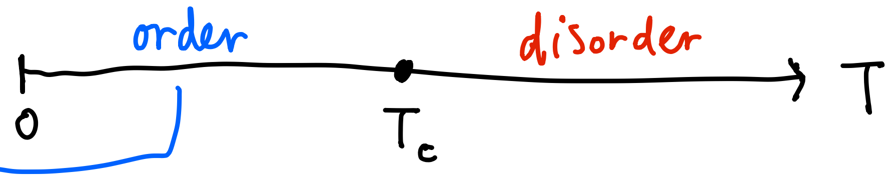
Entropic order

arXiv: 2503.22789

with: Han, Huang, Komargodski, Popov

Typical thermal phase diagram:

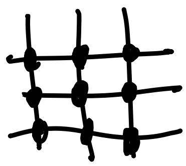


Spontaneous symmetry breaking (SSB)

But ordered phases are useful... e.g. high-T superconductors. Principles for stabilizing them?

As we'll see, a long history of trying to realize order at high T...

Example: 2d classical Ising model:



$$H = -\sum_{x\sim y} s_x s_y \qquad \text{where} \qquad s_x \in \{\pm 1\}$$

nearest neighbor on square lattice

Look at thermal state: $\mathbb{P}(\vec{s}) = \frac{1}{Z} e^{-\beta H(\vec{s})}$ with $\beta = \frac{1}{T}$.

temperature

$\mathbb{Z}_2$ symmetry of H: $H[\vec{s}] = H[-\vec{s}]$.

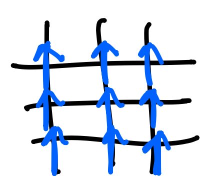
↳ broken at low T. How do we see this?

① thermodynamics:

system "minimizes" $F = E - TS$

free energy, energy, entropy

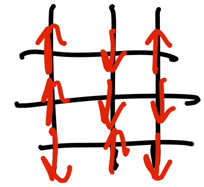
$T \to 0$: minimize $E = \langle H \rangle$:



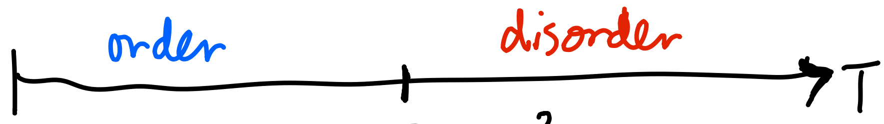
ferromagnet (break $\mathbb{Z}_2$)

$T \to \infty$: maximize $S \sim \log W$

"# of accessible configs"



paramagnet (have $\mathbb{Z}_2$)



$$T_c = \frac{2}{\log(1+\sqrt{2})}$$

Compute by finding nonanalyticity in

$$F = -T \log Z$$

$$Z = \sum_{\vec{s}} e^{-\beta H[\vec{s}]}$$

② Diagnose SSB via long-range order:

$$\langle s_x s_y \rangle - \cancel{\langle s_x \rangle \langle s_y \rangle}^{\,0} > 0 \quad \text{at large } |x-y|.$$

$$= \sum_{\vec{s}} s_x s_y \, \mathbb{P}(\vec{s}) \leftarrow \frac{e^{-\beta H(\vec{s})}}{Z}$$

Caveat: mathematically, $\langle s_x \rangle = -\langle s_x \rangle = 0$

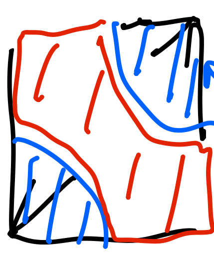
$\mathbb{Z}_2$ of ensemble...

return to this soon

Physics convention often takes $\langle s_x \rangle > 0$ at low T...

③ SSB is measure condensation of Gibbs ensemble

at low T, probability measure on $\{\pm 1\}^{L^2}$:



$\mathbb{P} \approx 1/2$

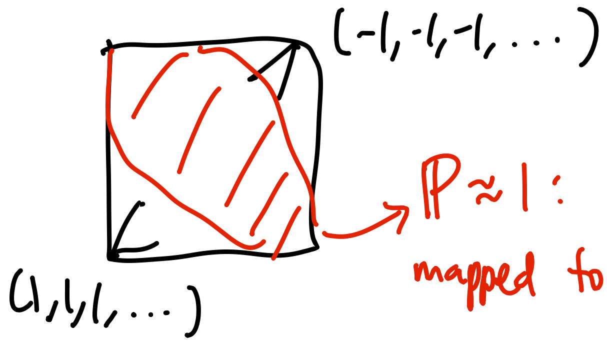
$\mathbb{P} \lesssim e^{-a \cdot L}$

Physics convention: restrict thermal average to (1,1,1...) pocket

SSB: $\mathbb{Z}_2$ maps one cluster ("replica") to another

<u>Thm</u>: thermal dynamics trapped in replica for time $t \gtrsim e^{a \cdot L}$

at high T,

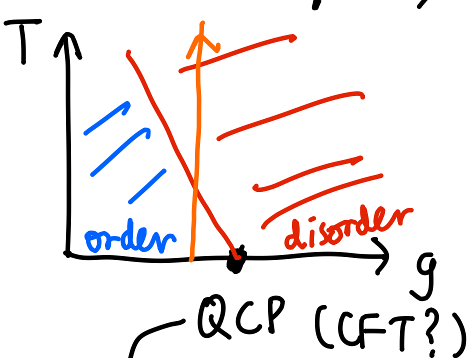


$\mathbb{P} \approx 1$: one cluster, mapped to self by $\mathbb{Z}_2$

Some systems (spin glass) best studied w/ perspective 3:
phase transitions w/o non-analytic F (LDPC code)

Q: is high T always disordered?

Physics: most systems look like:

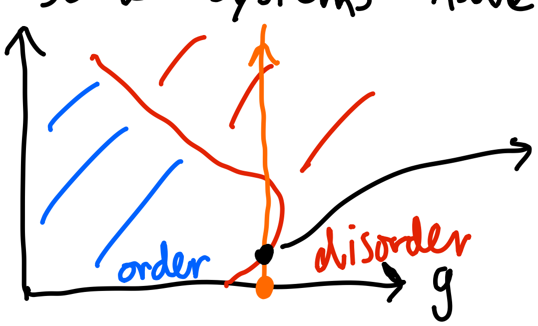


heat from order to disorder

can a CFT break symmetry at <u>all</u> T > 0?

an old puzzle going back to 1970s...

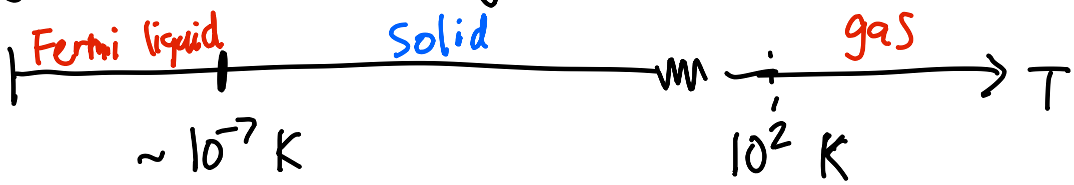
But some systems have "re-entrant" order:



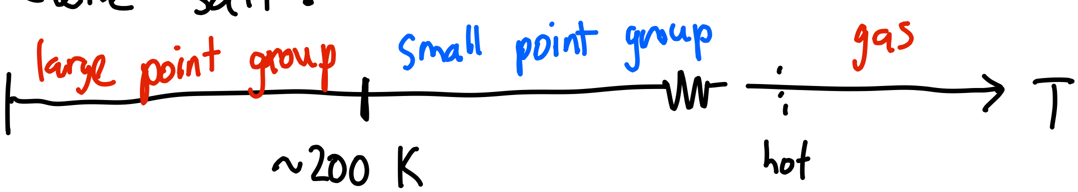
here, heat into an ordered phase!

Examples:

- Pomeranchuk effect (high-pressure ^3He): (1950's)



- Rochelle salt: (so old...)



- "order-by-disorder" in theoretical models of magnets...

All of these eventually order at high T.

↳ Q: Can we order as $\underline{T \to \infty}$?

Math: $\underline{\text{NO}}$, under 2 assumptions.

We're going to formalize a math problem... bear with me...

State space $\Omega = \{\text{any possible microstate}\}$ ← assume discrete

↳ 2d Ising: $\{\pm 1\}^{L^2}$

Energy function: $H : \Omega \to \mathbb{R}$

Gibbs ensemble: $\mathbb{P}(A \subseteq \Omega) = \sum_{\omega \in A} \frac{e^{-\beta H(\omega)}}{Z(\beta)}$, $\quad Z(\beta) = \sum_{\omega \in \Omega} e^{-\beta H(\omega)}$ (assume exists!)

