(Random) Quantum Circuits

(see first part of "Random Quantum Circuits", MPA Fisher, V. Khemani, A. Nahum, SV, ARCMP 2023, and references therein)

An introduction to discrete-time models of unitary quantum many-body dynamics, in order

(1.) To understand universal structures that emerge in a generic quantum many-body system driven far from its ground-state, particularly in "information-theoretic" observables of quantum correlations (e.g. measures of quantum entanglement) which are sensitive to the process of thermalization and which can quantify the hardness of classically simulating quantum many-body evolution.

(2.) To understand the dynamical regimes of quantum many-body evolution which can arise in platforms for quantum simulation/computation in which discrete time-evolution is naturally implemented ("gate-based" platforms, e.g. superconducting qubits, see Z. Minev & S. Girvin's lectures in week 4)

A natural starting point to investigate these questions is to study minimally-structured quantum many-body evolution which retain

(i.) locality in d spatial dimensions

(ii.) unitarity (if considering a closed q. system)

but no other symmetries that may arise in unitary evolution by a particular quantum many-body Hamiltonian.

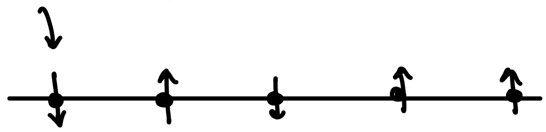
⊛ Instead of studying any one particular minimally-structured evolution, we can study an ensemble of quantum many-body dynamics (more theoretical tractability) and use this as a starting point to study typical instances

(The philosophy behind this approach should be familiar, e.g. random matrix theory, the SYK model)

I. Quantum Gates & Quantum Circuits (Introduction & Notation)

each site j hosts a two-level system

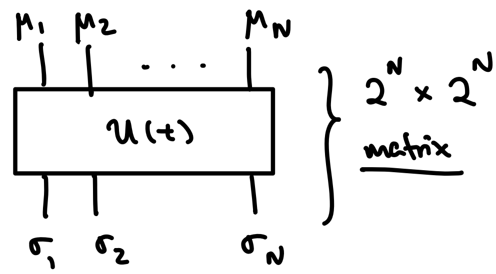
Consider a chain of N qubits



A time-independent Hamiltonian $\mathcal{H}$, defines unitary time evolution $\mathcal{U}(t) = e^{-i\mathcal{H}t}$

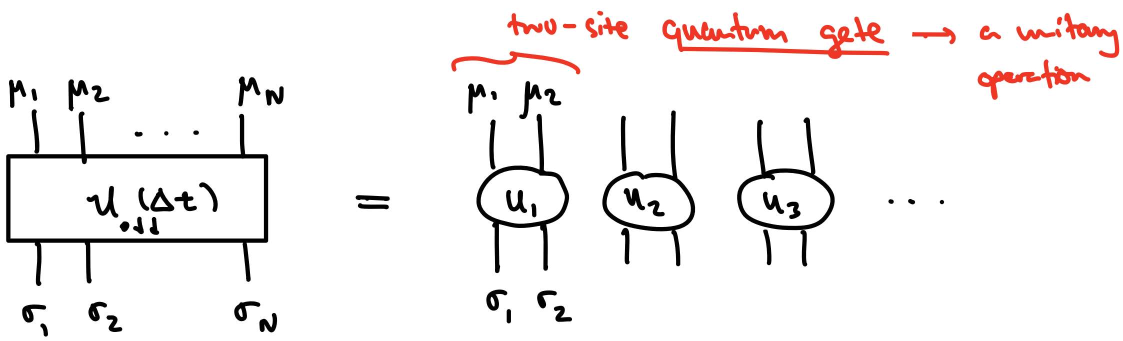
We will introduce a tensor network representation

$$\langle \mu_1, \ldots, \mu_N | \, \mathcal{U}(t) \, | \sigma_1, \ldots, \sigma_N \rangle =$$



Suppose the interactions were only on odd bonds $\mathcal{H}_{odd} = \sum_{j=1}^{N} h_{2j-1,2j}$

turned on for a time interval Δt

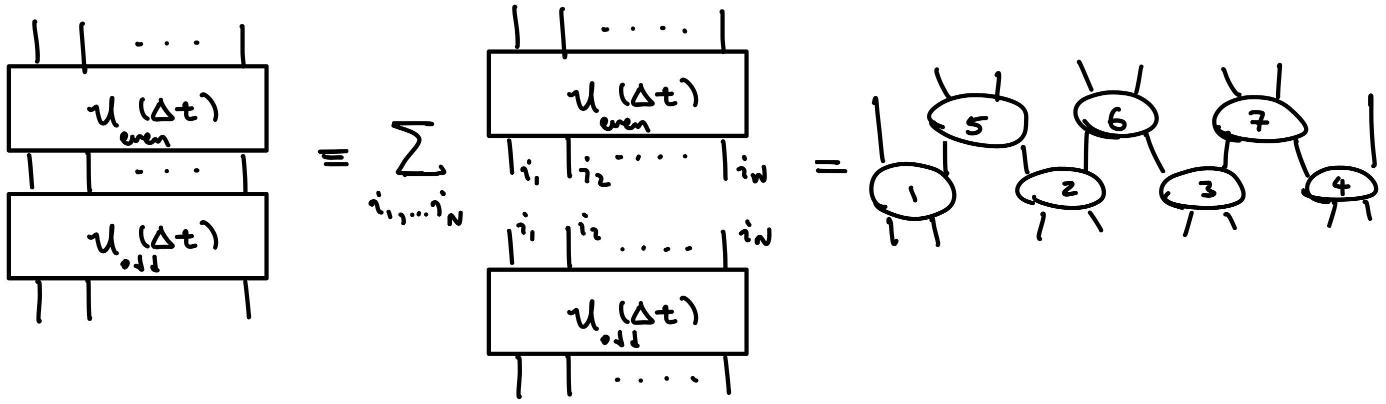


(1.)

Each $u_j = e^{-i h_{2j-1,2j} \Delta t}$ defines a two-site quantum gate, a unitary operation coupling some collection of qubits. A product of quantum gates defines a quantum circuit on the N-qubit system.

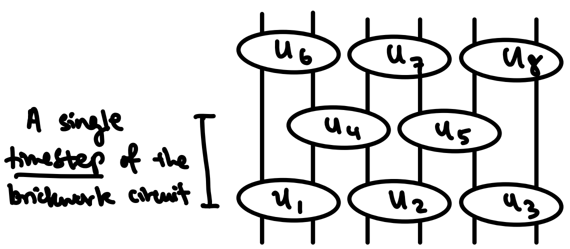
⊛ Note : u_j is not assumed to be close to the identity, so a quantum circuit does not necessarily describe a "Trotterized" Hamiltonian evolution.

Afterwards, we "switch off" these interactions and "switch on" interactions on even bonds

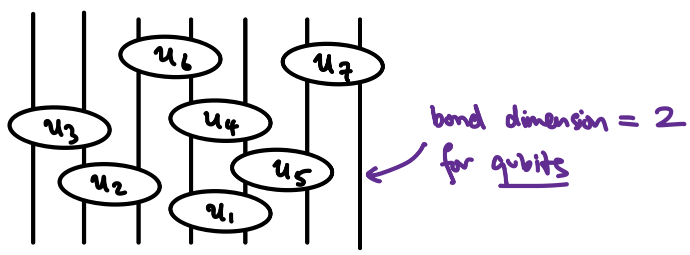


(2.)

