer per charge.}$$

$$\Delta Q = T\underbrace{\Delta S}_{\text{entropy}}$$

$$\Rightarrow \quad S = \frac{\Pi}{T} = \frac{\Delta S}{e} \quad : \text{(transport) entropy per charge.}$$

(slide 8-9)

Ballistic Conductors



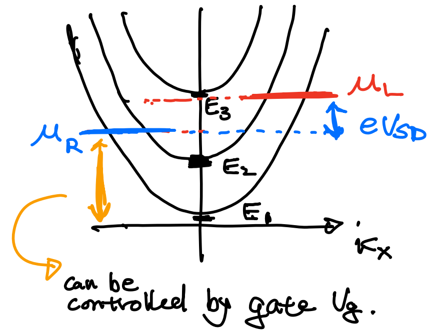
Consider a ballistic conductor connected between two electron reservoirs with chemical potential μ_L & μ_R.
The left reservoir is connected higher electron potential side of a battery so that

$$\mu_L = \mu_R + eV_{SD} \quad \text{— ①}$$

Fermi-Dirac distribution of each reservoir is given by

$$f_i(E) = \left[1 + \exp\{(E-\mu_i)/k_BT\}\right]^{-1} \quad i = R, L. \quad \text{— ②}$$

We now consider 1D sub bands whose minima are located at E_n as shown in the right



Then the current from the left reservoir traveling (+) direction (moving to the right) is given by

$$I^+ = \sum_n \underbrace{\sum_k}_{\frac{L}{2\pi}\int dk} \underset{\sim}{I_{nk}} = \sum_n \frac{2e}{L}\frac{1}{\hbar}\frac{L}{2\pi}\int dk \frac{\partial E_{nk}}{\partial k} f_L(E_{nk}) = \frac{2e}{h}\sum_n \int_{E_n}^{\infty} dE f_L(E)$$

$$\to I_{nk} = \frac{2e}{L}\frac{1}{\hbar}\frac{\partial E_{nk}}{\partial k}$$

Similarly $\quad I^- = \frac{2e}{h}\sum_n \int_E^{\infty} dE\, f_R(E)$

$$I_{tot} = I_+ - I_- = \frac{2e}{h}\sum_n \int_{E_n}^{\infty} dE\,(f_L(E) - f_R(E))$$

Assume V_{sd} is small (linear response regime)

$$f_L(E) - f_R(E) = \frac{\partial f_L}{\partial \mu_L}(\mu_L - \mu_R) = -\frac{\partial f_L(E)}{\partial E} eV_{sd}$$

$$I_{tot} = \frac{2e^2}{h}\sum_n \int_{E_n}^{\infty} dE\left(-\frac{\partial f_L(E)}{\partial E}\right)\cdot V_{sd} = \frac{2e^2}{h}\underbrace{\sum_n f_L(E_n)}_{\#\text{ of occupied mode}} \cdot V_{sd}$$

$$\text{''} \; M$$

$\Rightarrow \quad I_{tot} = \frac{2e^2}{h} M V_{SD} \quad$ or $\quad \boxed{G = \frac{2e^2}{h} M} \qquad M$: # of transporting mode.

$\mapsto$ This is called "Landauer formula (slide 10-11)

Now we do similar steps for thermal transport.

Then

$$I_Q = \frac{2}{h}\sum_n \int_{E_n}^{\infty} dE\,\{(E-\mu_L)f_L - (E-\mu_R)f_R\} \qquad \mu_L \simeq \mu_R$$

$$= \frac{2}{h} M \int_0^{\infty} \epsilon\,[f(\epsilon, T_L) - f(\epsilon, T_R)]\,d\epsilon \quad ; \quad f(\epsilon, T) = \frac{1}{e^{\epsilon/kT}+1}$$

$$\simeq \frac{2}{h} M \underbrace{\int_0^{\infty} \epsilon \left(\frac{\partial f}{\partial T}\right) d\epsilon}_{\frac{\pi^2}{6} k_B^2 T} \cdot \Delta T$$

$$K = M K_0 \quad ; \quad K_0 = \frac{\pi^2 k_B^2}{3h} T$$

Thermal Quanta.

How about thermal conduction by Bosonic degree of freedom?

$$I_Q = \frac{1}{h}\sum_m \int_0^\infty \frac{dk}{2\pi}\, \hbar\omega_m(k)\, v_m(k)\, [\eta_{hot} - \eta_{cold}] \qquad \eta(T) = \frac{1}{e^{\hbar\omega/kT}-1}$$

$$= \frac{M}{h} M \underbrace{\int_0^\infty \epsilon \left(\frac{\partial\eta}{\partial T}\right) d\epsilon}_{\frac{\pi^2}{3}k_B T} \Delta T$$

$$\underbrace{\kappa^{(ph)} = M\kappa_0}_{\text{thermal quantum}} \qquad \kappa_0 = \frac{\pi^2 k_B^2}{3h} T$$

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- Thermopower in Quantum Hall regime

$$S_{xx} = \frac{\text{entropy transport}}{\text{charge carrier}} = \frac{k_B}{e}\frac{\ln 2}{\nu}$$

$\ln 2$ → occupied or not

ν ← filling fraction (# of edge channels transporting)

$$\frac{k_B}{e}\ln 2 \simeq 60\mu V/K$$

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- Thermoelectric figure of merit : dimensionless # to show the efficiency thermoelectric application.

$$ZT = \frac{\sigma T S^2}{\kappa} = \frac{GT}{K} S^2$$

In the quantum limit, $G \to \frac{e^2}{h}$; $K \to \frac{\pi^2 k_B^2}{3h} T$

$$\frac{G^Q T}{K^Q} = \left(\frac{\pi^2 k_B^2}{3e^2}\right)^{-1} = \frac{1}{L_0}$$

Lorentz # !

$$\Rightarrow ZT^{(Q)} = \frac{S^2}{L_0} = \left(\frac{k_B}{e}\right)^2 \frac{1}{L_0} C^2$$

$$= \frac{3}{\pi^2} C = 0.304 \times C^2$$

How can we make C large ??

Electronic Hydrodynamic Transport.

Navier-Stokes eqn :
for imcompressible liquid

$$\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}}_{\frac{d}{dt}\vec{v}(\vec{r},t)} = -\underbrace{\frac{\overset{\text{pressure}}{\vec{\nabla} P}}{\rho}}_{\text{ignorable for incompressible electron gas.}} + \overbrace{\nu}^{\text{kinematic viscosity } (m^2/s)}\vec{\nabla}^2\vec{v} + \frac{e}{m}\vec{\nabla}\phi$$

With consideration of momentum relaxation,
we introduce the damping term $\gamma = \frac{1}{\tau_{mr}} = \frac{ne^2}{m\sigma_0}$,

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \nu\vec{\nabla}^2 v - \gamma\vec{v} - \left(\frac{e}{m}\right)(\vec{E} + \vec{v}\times\vec{B})}$$

Navier-Stoke's Ohm equation.

Here ν : electron-electron momentum diffusion by interaction

$$\nu \approx \left(\frac{v_F}{2}\right)^2 \cdot \tau_{ee}$$

Gurzhi length $\quad l_G = \sqrt{\nu\,\tau_{mr}} \rightarrow$ Compare with the size of the channel L.

$l_G \ll L$: Ohmic dominant.

$l_G \gg L$: hydrodynamic dominance

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