$\underline{\text{Thm}}$: if ① Ω is $\underline{\text{finite}}$

② $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$

(no hard-wired constraints)

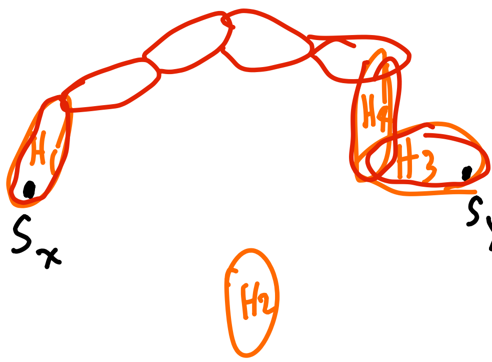
③ H is extensive (each DOF change H by $O(1)$ amount) then for $\beta < \beta_c = O(1)$, no long-range order / measure condensation.

<u>Proof sketch</u>: use <u>cluster expansion</u>:

$$\langle s_x s_y \rangle = \frac{1}{Z}\sum s_x s_y e^{-\beta H(\omega)} = \frac{1}{Z}\sum s_x s_y \left(1 - \beta H + \frac{1}{2}\beta^2 H^2 - \dots\right)$$

local observable

If H is a sum of local terms...



$S_x S_y (\beta H)^4$ contains

→ identical contribution to $\langle s_x \rangle \langle s_y \rangle$

$S_x S_y (\beta H)^{|x-y|}$ contributes...

But $\beta H_j \ll 1$ at small β by extensivity, $\lesssim n!$ clusters of size n.

So series converges <u>and</u>

$$\langle s_x s_y \rangle - \langle s_x \rangle \langle s_y \rangle \lesssim e^{-\lambda \cdot |x-y|} \quad \text{for some } \lambda > 0.$$

The same thing happens in <u>quantum systems</u>:

↳ Hilbert space $\mathcal{H} = L^2(\Omega)$

Hamiltonian $H : \mathcal{H} \otimes \mathcal{H} \to \mathbb{R}$ is Hermitian

quantum Gibbs state: $\rho_\beta = \frac{1}{Z} e^{-\beta H}$, $\quad Z = \mathrm{tr}(e^{-\beta H})$.

Under same assumptions, no long-range order <u>and</u>

$$\rho_\beta = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \qquad \text{where } |\psi_\alpha\rangle \text{ is a } \underline{\text{product state}}$$

<u>separable</u> state with <u>no entanglement</u>

It seems like math forbids high T order.

Let's try to avoid this theorem.

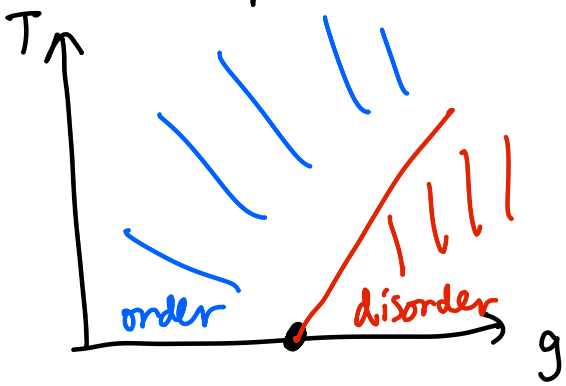
↳ – fun

– ordered states are good. how to stabilize?

could this be physically relevant?

– landscape of CFTs?

Goal:



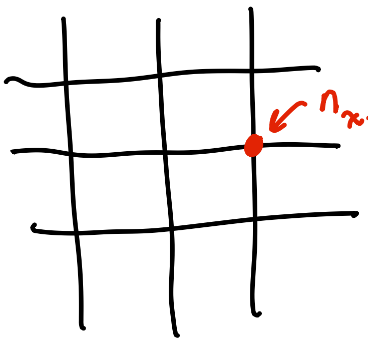
Formal assumptions: ① Ω finite ② no hard constraints ③ "extensive"

$\underbrace{\text{① } \Omega \text{ finite}}_{\text{relax}}$ $\underbrace{\text{② no hard constraints ③ "extensive"}}_{\text{don't touch?}}$

bosons have $\underbrace{\text{countably infinite } \Omega}$

breaks the cluster expansion. the $\beta=0$ ensemble does NOT exist.

