2 Basics of Floquet Theory

Periodically driven quantum systems

$$\boxed{\hat{H}(t+T) = \hat{H}(t)}$$

Solutions of time dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

$$\Rightarrow \boxed{|\psi(t)\rangle = e^{-i\varepsilon_\alpha t/\hbar} \, |\phi_\alpha(t)\rangle}$$

Floquet mode with same periodicity as the Hamiltonian

$$|\phi_\alpha(t+T)\rangle = |\phi_\alpha(t)\rangle$$

ε_α is called quasienergy

Inserting Ansatz $|\psi(t)\rangle$ into time dep. Schrödinger eq. yields:

$$\left(\hat{H}(t) - i\hbar \partial_t\right)|\phi_\alpha(t)\rangle = \varepsilon_\alpha |\phi_\alpha(t)\rangle$$

Fourier expansion: $\boxed{|\phi_\alpha(t)\rangle = \sum_\beta e^{i\beta\omega t} \, |u_\alpha^\beta\rangle}$

Time-dependent problem transformed into time-indep one

↳ The eigenvalue problem involves an infinite matrix

Quasienergies are only defined up to integer multiples of the driving energy quantum $n \cdot \hbar\omega$ with $n \in \mathbb{Z}$

⇒ Like spatially periodic systems, where quasimomentum is defined within the first Brillouin zone, the quasienergy can be defined in the range

$$\boxed{-\frac{\hbar\omega}{2} \leq \varepsilon_\alpha \leq +\frac{\hbar\omega}{2}}$$

Evolution Operator

Description in terms of the unitary time evolution operator $\hat{U}(t,t_0)$

$$|\psi(t_0)\rangle \xrightarrow{\hat{U}} |\psi(t)\rangle$$

which is a solution of the time dep. SEq

$$i\hbar\,\partial_t \hat{U}(t,t_0) = \hat{H}(t)\,\hat{U}(t,t_0)$$

$$\text{with } \hat{U}(t_0,t_0) = \hat{\mathbb{1}}$$

Periodic Systems

$$\hat{U}(t+T,0) = \hat{U}(t,0)\,\hat{U}(T,0)\;; \quad \hat{U}(t+T,T) = \hat{U}(t,0)$$

$$\Rightarrow \boxed{\hat{U}(nT,0) = [\hat{U}(T,0)]^n = (\hat{U}(T))^n} \qquad n \in \mathbb{N}$$

where $\hat{U}(T)$ is the time evolution operator over one period T

⇒ the long time behaviour of the system can be described by $\hat{U}(t)$ at times $t = n\cdot T$.

Floquet Theorem

Evolution of the system after one driving period can be described by an effective time-indep. Floquet Hamiltonian $\hat{H}_F$

$$\Rightarrow \qquad \hat{U}(nT) = [\hat{U}(T)]^n = e^{-i/\hbar \, nT \hat{H}_F}$$

or

$$\boxed{\hat{U}(T) = e^{-i/\hbar \, \hat{H}_F T}}$$

<u>Note:</u>
* effective Hamiltonian is not unique
* they are related via gauge transformations
* they share the same spectral and topological properties

In general it is difficult to find an analytic expression for the effective Floquet Hamiltonian !

<u>High-Frequency Limit</u>

$\omega = 2\pi/T$ large compared to other characteristic energy scales

$\Rightarrow$ <u>$\hat{H}_F$ can be computed perturbatively</u>

<u>Note:</u> Having computed $\hat{H}_F$, the theoretical discussion simplifies and for most cases we can study the dynamics of the system using the established techniques for undriven systems

e.g. topological invariant to characterize quasienergy bands $\rightarrow$ Chern number ...

Magnus Expansion useful for description of stroboscopic long-term dynamics

$$\boxed{\hat{H}_F = \sum_{n=0}^{\infty} \hat{H}_F^{(n)}}$$

with $\hat{H}_F^{(0)} = \frac{1}{T}\int_0^T \hat{H}(t)\, dt$

$$\hat{H}_F^{(1)} = -\frac{i}{2\hbar T}\int_0^T \int_0^{t_2} [\hat{H}(t_2), \hat{H}(t_1)]\, dt_1\, dt_2$$

$\vdots$

+ higher order terms scaling as $1/\omega^n$

Resonant Driving additional terms in the Hamiltonian $\hat{H}_0$ that diverge with $\omega \to \infty$ denoted as $\hat{H}_\omega$

$\Rightarrow$ perform a unitary transformation

$$\hat{R}(t) = e^{i/\hbar\, \hat{H}_\omega t}$$

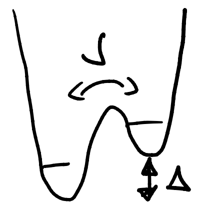
$$\rightsquigarrow \boxed{\mathcal{H}(t) = \hat{R}(t)\hat{H}(t)\hat{R}^+(t) - i\hbar\, \hat{R}(t)\,\partial_t \hat{R}^+(t)}$$