Repeating this gives rise to a "brickwork" array of two-site unitary gates. (Note that we haven't specified the unitary gates, themselves...) We will often study quantum circuits with a brickwork geometry but will consider other geometries as well.



"Brickwork" array



A more irregular geometry...

II. Random unitary gates: choose unitary gates drawn from some ensemble

A particularly natural choice is given by a probability distribution over $U \in U(d)$ unitary gates which is invariant under

$$U \longmapsto V U W \qquad (3.)$$

where V, W are unitary, and independent of U.

Eq. (2) is a defining property of a unique distribution over $U(q^2)$ known as the Haar measure. We will not prove this statement, but will provide some elementary examples before proceeding.

Example: Haar measure on $U(1)$. The group elements $e^{i\theta}$ parametrized by $\theta \in [0, 2\pi]$. Invariance requires that the probability density $p(\theta) = p(\theta + \varphi)$ for any φ so $p(\theta) = \text{constant}$.

To conclude, the Haar measure for $U(1)$ is $\frac{d\theta}{2\pi}$, the uniform measure on S^1.

Exercise: an $SU(2)$ matrix $U = x_0 \mathbb{1} + i x_1 \sigma^x + i x_2 \sigma^y + i x_3 \sigma^z$ where $\sum_{j=0}^{3} x_j^2 = 1$. Show that the Haar measure for $SU(2)$ is the uniform measure on S^3 by the invariance property of the Haar measure.

The Haar measure for the unitary group on a d-dimensional Hilbert space: Recall invariance of the Haar measure which requires that

$$\mathop{\mathbb{E}}_{U\sim \text{Haar}} \left[(U \otimes U^*)^{\otimes n} \right] = \mathop{\mathbb{E}}_{U\sim \text{Haar}} \left[VUW \otimes V^* U^* W^* \otimes \cdots \right]$$

for any $V, W \in U(d)$.

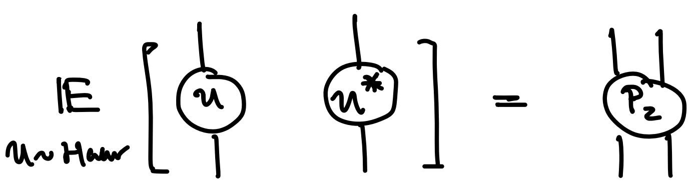
(Note $\mathop{\mathbb{E}}_{U\sim\text{Haar}} \left[U^{\otimes n} \otimes (U^*)^{\otimes m} \right] = 0$ if $n \neq m$ by invariance, Eq. (3))

More generally, define $P_n \equiv \mathop{\mathbb{E}}_{U\sim \text{Haar}} \left[(U \otimes U^*)^{\otimes n} \right]$ and note that P_n is Hermitian (prove this by uniqueness of Haar measure) and observe

$$\boxed{P_n^2 = P_n} \qquad (4.)$$

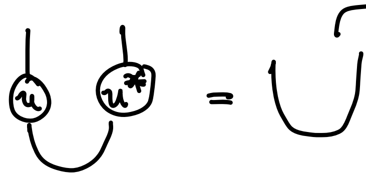
by the invariance of the Haar measure. And as a result, P_n is a projection operator in a d^{2n}-dimensional Hilbert space.

What is the Hilbert space that P_n projects onto? P_1 for example



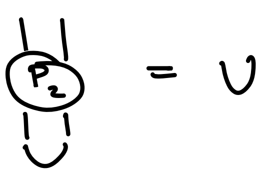
$\sum_{i=1}^{d} |ii\rangle$

Unitarity requires that



for every U.

And so



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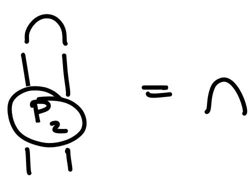
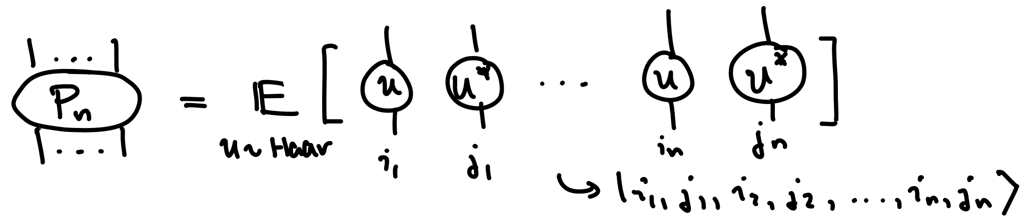


$P_1 = \frac{1}{d} \sum_{ij} |ii\rangle\langle jj|$ is therefore the projection operator onto this symmetric Hilbert space. As a consequence, $\mathbb{E}_{u\sim \text{Haar}} \left(u M u^\dagger \right) = \frac{\text{Tr}(M)}{d} \mathbb{1}_{d\times d}$

More generally,



Any $|\sigma\rangle \equiv \sum_{i_1,\dots,i_n} |i_1, i_{\sigma(1)}, i_2, i_{\sigma(2)}, \dots, i_n, i_{\sigma(n)}\rangle$ where $\sigma \in S_n$ is an element of the permutation group on n elements satisfies

$$P_n |\sigma\rangle = |\sigma\rangle \qquad (5.)$$

P_n projects onto the permutation-invariant ("symmetric") Hilbert space spanned by $\{|\sigma\rangle\}$. This result can be made rigorous by Schur-Weyl duality (see notes below in red box and reference therein). Ultimately,

$$P_n = \sum_{\sigma,\tau \in S_n} Wg^{(n)}(d; \sigma^{-1}\tau) \, |\sigma\rangle\langle\tau| \qquad (6.)$$

where the Weingarten function $Wg^{(n)}(d; \mu)$ only depends on the conjugacy class of $\mu \in S_n$.

A more rigorous form of the result above (see e.g. A. Harrow, arXiv: 1308.6595) is due to Schur-Weyl duality

Let $\mathcal{O}$ be an operator on the Hilbert space $\mathcal{H}^{\otimes n}$, where $\mathcal{H}$ is a d-dimensional Hilbert space. Let $\mathcal{U}(\mathcal{H})$ be the unitary group on $\mathcal{H}$.

Thm. (Schur-Weyl duality)

$$[\mathcal{O}, W^{\otimes n}] = 0 \quad \text{for } \underline{\text{any}} \quad W \in \mathcal{U}(\mathcal{H})$$

<u>if and only if</u> $\mathcal{O}$ is a linear combination of permutations i.e.

$\mathcal{O} = \sum_{\sigma \in S_n} c_\sigma \Pi_\sigma$ where $\Pi_\sigma |i_1, i_2, \dots, i_n\rangle = |i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}\rangle$.