Example: Take $\Omega = \{0, 1, 2, \ldots\}^{L^2}$



$n_x \geq 0$ particles on each site.

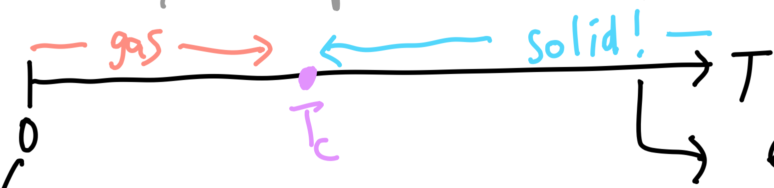
$$H = \sum_x n_x + U \sum_{x \sim y} n_x^2 n_y^2$$

sites   nearest-neighbors.

Problem is well-posed since

$$Z(\beta) \geq \sum e^{-\beta \sum n_x} = \prod_{j=1}^{L^2} \sum_{n_j=0}^{\infty} e^{-\beta n_j} = \left(\frac{1}{1-e^{-\beta}}\right)^{L^2} < \infty.$$

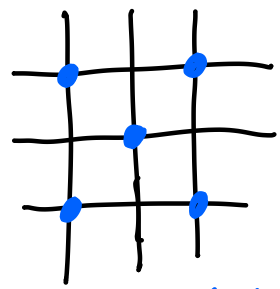
let's now try to predict the phase diagram!

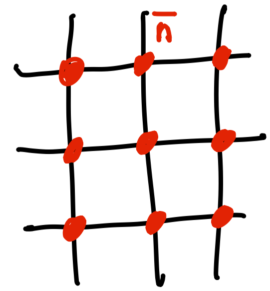


estimate contributions to Z...

<u>unique</u> ground state

$n_x = 0$





checkerboard (solid)

$$Z(\beta) \sim \left(\frac{1}{1-e^{-\beta}}\right)^{L^2/2}$$

so $Z \sim T^{L^2/2}$

disordered (gas)

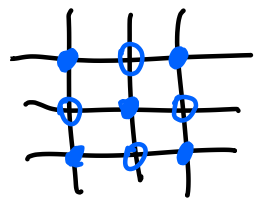
$$Z(\beta) \sim \bar{n}^{L^2}$$

so $Z \sim T^{L^2/4}$

Estimate $\bar{n}$ by: $\beta U \bar{n}^4 \sim 1$, so $\bar{n} \sim T^{1/4}$

Numerics: continuous transition in <u>2d Ising</u> universality class.

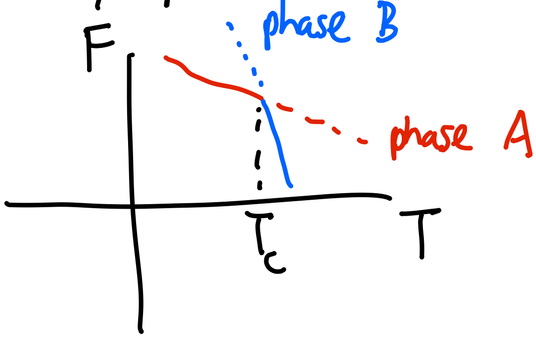
the solid SSB's a $\mathbb{Z}_2$:



The order persists to <u>arbitrarily high T</u>.

We call this <u>entropic order</u>, because...

at any phase transition,

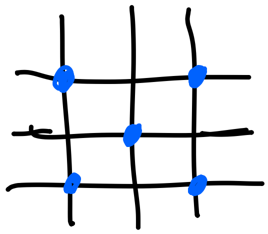


to <u>minimize F</u>, take phase w/ smaller $\frac{\partial F}{\partial T} = -S$

laws of thermo

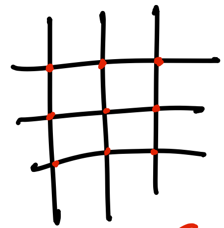
The high T phase has <u>more entropy</u>, and is also <u>ordered</u>.

Ordering one DoF can enable strong fluctuations in another!



$S \sim \frac{L^2}{2} \cdot \log T$

vs.



"no $\mathbb{Z}_2$ SSB" boson fluct.

