Finite-size scaling of rigidity percolation

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July 2024

Using data from a simulation of rigidity percolation on a triangular lattice, we will seek to determine critical exponents and the shape of nonlinear universal scaling functions. We will also understand the systematic effects of the reduction of system size from infinity, using simulations of small systems to understand the behavior in the infinite system.

The model we will focus on is that of an infinite triangular lattice with harmonic spring connections between nearest neighbor sites (Figure 1). The bonds are randomly occupied with probability p. Under an infinitesimal strain ε_{ij}, a material has energy costs quadratic in the strain $2E = \varepsilon_{ij} C_{ijk\ell} \varepsilon_{k\ell}$. The infinite model undergoes a continuous phase transition at a particular value of $p = p_c$. All components of the linear elastic tensor vanish continuously at p_c.

In a simulation, we measure all independent components of the elastic tensor $C_{ijk\ell}$ as a function of the bond occupation probability p and the linear system size L by applying different types of strain. These measurements of the moduli are averaged over many instances of the disorder, giving an estimate of $C_{ijk\ell}(p, L)$. For an infinite system, we expect all components $C_{ijkl}(p, \infty) = 0$ below p_c, and $C_{ijkl}(p, \infty) \propto |\delta p|^f$ for $p > p_c$, where f is a *critical exponent* governing the growth of the elastic moduli on the rigid side of the transition. From renormalization group-type arguments, we expect a systematic dependence near the transition of the form

$$C_{ijk\ell}(p, L) \sim L^{-f/\nu} \mathcal{C}^{\pm}_{ijk\ell} \left(\frac{1}{|\delta p|^{\nu} L} \right). \tag{1}$$

Here ν is the critical exponent governing the growth of the sizes of rigid clusters and the effects of the system size; ν and f characterize the particular *universality class* of our phase transition. The functions $\mathcal{C}^{\pm}_{ijk\ell}(X)$ are similarly universal, and $\pm$ refer to the two branches of the function for $\delta p \equiv p - p_c > 0$ and $\delta p < 0$.[1] For large system sizes ($L \gg |\delta p|^{-\nu}$) with $p < p_c$, the system is floppy, and so $\mathcal{C}^{-}_{ijkl} \to 0$. We shall discuss the form of $\mathcal{C}^{+}_{ijkl}$ in part (b) below.

[1] The existence of two branches is an artifact of our choice of scaling variable, but this choice is standard. We could instead choose $X' = \delta p \, L^{1/\nu}$ in which case there is only one branch and X' can be negative.

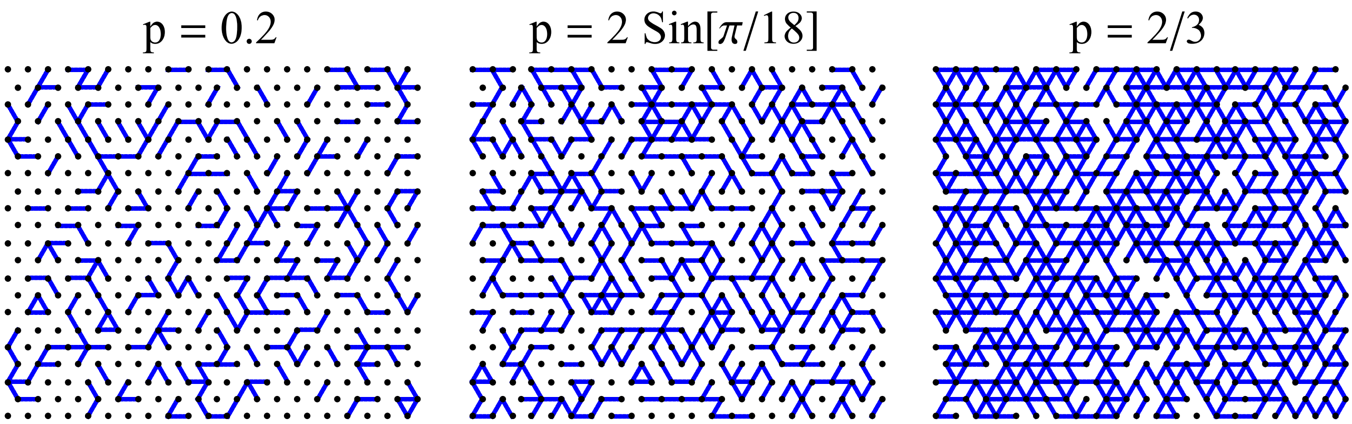

Figure 1: **A triangular lattice with its Hookean spring bonds randomly removed.** We seek to understand the rigidity of this model as we tune the fraction p of occupied bonds. (Left) At small values of p, the lattice is fragmented into sections that are not connected to each other. It deforms freely and hence has no linear elastic moduli. (Middle) At the geometric percolation threshold, we can (on average, in the infinite system) find a path from one side of the lattice to the other, which is a necessary condition for rigidity. However, this tenuously connected path can freely deform, and this lattice is still floppy. (Right) At a larger value of p, large clusters that can resist deformations are connected by thin trusses to other rigid clusters. The lattice is now macroscopically rigid.

Using the averaged data, we will seek to determine the nonuniversal location of the critical point p_c, together with the universal critical exponents f and ν and the shape of the universal scaling function $\mathcal{C}^{\pm}$. There is a famous argument due to Maxwell, which gives a rather good estimate for p_c by comparing the number of constraints in a system to the number of degrees of freedom.