Refs: N. Goldman, J. Dalibard, M. Aidelsburger, N.R. Cooper Phys. Rev. A 91, 033632 (2015) (resonant drive)

N. Goldman and J. Dalibard Phys. Rev. X 4, 031027 (2014)

S. Rahav, I. Gilary and S. Fishman Phys. Rev. A 68, 013820 (2003)

EXAMPLE: ① driven TLS !



e.g. spatial DW with offset

$$\hat{H} = J\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) + \Delta |1\rangle\langle 1|$$
$$+ V_0 \cos(\omega t + \varphi)\, |0\rangle\langle 0|$$

Resonant driving $\hbar\omega = \Delta$

↳ Perform transformation into rotating frame

$$\hat{R}(t) = e^{\,i\omega |1\rangle\langle 1| t + \frac{i}{\hbar}\int^t V(t')\,dt'\, |0\rangle\langle 0|}$$

↳ $$\boxed{\mathcal{H}(t) = J\, |0\rangle\langle 1|\, e^{i\eta(t)} + J\, |1\rangle\langle 0|\, e^{-i\eta(t)}}$$

→ CHECK ! as Exercise

with $\eta(t) = -\left(\omega t - \frac{V_0}{\hbar\omega}\sin(\omega t + \varphi)\right)$

Lowest-order Floquet Hamiltonian using Magnus expansion:

$$\mathcal{H}_F = \frac{1}{T}\int_0^T \mathcal{H}(t)\,dt$$

$$\underset{\tau = \omega t}{=} \frac{1}{2\pi}\int_0^{2\pi}\left[J\, |0\rangle\langle 1|\, e^{-i\left(\tau - \frac{V_0}{\hbar\omega}\sin(\tau+\varphi)\right)} + h.c.\right] d\tau$$

$$= J\, \mathcal{J}_1\!\left(\frac{V_0}{\hbar\omega}\right) \underbrace{e^{i\varphi}}_{\text{inherited from drive phase}}\, |0\rangle\langle 1| + h.c.$$

where $$\mathcal{J}_1(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(\tau - x\sin\tau)}\, d\tau$$

⇒ In the presence of resonant periodic modulation, the coupling between levels $|0\rangle$ and $|1\rangle$ is restored with an effective coupling strength

$$\boxed{J_{eff} = J\, \mathcal{J}_1\left(\frac{V_0}{\hbar\omega}\right) e^{i\varphi}}$$

phase of modulation appears as complex tunneling term

Multi-Photon Process

$$\hat{H}(t) = J\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) + \Delta |1\rangle\langle 1|$$
$$+ V_0 \cos(\omega t + \varphi)\, |0\rangle\langle 0|$$

where $\boxed{\Delta = \nu \cdot \hbar\omega}$ and $\nu \in \mathbb{Z}$

↳ same transformation to rotating frame as before $\hat{\Omega}(t)$

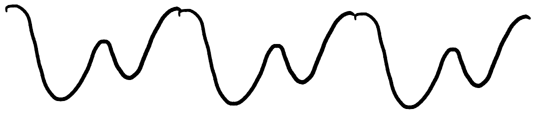
↳ effective Floquet Hamiltonian

$$\boxed{\hat{\mathcal{H}}_F = J \underbrace{\mathcal{J}_\nu\left(\frac{V_0}{\hbar\omega}\right)}_{} \left[e^{i\nu\varphi}\, |0\rangle\langle 1| + e^{-i\nu\varphi}\, |1\rangle\langle 0|\right]}$$

Bessel function of first kind of order ν

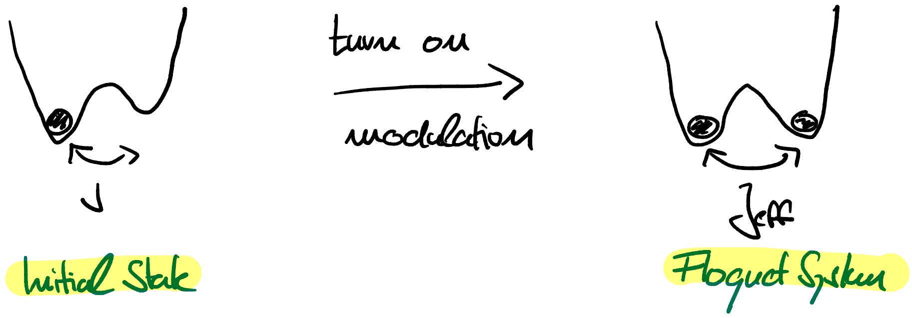
$$\mathcal{J}_\nu(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(\nu\tau - x\sin\tau)}\, d\tau$$