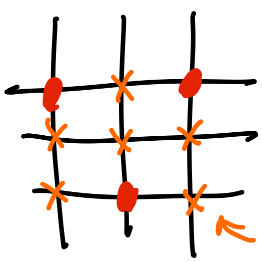
$S \sim L^2 \left[\log 2 + \log T^{1/4}\right]$

Take limit $U \to \infty$: this becomes a hardcore lattice gas:

$$Z(\beta) = \sum_{N=0}^{L^2} z^N C_N$$

C_N ← # of ways to occupy N non-adjacent sites

z ← fugacity $z = \frac{1}{e^{\beta}-1} = \sum_{n=1}^{\infty} e^{-\beta n}$



This stat-mech model is exactly solved!

For fugacity $z \gtrsim 3.7$ on square lattice, $\mathbb{Z}_2$ SSB in 2d Ising universality.

Usually have in mind:

— or • (ΔH = −μ) $\Rightarrow$ $z = e^{\beta\mu}$

But if we allow up to K particles...

— or • (ΔH = −μ) ... or ⋮ }K (ΔH = −Kμ)

$$z = \sum_{n=1}^{K} e^{\beta\mu n}$$

$\lim_{\beta \to 0} z = K$. If $K \geq 4$ we have order as $\beta \to 0$.

↳ So, <u>constrained</u> systems can also realize entropic high-T order

Recall: Thm on no high-T order required
① Ω finite ② Ω not constrained.
Both are necessary.

We'll soon explore how to engineer entropic order. But first...

Is entropic order robust? <u>NO</u>

$$H = \sum_x n_x + U \sum_{x\sim y} n_x^2 n_y^2 + \underbrace{\varepsilon \sum_x n_x^4}_{\text{at low } T, \text{ this is irrelevant}}$$

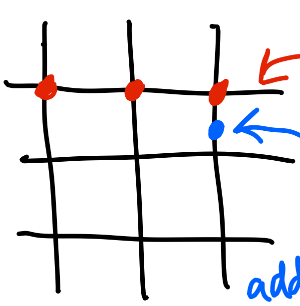
High T physics is sensitive to irrelevant, not relevant operators.

↳ <u>BUT</u>, entropic order would persist to $T \lesssim \frac{1}{\varepsilon^{1/3}}$ ↖ crude estimate.

Still learn useful design principles for stabilizing high-T order.

How can we engineer a model w/ entropic order?

Use bosons to "transmute" low ⟷ high T.



Ising spin $s_x \in \{\pm 1\}$

add: "link boson" $n_{xy} \in \{0, 1, 2 \dots\}$ for $x \sim y$

$$H = \sum_{x \sim y} (a - b s_x s_y) n_{xy} .$$

$\underbrace{a > b}$ > 0

So that $Z(\beta)$ well-defined...

$$Z(\beta) = \sum_{s_x} \sum_{n_{xy}} e^{-\beta H}$$

$$\hookrightarrow \sum_{n=0}^{\infty} e^{-\beta(a - b s_1 s_2) n} = \frac{1}{1 - e^{-\beta(a - b s_1 s_2)}}$$

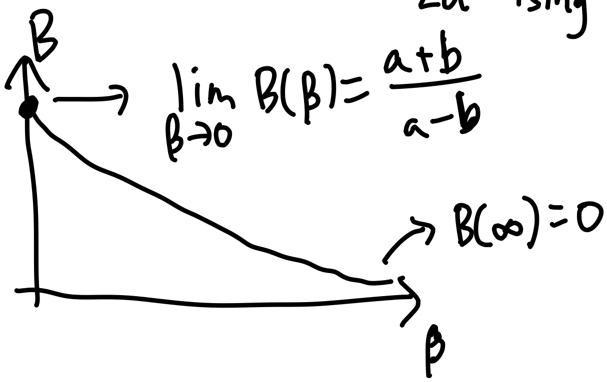
$$= \begin{cases} \dfrac{1}{1 - e^{-\beta(a-b)}} & s_1 s_2 = +1 \\ \dfrac{1}{1 - e^{-\beta(a+b)}} & s_1 s_2 = -1 \end{cases} = e^{A + B s_1 s_2} \quad \text{where:}$$

$$e^{2A} = \frac{1}{(1 - e^{-\beta(a-b)})(1 - e^{-\beta(a+b)})}, \quad e^{2B} = \frac{1 - e^{-\beta(a+b)}}{1 - e^{-\beta(a-b)}}$$

Up to overall prefactor: $Z(\beta) = \sum_s e^{B(\beta) \sum_{x \sim y} s_x s_y}$