Consider $F(M) \equiv \mathbb{E}_{\mathcal{U} \sim \text{Haar}} \left(\mathcal{U}^{\otimes n} M (\mathcal{U}^\dagger)^{\otimes n} \right)$, which satisfies the following properties

(i.) $[V^{\otimes n}, F(M)] = 0$ for any unitary V on the d-dim. Hilbert space by left invariance $\mathcal{U} \mapsto V \cdot \mathcal{U}$ of the Haar measure. And so,

$$F(M) = \sum_{\sigma \in S_n} c_\sigma(M) \cdot \Pi_\sigma$$

where $c_\sigma(M)$ are coefficients which may be written as $\mathrm{Tr}(M \cdot \mathcal{O}_\sigma)$ where $\mathcal{O}_\sigma$ is some operator which depends on the choice of $\sigma \in S_n$.

(ii.) Right-invariance $\mathcal{U} \mapsto \mathcal{U} V$ requires $c_\sigma(M) = c_\sigma(V^{\otimes n} M (V^\dagger)^{\otimes n})$ so that $[\mathcal{O}_\sigma, V^{\otimes n}] = 0$ for any unitary V. Therefore,

$$\mathcal{O}_\sigma = \sum_{\mu \in S_n} w_{\sigma\mu} \Pi_\mu$$

by Schur-Weyl duality and $\boxed{F(M) = \sum_{\sigma, \mu \in S_n} w_{\sigma\mu} \mathrm{Tr}(M \Pi_\mu) \Pi_\sigma}$

(ii.) Observe that $F(\Pi_\mu) = \Pi_\mu$ for all $\mu \in S_n$, and that

$$\mathrm{Tr}(\Pi_\sigma) = \sum_{\{i\}} \left(\prod_{k=1}^{n} \delta_{i_k, i_{\sigma(k)}} \right) = d^{\chi(\sigma)} \quad \text{where}$$

$\chi(\sigma)$ is the number of <u>cycles</u> in permutation $\sigma \in S_n$.

(iii.) $F(\Pi_\tau) = \Pi_\tau = \sum_{\sigma, \mu \in S_n} w_{\sigma,\mu} \, d^{\chi(\tau\mu)} \, \Pi_\sigma$. Therefore,

$$\boxed{\sum_{\mu \in S_n} d^{\chi(\tau\mu)} w_{\sigma,\mu} = \delta_{\sigma,\tau}}$$

Shift $\sigma \to \mu_1 \sigma \mu_2^{-1}$ and $\tau \to \mu_1 \tau \mu_2^{-1}$ so that

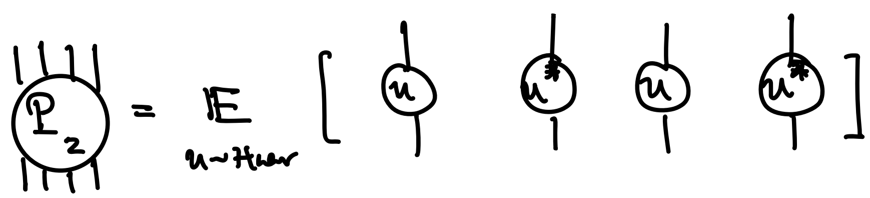
$$\sum_{\mu \in S_n} d^{\chi(\mu_1 \tau \mu_2^{-1} \mu)} w_{\mu_1 \sigma \mu_2^{-1}, \mu} = \sum_{\mu \in S_n} d^{\chi(\tau\mu)} w_{\mu_1 \sigma \mu_2^{-1}, \mu_1 \mu \mu_2^{-1}} = \delta_{\sigma,\tau}$$

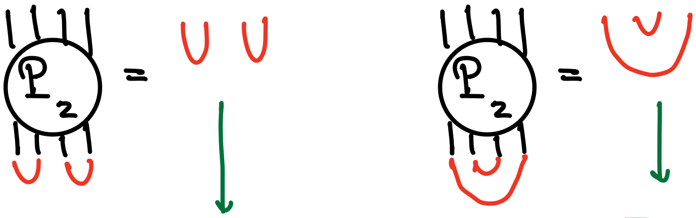
and

$$w_{\sigma,\tau} = w_{\mu_1 \sigma \mu_2^{-1}, \mu_1 \tau \mu_2^{-1}} \quad \text{for any } \mu_1, \mu_2 \in S_n$$

$w_{\sigma,\tau}$ only depends on the conjugacy class of $\sigma^{-1}\tau$. Two permutations belong to the same conjugacy class <u>iff</u> they have the same number of cycles i_k of length k, for each k. Therefore $w_{\sigma,\tau}$ (Weingarten function) depends only on this <u>cycle decomposition</u> of $\sigma^{-1}\tau$.

<u>Example</u> :

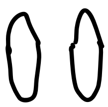
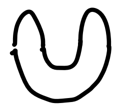




$$|\uparrow\rangle \equiv \sum_{ij} |iijj\rangle \qquad |\downarrow\rangle \equiv \sum_{ij} |ijji\rangle \qquad (7.)$$

So P_2 is a projector onto span$(|\uparrow\rangle, |\downarrow\rangle)$.

These states are <u>unnormalized</u>, and <u>not orthogonal</u>

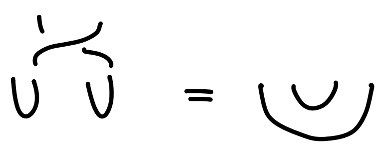
$\langle\uparrow|\uparrow\rangle =$  $= d^2$, $\quad \langle\uparrow|\downarrow\rangle =$  $= d$. (8.)

(1) and (2) together can be used to determine that (<u>exercise!</u>)

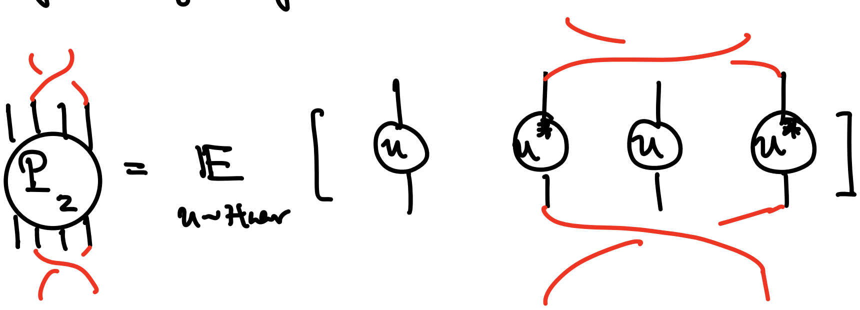
$$P_2 = \frac{1}{d^2-1}\left[\,|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| - \frac{1}{d}\left(|\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow|\right)\right]. \qquad (9.)$$

Note that $\begin{matrix}|\uparrow\rangle \to |\downarrow\rangle \\ |\downarrow\rangle \to |\uparrow\rangle\end{matrix}$ leaves P_2 invariant. This Ising symmetry

is implemented by a <u>swap operation</u> SWAP_{24}

$\mathrm{SWAP}_{24}|\uparrow\rangle =$  $= |\downarrow\rangle$ (10)

which is clearly a symmetry of P_2 since



(11)

this simply exchanges two copies of u^* which leaves the Haar average invariant.

