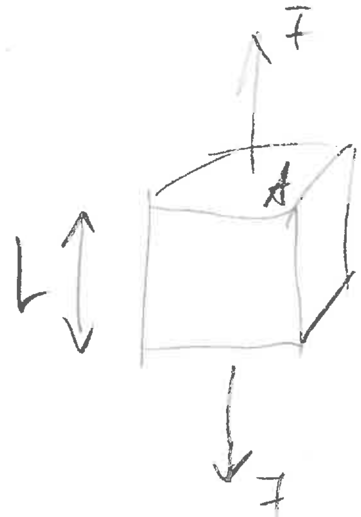
1d (or scalar) elasticity → 3d elasticity



mechanical test

$$\sigma = \frac{F}{A} \qquad \varepsilon = \frac{\Delta L}{L} \qquad \sigma = E \cdot \varepsilon$$

↓ ↓ ↓ LET

$$\sigma_{ij} \qquad \varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \qquad \sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

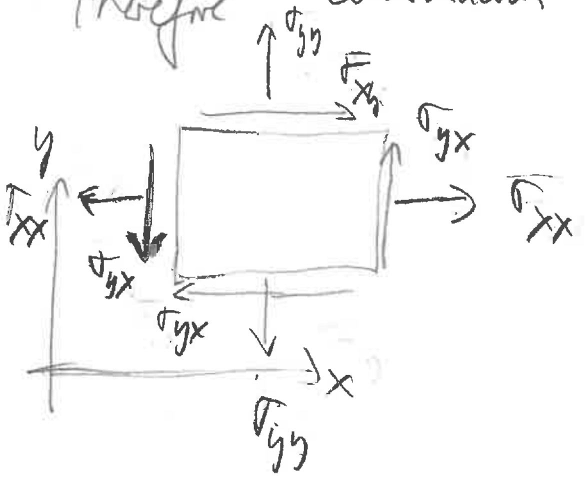
Force balance: $\dfrac{dN}{dx} = \dfrac{d}{dx}\left(C \cdot \dfrac{du}{dx}\right) = C\, \dfrac{d^2u}{dx^2} = 0$

$$\rightarrow \quad \underline{\nabla}\, \underline{\underline{\sigma}} = (\lambda+\mu)\, \underline{\nabla}(\underline{\nabla}\, \underline{u}) + \mu\, \Delta \underline{u} = 0$$

PDE coupling all components

3D continuum mechanics

Forces $\underline{F}(\underline{r})$ and displacements $\underline{u}(\underline{r})$ are 3d vectors.
Therefore continuum mechanics is a tensorial theory.



free body diagram in 2d

<u>Stress tensor</u> σ_{ij}

i: direction of force; j: normal of surface element

$\sigma_{ij} > 0$ if i, j have same direction

Force and moment balance lead to (same result in 3d)

$$\partial_j \sigma_{ij} = 0 , \qquad \sigma_{ij} = \sigma_{ji}$$

This is the generalization of $\partial_x \sigma = 0$ in 1d.

With inertia and body forces, we get Cauchy's equation:

$$\boxed{\rho \frac{\partial^2 u_i}{\partial t^2} = \rho g_i + \partial_j \sigma_{ij}}$$

<u>Strain tensor</u>: describes deformations in all directions at once
for a given direction $\underline{n}$, it should give us
the strain as $\frac{\Delta L}{L} = \underline{n} \, \underline{\underline{\varepsilon}} \, \underline{n}$

linear approx.

$$\Rightarrow \varepsilon_{ij} = \frac{1}{2} \left(\partial_i u_j + \partial_j u_i \right)$$

note $\varepsilon_{ij} = \varepsilon_{ji}$ | trace $\varepsilon_{ll} = \frac{\Delta V}{V}$

For linear isotropic material, we write (13)

$$\sigma_{ij} = -p\,\delta_{ij} + \sigma^d \leftarrow$$ deviatoric parts

$$\varepsilon_{ij} = \tfrac{1}{3}\,\varepsilon_{kk} + \varepsilon^d$$

p pressure, $\varepsilon_{ll} = \frac{dV}{V}$ relative volume change

We thus need two elastic constants:

$$p = K\,\varepsilon_{kk}, \quad \sigma^d = 2G\,\varepsilon^d$$

K compression modulus, G shear modulus

$$\Rightarrow \sigma_{ij} = K\,\varepsilon_{kk}\,\delta_{ij} + 2G\left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right)$$

$$= \underbrace{\left(K - \tfrac{2}{3}G\right)}_{=:\lambda}\varepsilon_{kk}\,\delta_{ij} + 2\underbrace{G}_{=:\mu}\,\varepsilon_{ij}$$