$\downarrow$

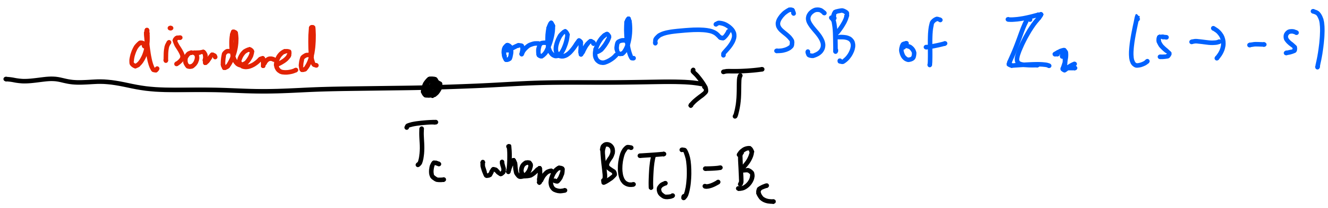
2d Ising at new effective $\beta_{eff} = B(\beta)$.



2d Ising has phase transition at

$$B_c = \frac{\log(1+\sqrt{2})}{2}$$

So we want $\dfrac{a+b}{a-b} > 1 + \sqrt{2} \longrightarrow 1 < \dfrac{a}{b} < 1 + \sqrt{2}$



To deduce entropic order:

$$\frac{S_{order}}{L^2} \sim \log \bar{n}_{order} \sim \log \frac{T}{a-b} \qquad \frac{S_{disorder}}{L^2} \sim \underbrace{\log 2}_{\text{disorder}} + \underbrace{\log \frac{T}{a+b}}_{\text{boson fluct.}}$$

$S_{order} > S_{disorder}$ if a/b sufficiently small.

Most configurations at fixed energy have broken $\mathbb{Z}_2 \rightarrow$ more n_{xy} fluct.

We can use this trick to build other phases at high T:

Superfluid (SSB of $U(1)$) in XY model in 3d:

$$H = \sum_{x\sim y} e^{-K\cos(\theta_x - \theta_y)} n_{xy} \qquad \begin{array}{l} \theta_x \sim \theta_x + 2\pi \\ n_{xy} \in \{0, 1, 2, \dots\} \end{array}$$

$$Z(\beta) = \int d\theta_x \sum_{n_{xy}} e^{-\beta H} = \int d\theta_x \prod_{x\sim y} \frac{1}{1 - e^{-\beta \exp[-K\cos(\theta_x - \theta_y)]}}$$

$$\underset{\beta\to 0}{\approx} \# \int d\theta_x \, \exp\Big[\sum_{x\sim y} K\cos(\theta_x - \theta_y)\Big]$$

which looks like usual XY at effective inverse temperature K.

Choose $K > \underset{\text{for XY}}{K_c}$ to have superfluid as $T \rightarrow \infty$.

Similar constructions for 4d toric code

$\rightarrow$ quantum entanglement & topological order as $T \rightarrow \infty$

Now let's turn to quantum field theories.

Thermal quantum field theory:

$$Z(\beta) = \mathrm{tr}\left(e^{-\beta H}\right) = e^{-\beta F} = \int \underset{\text{fields}}{D\phi} \; e^{-S_E[\phi]}$$

Euclidean action: $S_E[\phi] = \int_0^\beta d\tau \int d^d x \;\; \mathcal{L}(\phi, \partial_\mu \phi \ldots)$

Approximate: small fluctuations around homogeneous vacuum

$$\phi(x,\tau) = \bar{\phi} + \delta\phi(x,\tau)$$

Expand:
$$S_E[\phi] = S_E[\bar{\phi}] + \frac{1}{2} \int_{\substack{x_1, x_2 \\ \tau_1, \tau_2}} \frac{\delta^2 S_E}{\delta\phi_1 \delta\phi_2} \cdot \delta\phi(x_1, \ldots)\,\delta\phi(x_2 \cdots)$$

$$S_E[\bar{\phi}] = \beta V_d \cdot \mathcal{L}(\bar{\phi}) \qquad + \quad \text{quadratic}$$

volume of d-dim space "background potential energy"

Now:
$$\int D\,\delta\phi \;\; e^{\text{"}-\frac{1}{2}\frac{\delta^2 S}{\delta\phi^2}\delta\phi^2\text{"}} = e^{-\beta \mathcal{F}_1(\bar{\phi})}$$
1-loop thermal potential

Entropic order $\rightarrow$ $\mathcal{F}_1$ destabilizes minimum of $\mathcal{L}(\bar{\phi})$.