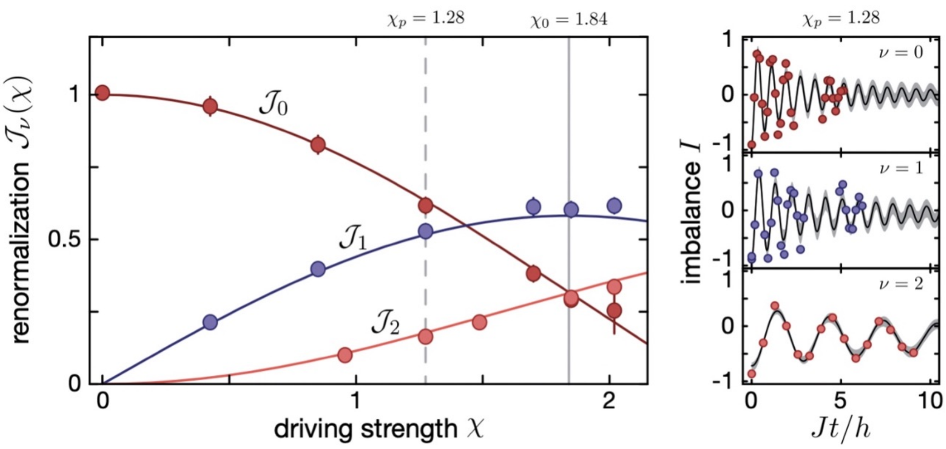
Experiments in isolated double wells



↳ tune modulation amplitude and modulation frequency

↳ prepare initial state, where atoms are localized on lattice site with lower energy





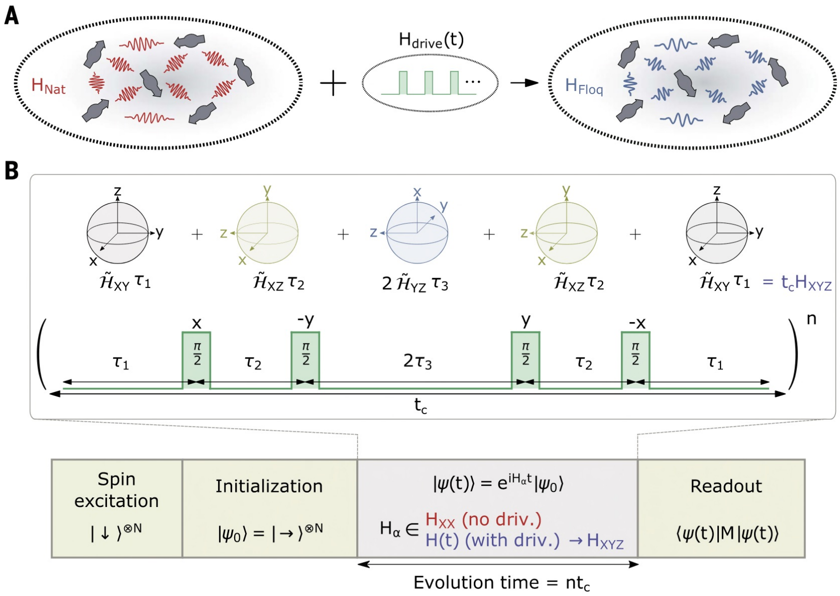
C. Schweizer et al. Nat. Phys. 15, 1168 (2019)

② Engineering XYZ Hamiltonian from XX + drive

$$H(t) = H_{XX} + H_{drive}(t)$$

$$= \sum_{ij} J_{ij}/\hbar \left(\hat{S}_x^i \hat{S}_x^j + \hat{S}_y^i \hat{S}_y^j \right) + \sum_i \Omega_i(t) \left[\cos\phi(t) \hat{S}_x^i + \sin\phi(t) \hat{S}_y^i \right]$$

S. Geier et al. Science 374, 1149 (2021)



$$\hookrightarrow \quad \boxed{H_F^{(0)} \sim \frac{2}{3} \sum J_{ij}/\hbar \left(\delta_x \hat{S}_x^i \hat{S}_x^j + \delta_y \hat{S}_y^i \hat{S}_y^j + \delta_z \hat{S}_z^i \hat{S}_z^j \right)}$$

$$\delta_x = 1 - v + u \qquad \delta_y = 1 + w - u \qquad \delta_z = 1 - w + v$$

$$\tau_1 = \tau(1 - 2v + 2w), \quad \tau_2 = \tau(1 + 2u - 2w), \quad 2\tau_3 = 2\tau(1 - 2u + 2v)$$

SUMMARY: * Floquet Systems powerful way for Hamiltonian Engineering

- Artificial Gauge Fields
- XXZ Tuning

* **HOWEVER:** There are cases where physics of driven system cannot be captured by effective Hamiltonian. Example: Anomalous Floquet System

see: M. Rudner, N. Lindner, E. Berg, M. Levin PRX 3, 031005 (2013)