Lamé coefficients

Cauchy eq.

$$\Rightarrow \partial_j \sigma_{ij} = \lambda\,\partial_i\,\varepsilon_{kk} + 2\mu\,\partial_j\,\varepsilon_{ij}$$

$$= \lambda\,\partial_i\,\partial_k u_k + \mu\,\partial_j\left(\partial_i u_j + \partial_j u_i\right)$$

$$= (\lambda+\mu)\,\partial_i(\partial_k u_k) + \mu\,\partial_j\partial_j u_i$$

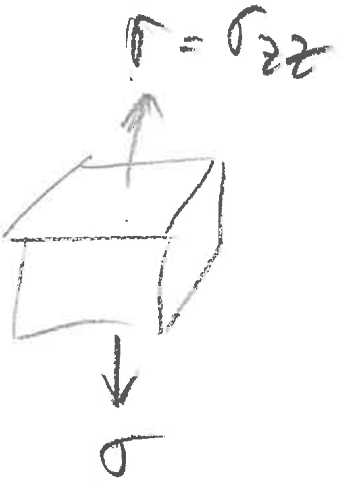
$$\Rightarrow \boxed{\underline{\nabla}\,\underline{\underline{\sigma}} = (\lambda+\mu)\,\underline{\nabla}(\underline{\nabla}\,\underline{u}) + \mu\,\Delta\,\underline{u} = \underline{0}}$$

force balance

As alternative to (K, G) or (λ, μ), one can also use (E, ν): $E = \frac{\mu(3\lambda + 2\mu)}{(\lambda+\mu)}$ Young's modulus

$\nu = \frac{\lambda}{2(\lambda+\mu)}$ Poisson ratio

(14)



A fully 3d analysis of uniaxial stretch gives

$$\underline{u} = \frac{\sigma}{E}\begin{pmatrix} -\nu x \\ -\nu y \\ z \end{pmatrix}$$

$$\Rightarrow \underline{\underline{\varepsilon}} = \frac{\sigma}{E}\begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \varepsilon_{ll} = \frac{\sigma}{E}(1-2\nu)$$

$\Rightarrow \nu = \frac{1}{2}$ incompressible

$$\varepsilon_{zz} = \frac{\sigma}{E}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu\sigma}{E}$$

We now can convert the 1d active continuum models into 3d:

Active gel / Maxwell

$$F = F_p + F_a$$

$$\dot{F}_p + \frac{1}{\tau}F_p = C\dot{\varepsilon}$$

$$\downarrow$$

$$(\dot{\sigma}_{ij} - \dot{\sigma}_{ij}^a) + \frac{(\sigma_{ij} - \sigma_{ij}^a)}{\tau} = \partial_t(\lambda\,\varepsilon_{ll}\,\delta_{ij} + 2\mu\,\varepsilon_{ij})$$

Active solid / Kelvin-Voigt

$$F = F_p + F_a$$

$$C\varepsilon + C_\eta\dot{\varepsilon} = F_p$$

$$\downarrow$$

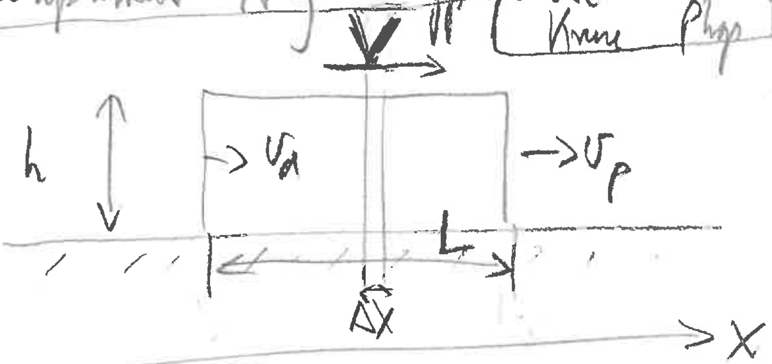
$$(\sigma_{ij} - \sigma_{ij}^a) = (1 + \tau\partial_t) \times (\lambda\,\varepsilon_{ll}\,\delta_{ij} + 2\mu\,\varepsilon_{ij})$$

$$\partial_j \sigma_{ij} = Y u_i \text{ (elastic foundation) or } = \xi \dot{u}_i \text{ (friction)}$$

Migrating cells as a slab of an active & polymerizing gel (15)

Active gel Theory: review Prost, Jülicher & Joanny Nat Phys 2015
Focus on slow (hydrodynamic) modes and symmetries,
write equations for thermodynamic pairs of fluxes and forces,
complement by appropriate BC

cell migration model Kruse Phys Biol 2006, IFF Spring School 2018



V cell velocity
v_p polymerization velocity
v_d depolymerization velocity
in x-direction