Example: O(N) model:

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{\lambda}{4N}(\phi^a\phi^a)^2$$
$a = 1, \ldots, N$

Guess $\bar{\phi}^a = (\sqrt{N}\,\bar{\Phi}, 0, 0, \ldots, 0)$

Keeping fluctuations for $\tilde{a}=2,\dots,N$:

$$\mathcal{L}_E \approx \frac{\lambda N}{4}\bar{\phi}^4 + \frac{\lambda}{2}\bar{\phi}^2\,\delta\phi^{\tilde{a}}\delta\phi^{\tilde{a}} + \frac{1}{2}(\partial_\mu\phi^{\tilde{a}})^2$$

which will give $\mathcal{F}_1$ accurate to leading order in $1/N$:

$$\beta\mathcal{F}_1 = \int_0^\beta d\tau \int d^dx\, \frac{N-1}{2}\log\left(-\partial_\mu\partial^\mu + \lambda\bar{\phi}^2\right)$$

$$\approx \frac{N}{2}\, V_d \underbrace{\sum_{n\in\mathbb{Z}}}_{\text{Matsubara frequencies}} \int d^dk\, \log\left(k^2 + (2\pi T n)^2 + \lambda\bar{\phi}^2\right)$$

Each bosonic mode is a quantum harmonic oscillator, so...

$$-\sum_{n\in\mathbb{Z}} \log\left((2\pi T n)^2 + \Omega^2\right) = \log \mathrm{tr}\left(e^{-\beta\Omega(a^\dagger a + \frac{1}{2})}\right)$$

$$= \log\left[e^{-\beta\Omega/2}\,\frac{1}{1-e^{-\beta\Omega}}\right]$$

So: $$\beta\mathcal{F}_1 = \frac{N}{2}V_d\int d^dk\left[\cancel{\frac{\beta}{2}\sqrt{k^2+\lambda\bar{\phi}^2}} + \log\left(1-e^{-\beta\sqrt{k^2+\lambda\bar{\phi}^2}}\right)\right]$$

Divergent constants $\rightarrow$ counterterms. Physical effect:

$$\mathcal{F}_1 = \frac{N}{2}V_d\cdot T\int d^dk\, \log\left(1-e^{-\beta\sqrt{k^2+\lambda\bar{\phi}^2}}\right)$$

$$\rightarrow \frac{N}{2}V_d T\cdot\left[\#\left(-T^d + \lambda\bar{\phi}^2 T^{d-2} + \dots\right)\right]$$

Crucial point here is the + sign.

Thus $$\mathcal{F} = N\, V_d\cdot\left[\frac{\lambda}{4}\bar{\phi}^4 + c\cdot\lambda\, T^{d-1}\bar{\phi}^2 + \dots\right]$$

The minimum of $\mathcal{F}$ is at $\bar{\phi}=0$, so $O(N)$ unbroken.
This is generic outcome. Thermal mass is usually positive.

Example: $O(N) \times \mathbb{Z}_2$ "biconical" model: Weinberg (1974)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{1}{2}(\partial_\mu \psi)^2 + \frac{\lambda}{4N}(\phi^a\phi^a - \psi^2)^2 + \frac{g}{6!N^2}\psi^6 + \cdots$$

irrelevant in QFT

Work in 2+1 dimensions. If $\bar{\phi}^a = (\sqrt{N}\bar{\phi}, 0, \ldots)$, $\bar{\psi} \to \sqrt{N}\bar{\psi}$:

Now... $$\delta\mathcal{L} = \frac{1}{2}(\partial_\mu \delta\phi^{\tilde{a}})^2 + \frac{1}{2}(\partial_\mu \delta\psi)^2 + \frac{\lambda N}{2}\bar{\phi}^2(\delta\phi^{\tilde{a}}\delta\phi^{\tilde{a}} - \delta\psi^2) + \frac{g}{2N^2}\bar{\psi}^4\delta\psi^2 - \frac{\lambda N}{2}\bar{\psi}^2(\delta\phi^{\tilde{a}}\delta\phi^{\tilde{a}} - \delta\psi^2)$$

crucial sign change!