P_n is clearly invariant under permutations of U^*, which may be written as

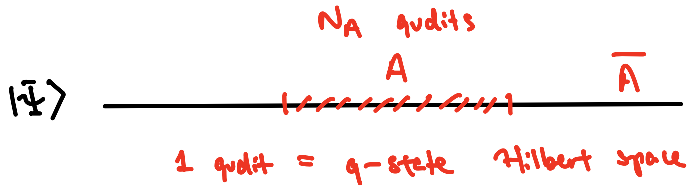
$$\Pi^{(1)}_{\mu} P_n \Pi^{(1)}_{\mu^{-1}} = P_n .$$

where $\underbrace{\Pi^{(1)}_{\mu} |\sigma\rangle}_{\text{Left multiplication by } \mu.} = \sum_{\{i\}} | i_1, i_{\mu(\sigma(1))}, i_2, i_{\mu(\sigma(2))}, \dots \rangle = |\mu\sigma\rangle$

P_n is also invariant under permutations of U, $\Pi^{(2)}_{\mu} P_n \Pi^{(2)}_{\mu^{-1}} = P_n$, where $\Pi^{(2)}_{\mu} |\sigma\rangle = \sum_{\{i\}} | i_{\mu(1)}, i_{\sigma(1)}, i_{\mu(2)}, i_{\sigma(2)}, \dots \rangle$. Easy to show that $\Pi^{(2)}_{\mu} |\sigma\rangle = |\sigma\mu^{-1}\rangle$.

Generally $(S_n \times S_n) \rtimes \mathbb{Z}_2$ symmetry where $\mathbb{Z}_2$ is $U \longleftrightarrow U^*$ followed by complex conjugation, which takes $\sigma \longrightarrow \sigma^{-1}$.

III. Quantifying Bipartite Entanglement: the von Neumann entanglement entropy of a reduced density matrix on a subregion A.



$$\rho_A \equiv \mathrm{Tr}_{\bar{A}}\left[|\psi\rangle\langle\psi|\right]$$

The reduced density matrix ρ_A contains information about all of the correlations within A. ρ_A is Hermitian, has non-negative eigenvalues and unit trace $\mathrm{Tr}_A\, \rho_A = 1$.

The Neumann entropy of ρ_A is a measure of quantum correlations in A, and is defined as

$$S[\rho_A] = -\mathrm{Tr}\left[\rho_A \log \rho_A\right] = -\sum_{i=1}^{D_A} \lambda_i \log \lambda_i \qquad (12)$$

where $\lambda_i \geq 0$ are eigenvalues of ρ_A.

(i.) Basis-independent: $S[\rho_A] = S[V \rho_A V^\dagger]$ for any V acting exclusively within A.

(ii.) Measures the cost of storing the state on a classical computer

(iii.) $0 \leq S[\rho_A] \leq N_A \cdot \log q$. Max. value attained iff $\rho_A \sim \mathbb{1}_A$.

(Many other nice properties enjoyed by S which we will use as needed...)

For calculational convenience we will often study the Rényi entanglement entropies

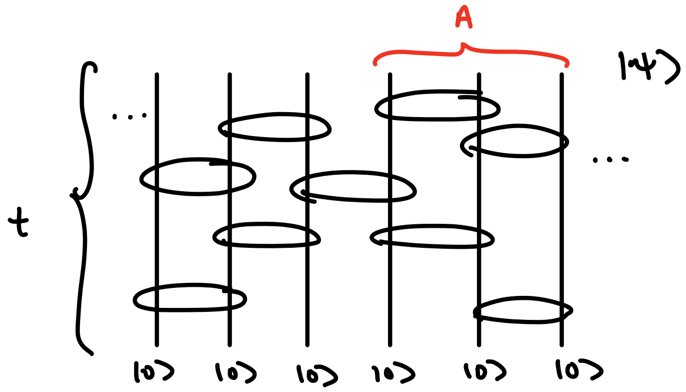
$$S_A^{(n)} \equiv \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n \qquad (13)$$

The von Neumann entropy is formally recovered as $S_A = \lim_{n \to 1} S_A^{(n)}$. Since ρ_A has non-negative eigenvalues which sum to 1,

$$S_A^{(m+1)} \leq S_A^{(m)} \quad \text{for all } m. \qquad (14)$$

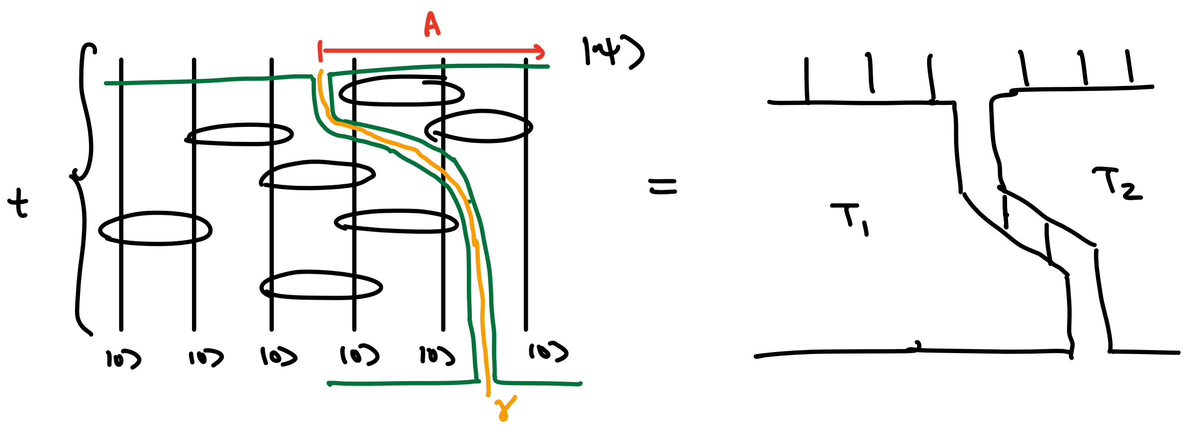
IV. Bounding Entanglement: the "Minimal Cut" Prescription

How does the entanglement of a product initial state grow under evolution by a quantum circuit?



$$\boxed{S_A(t) = ?}$$

Let us represent the wavefunction $|\Psi\rangle$ as a product of two tensors T_1 & T_2:



so $$|\Psi\rangle = \sum_{i,j,\alpha} (T_1)_{i\alpha} (T_2)_{\alpha j} \underbrace{|i\rangle_A}_{\text{complete basis in } A} \otimes \underbrace{|j\rangle_B}_{\text{complete basis in } B}$$

and $$\rho_A = \sum_{ij\alpha j'\beta} (T_1)_{i\alpha} (T_2)_{\alpha j} (T_1^*)_{i\beta} (T_2^*)_{\beta j'} |j\rangle_B \langle j'|$$

$$= T_2^\dagger T_1^\dagger T_1 T_2$$

The rank of ρ_A is at most the minimum bond dimension appearing in this product, i.e.