Again we perform the force balance for a thin slab:

$$-\int_0^h \sigma_{xx}(x,z)\,dz + \int_0^h \sigma_{xx}(x+\Delta x,z) - \int_x^{x+\Delta x} \sigma_{xz}(x',0)\,dx' + \int_x^{x+\Delta x} \underbrace{\sigma_{xz}(x',h)}_{=0 \text{ free BC}}\,dx' = 0$$

$\sigma_{xz}(x',0) = \xi v_x(x,0)$

$$\sigma(x) = \frac{1}{h}\int_0^h \sigma_{xx}(x,z)\,dz$$

$$\Rightarrow -h\,\sigma(x) + h\,\sigma(x+\Delta x) - \xi\, v(x)\,\Delta x = 0$$

$$\Rightarrow \partial_x \sigma = \frac{\xi}{h} v$$

$P > 0$

$$\sigma = \sigma_p + \sigma_a, \quad \sigma_p = \eta\dot\varepsilon = \eta\,\partial_x v, \quad \sigma_a = P = \text{const}$$

$$\Rightarrow \sigma = \eta\,\partial_x v + P = \frac{\eta h}{\xi}\partial_x^2\sigma + P$$

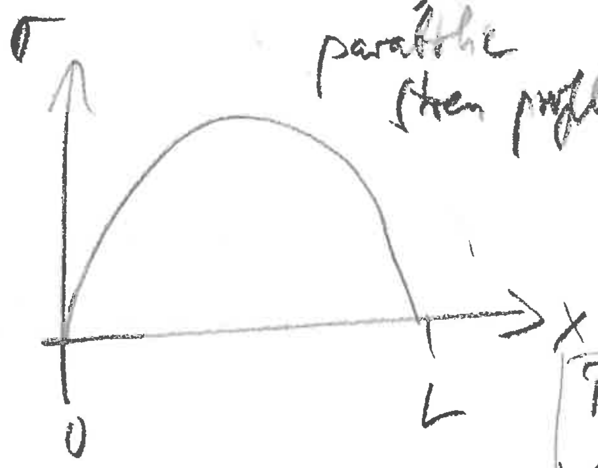
$$\Rightarrow \boxed{\lambda^2 \partial_x^2 \sigma - \sigma = -P} \qquad \lambda = \sqrt{\frac{\eta h}{\xi}} \text{ hydrodyn length scale}$$

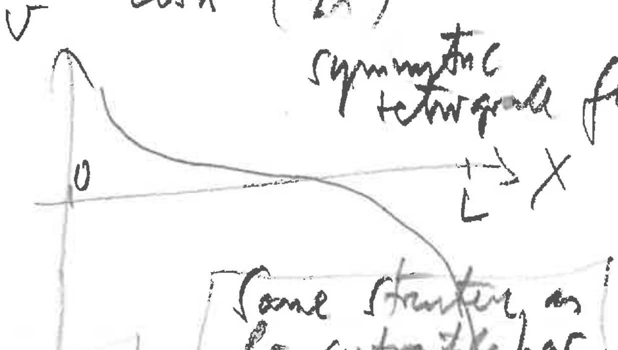
$\Rightarrow \sigma(x) = A e^{x/\lambda} + B e^{-x/\lambda} + P$

BCs : $\sigma(0) = \sigma(L) = 0$ free boundaries

$$\Rightarrow \sigma(x) = P \left(1 - \frac{\cosh\left(\frac{2x-L}{2\lambda}\right)}{\cosh\left(\frac{L}{2\lambda}\right)} \right)$$

$$v(x) = \frac{h}{\xi} \sigma' = \frac{-h P}{\xi \lambda} \frac{\sinh\left(\frac{2x-L}{2\lambda}\right)}{\cosh\left(\frac{L}{2\lambda}\right)}$$





$\boxed{\text{Both depend linearly on } P!}$

This solution is in the cell frame. Like how change to the substrate frame:

$v_p + v(L) = V, \quad v_d + v(0) = V$

because $v(0) = -v(L)$

v_p, v_d given $\Rightarrow$ $V = \frac{1}{2}(v_p + v_d)$

$$L = 2\lambda \, \operatorname{artanh}\left(\frac{\xi \lambda (v_p - v_d)}{2 h \sigma_a} \right)$$

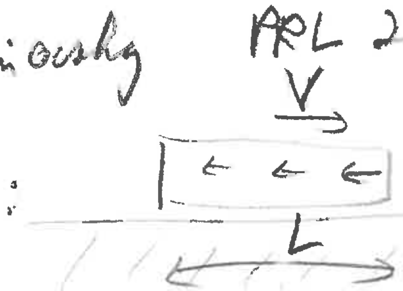
Problems: L cannot be fixed independently