(a) **Maxwell counting**: A point mass in two dimensions has two translational degrees of freedom. Attaching an inextensible rigid rod to the mass reduces the number of translational degrees of freedom by one, since it cannot move infinitesimally in a way that extends the rod. When the average site has no translational degrees of freedom, the triangular lattice is rigid. Use this to estimate that $p_c \approx 2/3$ for the triangular lattice. (Hint: in the fully occupied triangular lattice, each site has 6 attached bonds. However, there are 3 bonds per site.)

There are more detailed arguments that correct for redundant constraints, states of self-stress, additional zero modes, and second-order rigidity (lines of bonds at a buckling transition). But Maxwell counting does quite well, and the location of the transition is found through simulation to be only very slightly below $p = 2/3$. We now turn to the elastic moduli near the transition.

(b) **Critical exponents**: Suppose that the finite-size scaling in Equation (1) holds close to the transition. In the infinite system, $C_{ijk\ell} \sim |\delta p|^q$ as δp grows from 0. Write q in terms of ν and f. (Hint: Call X the scaling variable $(|\delta p|^{\nu} L)^{-1}$, and write the right-hand side of Equation (1) in terms of X and $|\delta p|$ alone. Define a new scaling function $\mathcal{C}^{+'}_{ijk\ell}(X) \equiv X^{f/\nu}\mathcal{C}^{+}_{ijk\ell}(X)$. Then fix $\delta p > 0$ and send $L \to \infty$. Assume $\mathcal{C}^{+'}(0) > 0$ is finite.)

We shall now do a *finite-size scaling collapse*, using Equation (1) to analyze numerical data for different system sizes. Performing the finite-size scaling collapse 1) allows us to extrapolate to determine physical quantities in the infinite system, 2) allows us to quantify the divergence of certain correlation lengths in the lattice upon approaching the critical point $\xi_1 \sim |\delta p|^{-\nu}$, and 3) allows us to determine interesting critical exponents relevant in the infinite system like f.

(c) **Scaling collapse**: We will perform a scaling collapse for the modulus C_{xxxx}. Write a routine to import data from the provided .csv files. The system sizes L are the leading numbers in the file names. We have provided you with the averaged data for $L = \{15, 30, 40, 50, 75, 100, 125, 500\}$, with fewer lattice instances for the larger system sizes. The first two columns contain the values of p and C_{xxxx} at that system size.

[i] Plot and admire the raw data (you should see some systematic dependence on system size that justifies trying a scaling collapse).

[ii] Pick values for $\{p_c, f, \nu\}$ and plot all of the data for $C_{xxxx}(p, L) \times L^{f/\nu}$ against $X \equiv (|p - p_c|^{\nu} L)^{-1}$ (log-log plots are useful here). Fiddle with the three parameters p_c, f, ν until the data are convincingly collapsed onto a single function $\mathcal{C}^{\pm}_{xxxx}(X)$. Is the Maxwell counting estimate of $p_c = 2/3$ compatible with your best collapse? See Figure 2 for an example of poorly collapsed and nicely collapsed data.

[iii] Try to quantify your errors in p_c, f, and ν based on your fiddling.

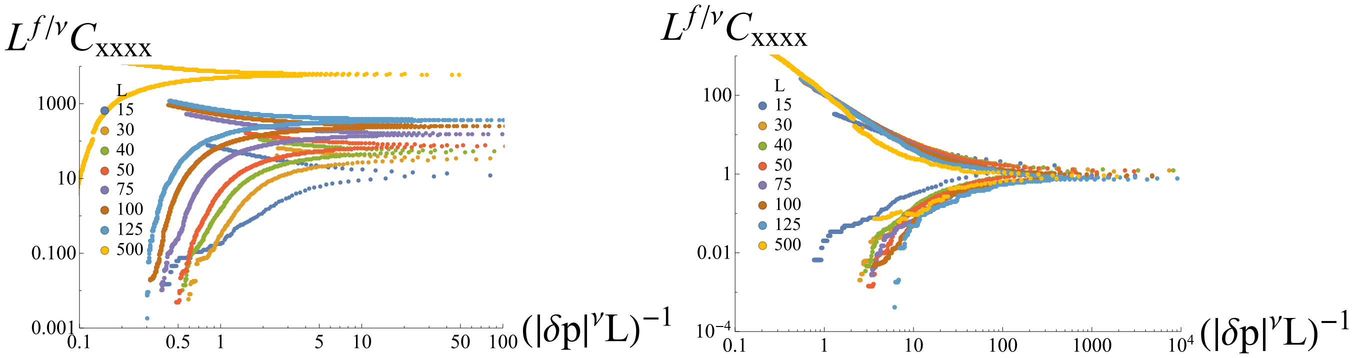

Figure 2: **Examples of collapse plots, using Equation** (1)**.** (Left) Poorly collapsed data. There is no clear pair of curves that could be identified as the upper and lower branches of a universal scaling function $\mathcal{C}^{\pm}_{xxxx}(X)$. There is also a strong dependence on L; we have not removed the leading-order systematic dependence on L with these scaling variables. (Right) Nicely collapsed data. We can clearly identify the two branches of the universal scaling function. There is, in principle, a small systematic dependence on L that remains; this is understood as a correction to scaling that enters as a higher power of L^{-1} than the leading-order scaling given by Equation (1). This is most noticeable at the smallest lattice sizes.