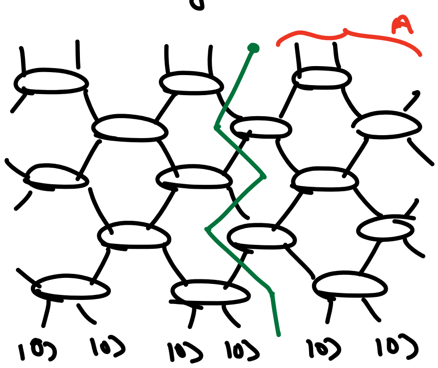
$$\mathrm{rank}(\rho_A) \leq \min\left(D_A, D_{\bar{A}}, q^{N_{\text{bonds}}(\gamma)}\right) \tag{15}$$

Best bound on $\mathrm{rank}(\rho_A)$ obtained by a minimal cut γ which intersects the fewest bonds and bipartitions the state $|\psi\rangle$ between A, $\bar{A}$.

$$S_A(t) \leq \min\left(N_A, N_{\bar{A}}, N_{\text{bonds}}(\gamma)\right) \times \log q \tag{16}$$

Cut becomes a d-dimensional surface in higher dimensions

Brickwork unitary circuit : all cuts have the same cost

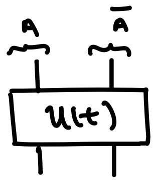


$N_\gamma = 2t$ so $S_A(t) \leq 2t \cdot \log q$ for a semi-infinite region A.

V. Entanglement Dynamics in the Haar-Random Circuit

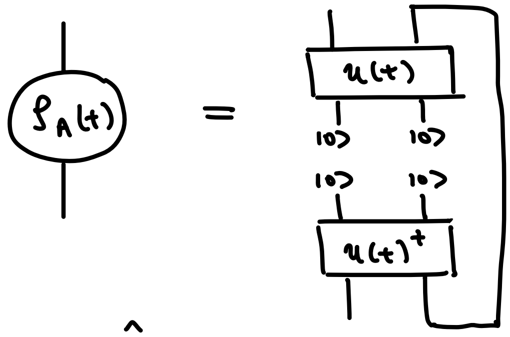
Purity of a sub-region $\mathrm{Tr}\, \rho_A(t)^2 = e^{-S_A^{(2)}}$. Note that ρ_A is

pure $\rho_A = |\varphi\rangle\langle\varphi|$ iff $\rho_A^2 = \rho_A$ so $\mathrm{Tr}\, \rho_A^2 = 1$.

Let  be the brickwork array of random unitary

gates for t timesteps, starting with a product initial state on $N_A + N_{\bar{A}}$

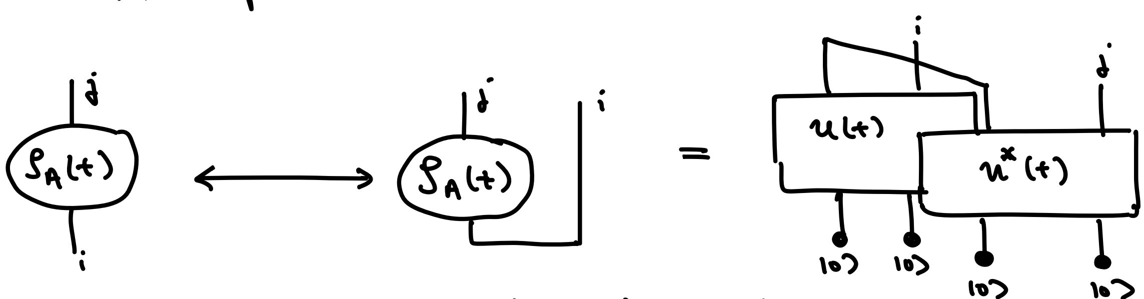
qudits $|0\rangle^{\otimes N}$:



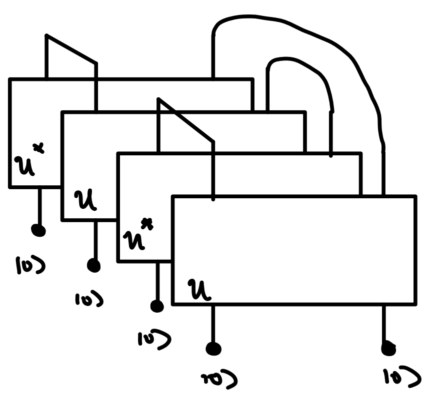
(17.)

Convenient to represent $\hat{\rho}_A(t)$ as an unnormalized wavefunction in

a doubled Hilbert space



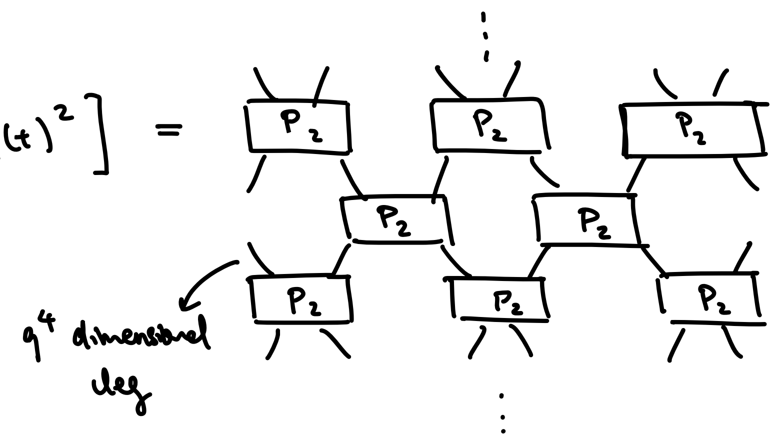
$$\sum_{ij} (\rho_A(t))_{ij} |i\rangle\langle j| \longleftrightarrow \sum_{ij} (\rho_A(t))_{ij} |i\rangle |j\rangle$$

$\mathrm{Tr}_A\, \rho_A(t)^2 =$ 

(18)

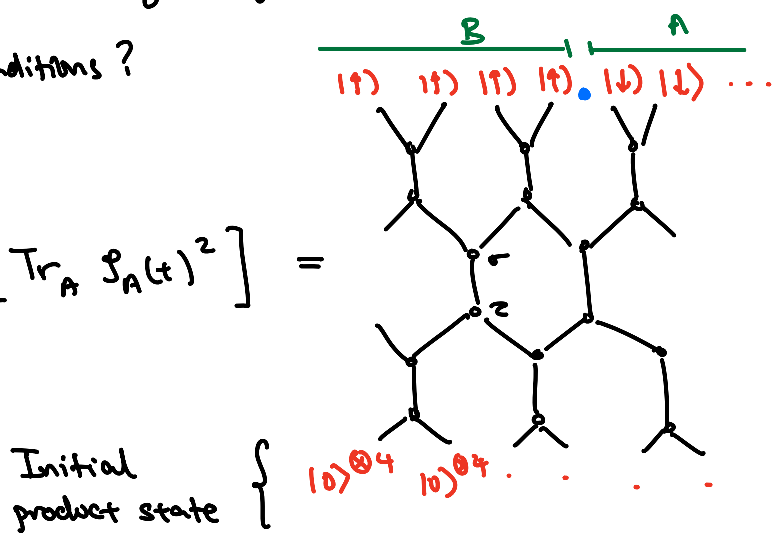
The Haar average can now be done independently for each two-site unitary, which converts $\mathrm{Tr}_A \rho_A^2(t)$ into

$$\mathop{\mathbb{E}}_{\text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t)^2 \right] = \tag{19}$$



Two Ising degrees of freedom per unitary. What about boundary conditions?