Cell migration driven "by hand" through polymerisation at the edges

Minimal active gel model for contractility-based cell migration (17)

Drozdowski, Ziebert + Schwarz PRE 2021

Compare also Recho, Putelat + Truskinovsky PRL 2013



Constitutive relation for Maxwell model:

elastic, frame invariance

$$\left(1+\tau\partial_t+\tau v\partial_x\right)\left(\sigma-\sigma_a\right)=\eta\,\partial_t\varepsilon=\eta\,\partial_x v$$

Force balance: $\partial_x\sigma=\xi v$ friction

$$\Rightarrow \boxed{\left(1+\tau\partial_t+\overbrace{\frac{\tau}{\xi}\left(\partial_x\sigma\right)\partial_x}^{NL}\right)\left(\sigma-\sigma_a\right)=\lambda^2\,\partial_x^2\sigma}$$

$\lambda=\sqrt{\frac{\eta}{\xi}}$ hydrodyn. length scale

σ_a = const active stress

Elastic BCs for cell edges: $\sigma(l_\pm(t),t)=-k\,\frac{L(t)-L_0}{L_0}$

$$\dot{l}_\pm(t)=v(l_\pm(t))=\frac{1}{\xi}\,\partial_x\sigma\left(l_\pm(t),t\right)$$

Non-dimensionless equations: length L_0, time $\frac{\xi L_0^2}{k}$, stress k

$$\Rightarrow \mathcal{L}=\sqrt{\frac{\eta}{\xi L_0^2}},\quad \mathcal{T}=\frac{k\tau}{\xi L_0^2}$$

$\boxed{\mathcal{T}=0 \text{ purely viscous case}}$

rather $\tau=\eta/E$, $E\to\infty$

$$\Rightarrow \boxed{\mathcal{L}^2\,\partial_x^2\sigma-\mathcal{T}\,\partial_t\sigma-\overbrace{\mathcal{T}\left(\partial_x\sigma\right)^2}^{NL}-\sigma=-\sigma_{act}}$$

$$\sigma(l_\pm(t),t)=-\left(L(t)-1\right)$$

$$\dot{l}_\pm=\partial_x\sigma\left(l_\pm(t),t\right)$$

For steady state: $V = \text{const}$, $\dot{L} = 0$ (18)

$\Rightarrow$ map coordinate x to $u = \frac{x - l_-}{L}$, $0 \le u \le 1$

Stress deviation from RC: $s = \sigma + (L-1)$

$$\Rightarrow \frac{\ell^2}{L^2} \partial_u^2 s + J \frac{V}{L} \partial_u s - \frac{J}{L^2} (\partial_u s)^2 - s + \underbrace{(L-1)}_{= -\sigma_{act}}$$

$s(u_\pm) = 0$, $\partial_u s(u_\pm) = V \cdot L$

Rewriting with two dynamical variables ($\ell = L = J = 1$)

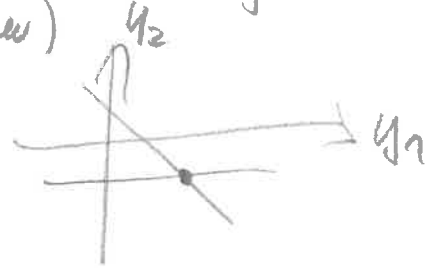
$y_1(u) = s(u)$, $y_2(u) = \partial_u s(u) - V$

gives $\partial_u y_1 = y_2 + V$ NL

$\partial_u y_2 = V y_2 + y_2^2 + y_1 - \sigma_{act}$

BC $y_1(0) = y_1(1) = 0$, $y_2(0) = y_2(1) = 0$

Nullclines (linear): $y_2 = -V$ $\quad y_2 = \frac{1}{V}(\sigma_{act} - y_1)$



fixed point not at $(0,0)$

$\Rightarrow$ period orbit required including $(0,0)$

However, we have a gradient system with

$$V = -\tfrac{1}{2} V y_2^2 - \tfrac{1}{3}(y_2)^3 - y_1 y_2 - V y_1 + \sigma_{act} y_2$$

$\Rightarrow$ no period orbit exists

$\Rightarrow$ no solution with $V \neq 0$ possible!

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Several mechanisms to get solutions with finite V:

→ include myosin concentration field
(Recho PRL 2013, Drozdowski Comm. Phys. 2023)

→ gradients generated by optogenetics
(PRE 2021)

→ polymerization at edges like Kruse
(PRE 2021)

→ adhesion dynamics (Wössner NJP 2024)

↳ compare open exercise and slides