Now $$\mathcal{F}(\bar{\phi},\bar{\psi}) = \frac{\lambda N}{4}\bar{\phi}^4 + \frac{gN}{6!}\bar{\psi}^6 + \frac{\lambda N}{2}T\left[\#\bar{\phi}^2 - \#\bar{\psi}^2\right] + \cdots$$

Minimized at $\bar{\phi}=0$, $\bar{\psi} = a\cdot\sqrt{T}$.
↳ (Need $1/N$ corrections to confirm...)

We think this is an actual CFT w/
- unbroken $\mathbb{Z}_2$ at $T=0$
- broken $\mathbb{Z}_2$ at $T>0$ (all $T\neq 0$ "equivalent")

Numerics: occurs if $N \gtrsim 10$

Why order? If $\bar{\psi}>0$, reduce thermal mass of ϕ^a...
→ stronger fluctuations of ϕ^a
= entropic order!

Example: superconductivity.

BCS mean-field theory w/ $N \gg 1$ fermion "flavors":

$$\mathcal{L}_E = \bar{\psi}_{\sigma,a}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\bar{\psi}_{\sigma,a} - \frac{g}{N}\bar{\psi}_{\uparrow a}\bar{\psi}_{\downarrow a}\psi_{\downarrow b}\psi_{\downarrow a}$$

$(a=1,\dots,N)$ & $\sigma = \uparrow,\downarrow$

$$\int D\psi D\bar{\psi}\, e^{-S_E} = \underbrace{\int}_{\text{Hubbard-Stratonovich}} D\psi D\bar{\psi} D\Delta D\bar{\Delta}\, e^{-S_E'}$$

where $S_E' = \int_0^\beta d\tau \int d^dx \left[\bar{\psi}_{\sigma,a}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\bar{\psi}_{\sigma,a} + \frac{N}{g}|\Delta|^2 - \Delta\bar{\psi}_{\uparrow a}\bar{\psi}_{\downarrow a} - \bar{\Delta}\psi_{\downarrow a}\psi_{\uparrow a}\right]$

Integrate out $\psi, \bar{\psi}$:

$$\beta F(\Delta,\bar{\Delta}) = \iint \left[\frac{N}{g}|\Delta|^2 - \mathrm{tr}\log\begin{pmatrix}\partial_\tau\cdots & \Delta \\ \bar{\Delta} & -(\partial_\tau\cdots)\end{pmatrix}\right]$$

Saddle point calculation leads to:

$$\frac{1}{N\beta}\frac{\partial F}{\partial\bar{\Delta}} = 0 = \frac{\Delta}{g} - \nu\Delta\int_0^{\beta\Lambda} dx\, \frac{\tanh\left[\frac{1}{2}\sqrt{x^2+(\beta\Delta)^2}\right]}{\sqrt{x^2+(\beta\Delta)^2}}$$

ν: DOS at Fermi surface

BCS gap equation.

Couple this superconductor to N fluctuating bosons!

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2}(\partial_\mu\phi^a)^2 + \frac{1}{2}\sigma(\phi^a\phi^a - t|\Delta|^2) + \frac{\sigma^2 N}{2\lambda}$$

Now BCS gap equation:

$$\frac{1}{g} - \sigma t = \nu\int\cdots$$

and at $\mathcal{F}$ minimum: $\bar{\phi} = 0$ and

$$t|\Delta|^2 = T\sum_n \int d^dk \, \frac{1}{(2\pi T n)^2 + k^2 + \sigma}.$$

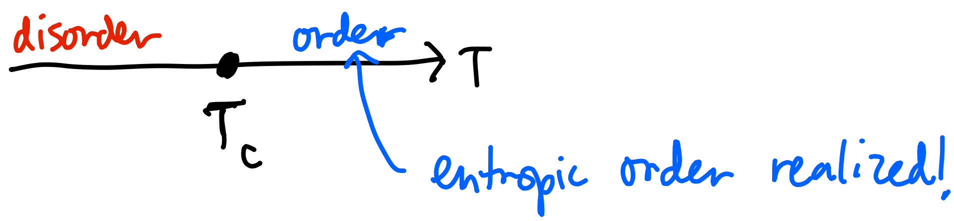
$\hookrightarrow$ $\Delta \neq 0$ always! entropic ordered SC.

ϕ^a fluctuations can stabilize $\Delta \neq 0$, rather than fermions.

Can this ever be relevant for experiment??

Summary: higher T phase $\Rightarrow$ higher S

if :



What's new? order as $T \to \infty$ possible with bosons
- explicit lattice models
- QFT

TBD: are design principles here ever helpful for experiments?