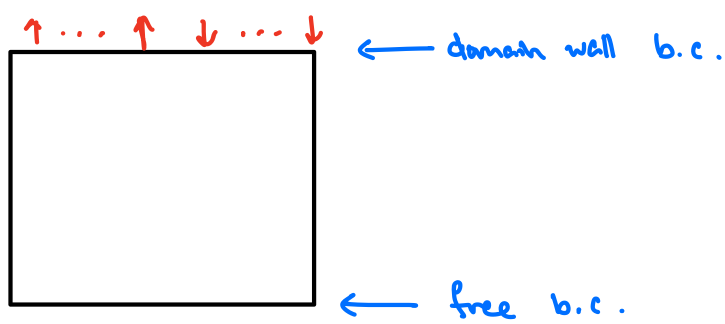
$$\mathop{\mathbb{E}}_{\text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t)^2 \right] =$$

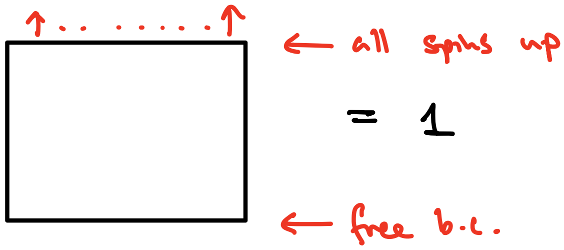


Boundary conditions are fixed by contracting the external legs with the appropriate states in the quadrupled Hilbert space

Note that $\langle \uparrow | \left(|0\rangle^{\otimes 4} \right) = \langle \downarrow | \left(|0\rangle^{\otimes 4} \right) = 1$ so we find

$$\mathop{\mathbb{E}}_{\text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t)^2 \right] =$$



Note that $\mathbb{E}_{u \sim \text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t) \right]^2 =$ 

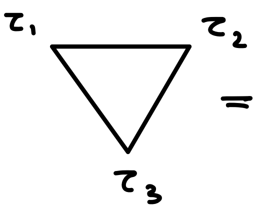
$= 1$ (20)

So the Ising magnet is deep within the ordered phase of the Ising model.

$$\mathbb{E}_{\text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t)^2 \right] = \frac{\mathbb{E}_{\text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t)^2 \right]}{\mathbb{E}_{u \sim \text{Haar}} \left[\mathrm{Tr}_A \, \rho_A(t) \right]^2} \sim e^{-\Delta F_{dw}}$$

where $\Delta F_{dw} \sim t$ is the free energy cost of the Ising domain wall.

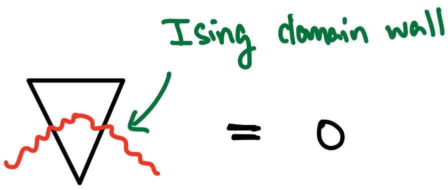
A technical point: Integrate out the $\underline{\sigma \text{ spins}}$ to obtain an Ising model on a triangular lattice with weights

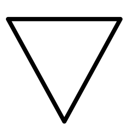
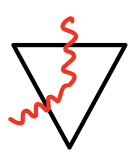
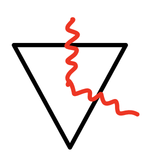


$$= \sum_{\sigma \in \uparrow, \downarrow} Wg^{(2)}(q^2; \sigma \tau_3) \langle \tau_1 | \sigma \rangle \langle \tau_2 | \sigma \rangle \qquad (21)$$

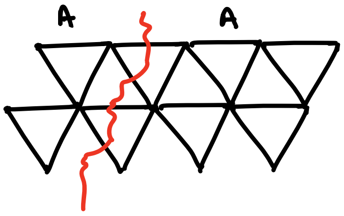
where
$$\begin{cases} Wg^{(2)}(q^2; +) = \dfrac{1}{q^4 - 1} & \text{as derived in Eq. (7).} \\[2ex] Wg^{(2)}(q^2; -) = \dfrac{-1}{q^2(q^4-1)} \end{cases}$$

on each vertex. Observe that the weights are invariant under the Ising symmetry transf. applied to τ. Show that



$= 0$  $= 1$  $=$  $= \dfrac{q}{q^2+1}$.

Using these weights, and the boundary conditions, it is easy to see that



$$\underbrace{\left(\frac{q}{q^2+1}\right)^{2t}}_{\text{weight at each downward triangle w/ a domain wall}} \cdot \underbrace{2^{2t}}_{\text{Number of trajectories of the dw}}$$

and so

$$\boxed{\mathop{\mathbb{E}}_{V\sim \text{Haar}}\left[\operatorname{Tr}_A \rho_A(t)^2\right] = \left(\frac{2q}{q^2+1}\right)^{2t}} \qquad (22)$$

What about $\displaystyle\mathop{\mathbb{E}}_{V\sim \text{Haar}} S^{(2)}[\rho_A(t)] = \mathop{\mathbb{E}}_{V\sim \text{Haar}} \left(-\log \operatorname{Tr}_A \rho_A(t)^2\right)$?

By convexity of $f(x) = -\log x$, i.e.

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2)$$

for all $t \in [0,1]$, we observe

$$-\log_q\left(\mathop{\mathbb{E}}_{u\sim \text{Haar}} \rho_A(t)^2\right) = 2t \log\left(\frac{q^2+1}{2q}\right) \overset{\text{convexity}}{\leq} \mathop{\mathbb{E}}_{u\sim \text{Haar}} S^{(2)}[\rho_A(t)]$$

$$\leq \mathop{\mathbb{E}}_{u\sim \text{Haar}} \underbrace{S[\rho_A(t)]}_{\text{von Neumann entropy}} \overset{\text{min-cut bound}}{\leq} 2t \log q \qquad (23)$$

So the von Neumann entanglement grows at long times as

$$\mathbb{E}_{u\sim \mathrm{Haar}}\, S[\rho_A(t)] = \log q \cdot v(q)\cdot t + (\text{sub-leading}) \qquad (24)$$

where $v(q)$ is an "entanglement velocity" which satisfies

$$2\log_q\left(\frac{q^2+1}{2q}\right) \le v(q) \le 2 \qquad (25)$$

So the Haar-averaged von Neumann entropy grows <u>ballistically</u> and — in the $q\to\infty$ limit — at the maximal rate. The ballistic growth of entanglement often holds, even when there are conserved quantities which propagate more slowly (H. Kim, D. Huse, PRL 111, 127205 (2013)).

VI. The Entanglement Domain Wall

What happens (quantitatively) away from $q \to \infty$? Here, we will argue that in any realization of the brickwork circuit,

$$S_A^{(2)}(t) = \log q \left(v_E^{(2)}(q)\, t + b\, \chi_A(t)\, t^{1/3} \right) \tag{26}$$

where $\chi_A(t)$ is a random variable of $O(1)$ size and the exponent $1/3$ is universal.

Consider the replica trick for the second Rényi entropy

$$S_A^{(2)} = \lim_{k \to 0} \frac{1 - \left[\mathrm{Tr}\, \rho_A^2\right]^k}{k} \tag{27}$$

We wish to calculate $\mathcal{Z}_k = \mathbb{E}_{\text{Haar}} \left[\left(\mathrm{Tr}\, \rho_A^2 \right)^k \right]$ which in principle describes the statistical mechanics of spins valued in S_{2k}. However, we have shown

$$\lim_{q \to \infty} \mathbb{E}_{\text{Haar}} \left(-\log \mathrm{Tr}\, \rho_A^2 \right) = \lim_{q \to \infty} \left(-\log \mathbb{E}_{\text{Haar}} \mathrm{Tr}\, \rho_A^2 \right) \tag{28}$$

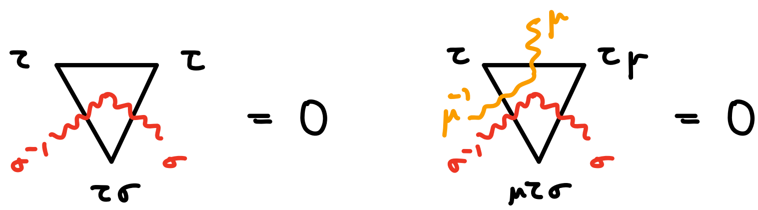
And therefore,

$$\lim_{q \to \infty} \mathcal{Z}_k = \left[\mathbb{E}_{u \sim \text{Haar}} \left(\mathrm{Tr}\, \rho_A^2 \right) \right]^k = \mathcal{Z}_{\text{Ising}}^k \tag{29}$$

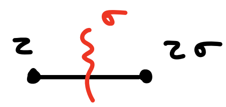
so that $\mathcal{Z}_k$ describes k decoupled Ising magnets when $q \to \infty$.

When $\frac{1}{q}$ is non-zero (and small) one may expect interactions between the k replicas. It is possible, though technically challenging, to show that this is the case.

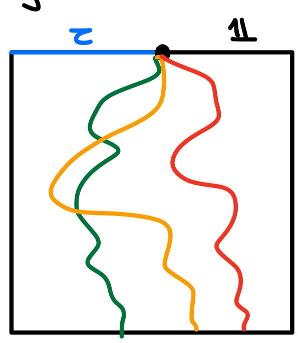
Specifically, one can show that the weights (see T. Zhou, A. Nahum PRB 99, 174205)



for any $\sigma \in S_n$ and elementary transposition μ i.e. so that $\chi(\mu) = n-1$. These weights are exact for any q. Here σ & μ are also used to label the domain walls in this lattice magnet



The leading interaction in the large-q limit comes from a pair of Ising domain walls. The boundary conditions for Z_k are

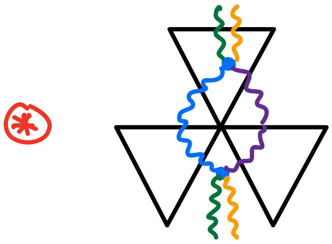


} dw b.c. where $\tau = (1\,2)(3\,4)\cdots(2k-1\;\,2k)$

At $q \to \infty$, the elementary, commuting transpositions that make up the τ domain wall do not interact

} free b.c.

At large but finite q, the elementary transpositions interact weakly. To leading order, there is an attractive interaction, see the diagram below for the interaction of a pair of commuting transpositions



$(12)(34) = (14)(23) \times (24)(13)$

$\sim \mathcal{O}(q^{-8})$ in the large-q limit

giving rise to an overall weight $\left(\frac{q}{q^2+1}\right)^4 \cdot 4 \cdot \left(1 + \underbrace{\mathcal{O}(q^{-4})}_{\text{leading correction from } \circledast}\right)$

As a result, $\mathcal{Z}_k$ describes k directed paths with pairwise attractive interactions of $\mathcal{O}(q^{-4})$ in the large-q limit. Compare this with a directed polymer in a random environment (DPRE), describing the behavior of a directed path in a potential landscape:

$$\mathcal{Z}[V] \equiv \int Dx(\tau) \; e^{-S[x,V]} \tag{30}$$

$$\text{where } S[x,V] \equiv \int_0^t d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V[x(\tau),\tau]\right]$$

Here, the potential $V(x,z)$ is random and short-range correlated. Often can take $\overline{V(x,z)} = 0$, $\overline{V(x,z)\,V(x',z')} = \sigma^2 \delta(x-x')\,\delta(z-z')$.

Disorder averaged free energy

$$\overline{F(t)} = -\lim_{k\to 0} \frac{\overline{\mathcal{Z}^k} - 1}{k} \tag{31}$$

where $\overline{\mathcal{Z}^k} = \int Dx_1 \cdots Dx_k \; e^{-S_{\text{eff}}[x_1,\dots,x_k]}$ and the action

$$S_{\text{eff}}[x_1,\dots,x_k] = \int_0^t d\tau \left[\frac{1}{2}\sum_{j=1}^{k}\left(\frac{dx_j(\tau)}{d\tau}\right)^2 - \underbrace{\frac{\sigma^2}{2}\sum_{i\neq j}}_{\text{Attractive interaction}} \delta(x_i(\tau)-x_j(\tau))\right] \qquad (32)$$

So at large, but finite q, one can conjecture that the DPRE fixed-point controls the scaling behavior of $S_A^{(2)}(t)$

This behavior is believed to hold for the von Neumann entropy, and the other Rényi entropies as well

From known results for the DPRE (see e.g. M. Kardar, Y.-C. Zhang, PRL 58, 2087) we conclude that

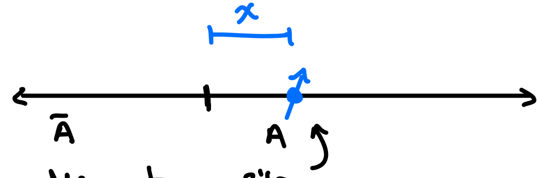
$$S_A(t) = \log q \cdot \left(v_E t + \chi\, t^{1/3}\right) \qquad (33)$$

Fluctuations grow as $\sim t^{1/3}$ and are parametrically smaller than $v_E t$.

Subleading corrections $\sim t^{1/3}$ are present in a typical Haar circuit (T. Zhou, A. Nahum PRX 10 031066 (2020))

Similar methods have been used (with important technical differences) to show that in the volume-law-entangled phase of quantum many-body evolution with a weak rate of measurements, the von Neumann entropy of a region with L_A qudits has the behavior

$$\underbrace{\langle S_A \rangle}_{\substack{\text{averaged over}\\ \text{pure-state trajectories}}} = s_{eq} \cdot L_A + \underbrace{\chi \cdot L_A^{1/3}}_{\text{universal power}} + (\text{sub-leading}) \qquad (34)$$



and the mutual information between a qudit at a site a distance x away from $\bar{A}$, and $\bar{A}$ is $\langle I(x) \rangle \sim x^{-\Delta}$, which is also related to the DPRE behavior of the entanglement domain wall in this setting. See [Y. Li, SV, M.P.A. Fisher, PRX Quantum 4 010331 (2023)]

Recap : Calculation of purity $e^{-S_A^{(2)}} \sim$ Ising magnet in ordered phase
Free energy cost of an Ising dw

Boundary conditions for Ising magnet determined by the quantity (purity) of interest but the bulk is unaffected by the choice of observable.

Unitarity requires that Ising domain walls cannot be created in the bulk ($\rho \sim \mathbb{1}$ is a fixed-pt. of any unitary dynamics)

We may be interested in a replica limit (e.g. in the calculation of entanglement entropies) but the qualitative regimes of many-body evolution can often be understood in the Ising "two replica" picture in random circuits w/o conservation laws.

VII. Gaining (some) intuition from the Ising domain wall

(i.) Probing the ordered phase → entanglement growth, measures of operator spreading → (Ising) domain wall line tension

"Entanglement domain" also appears in more generic models, to capture leading-order growth of entanglement; can be consistently given a "line tension", (see T. Zhou, A. Nahum PRB 99, 174205)

See also A. Nahum, J. Ruhman, SV, J. Haah PRX 7 031016 (2017)

(ii.) <u>Order / disorder phase</u> <u>transition</u> → <u>measurement-driven</u> entanglement phase transition; entanglement domain wall has vanishing line tension in the disordered (area-law phase), though transition <u>is</u> <u>not</u> in the Ising universality class!

(see Romain Vasseur's lectures!)

(iii.) <u>External magnetic field</u> → coupling to an external environment (dephasing / decoherence)

(iv.) <u>"Pinning" of the Ising</u> <u>domain wall (localization transition)</u> → phase transition in the ability to <u>recover</u> initially locally-accessible quantum information which undergoes

[I. Lovas, U. Agrawal, SV, arXiv: 2304.02664] chaotic unitary evolution in in the presence of boundary dissipation see also [Z. Weinstein, S. Kelly, J. Marino, E. Altman, arXiV: 2210.14242]

where bulk dissipation drives a scrambling / coding transition

(v.) Bound-state phase transition for multiple directed paths $\longrightarrow$ phase transition in "histories" of operator $O(x,t)$ which contribute to real-time correlations

[A. Nahum, S. Roy, SV, T. Zhou, PRB 106, 224310 (2022)]

VIII. Adding conservation laws (an example...)

More structures emerge w/ conservation laws, e.g. time-translation symmetry. Here we study one such problem — the emergence of random matrix level statistics in ergodic Floquet dynamics.

The spectral form factor $K(t) \equiv |\mathrm{Tr}\, \mathcal{U}(t)|^2$ characterizes the emergence of random matrix level statistics in chaotic quantum evolution. For example, if $\mathcal{U}(t) = V^t$ where V is a Haar-random unitary on a D-dim. Hilbert space, then

$$\mathbb{E}_{U\sim \mathrm{Haar}}\left(|\mathrm{Tr}\, V^t|^2\right) = \begin{cases} D^2 & t = 0 \\ t & 0 < t \leq t_H \\ D & t = t_H \end{cases} \qquad (35)$$

where the Heisenberg time $t_H = D$, see e.g. [P. Kos, M. Ljubotina, T. Prosen PRX 8, 021062]. The linear growth in $\mathbb{E}_{\mathrm{Haar}}\, K(t)$ when $t < t_H$ is due to level repulsion. Note

$$K(t) = \sum_{n,m} e^{i(\theta_n - \theta_m)t}$$

where $\{e^{i\theta_n}\}$ are eigenvalues of V and $\mathcal{U}(t) = V^t$.

Then $\mathbb{E}_{\text{Haar}}\, K(t) \overset{t\neq 0}{=} D + D(D-1)\int d\theta\, d\theta'\, P_2(\theta,\theta')\, e^{i(\theta-\theta')t}$

where $P_2(\theta,\theta')$ is the joint distribution of pairs of eigenvalues of a Haar-random unitary $\mathcal{U}$. <u>Level repulsion</u> between eigenvalues can be used to derive the ramp behavior.

An alternate way to understand this result is via the Haar average and the "pairing" field in the limit $D \gg 1$. Here,

we recall that

$$\mathbb{E}_{\text{Haar}} \underbrace{[V \otimes V^* \otimes V \otimes \dots]}_{2t} = \sum_{\sigma,\tau \in S_t} Wg^{(t)}(D;\sigma^{-1}\tau)\, |\sigma\rangle\langle\tau| \qquad (36)$$

& we will use the result that when $D \gg 1$, the dominant contribution comes from $\sigma = \tau$ in the above sum, and

$$Wg^{(t)}(D;\mathbb{1}) = \frac{1}{D^t}\left(1 + \mathcal{O}(D^{-2})\right) \qquad (37)$$

All $Wg^{(t)}(D;\mu)$ for $\mu \neq \mathbb{1}$ (the identity permutation) are suppressed by additional factors of $1/D$.

The relation $P_n^2 = P_n$ requires that

$$\sum_{\sigma \in S_n} D^{\chi(\sigma\mu^{-1})}\, Wg^{(n)}(D;\sigma^{-1}\tau) = \delta_{\mu\tau}$$

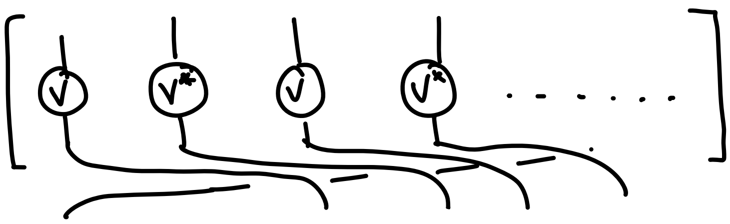
where $\chi(\mu)$ denoting the number of cycles in permutation $\mu \in S_n$.

So in the $D \to \infty$ limit, $\mathrm{Wg}^{(n)}(D; \sigma^{-1}\tau) = \frac{1}{D^n} \delta_{\sigma\tau} + \ldots$

(where $\ldots$ denotes subleading corrections in $1/D$).

Now, we can write

$$\mathbb{E}_{\text{Haar}}\, k(t) = \mathbb{E}_{U \sim \text{Haar}} \mathrm{Tr} \qquad (38)$$



$$= \sum_{\sigma, \tau \in S_t} \mathrm{Wg}^{(t)}(D; \sigma^{-1}\tau)\, \mathrm{Tr}\left[|\sigma\rangle\langle \mu^{-1}\tau\mu| \right] \overset{D \gg 1}{=} \frac{1}{D^t} \sum_{\sigma \in S_t} \langle \sigma | \mu^{-1}\sigma\mu \rangle$$

where $\mu = (1\ 2\ 3 \cdots t)$ is a cyclic permutation

$$= \frac{1}{D^t} \sum_{\sigma \in S_t} D^{\chi(\sigma^{-1}\mu^{-1}\sigma\mu)}$$

with $\chi(\tau)$ denoting the number of cycles in a permutation $\tau \in S_t$.

The number of cycles $\chi(\sigma^{-1}\mu^{-1}\sigma\mu)$ is maximized when $\sigma^{-1}\mu^{-1}\sigma\mu = \mathbb{1}$ i.e. iff $\sigma\mu = \mu\sigma$ which requires that $\sigma = \mu^k$ for $k = 1, \ldots, t$ i.e. the only elements of S_t commuting with the cyclic shift μ are elements of the Abelian group generated by μ.

$$\therefore \quad \mathbb{E}_{\text{Haar}}\, k(t) = \frac{1}{D^t} \sum_{\sigma \in S_t} D^{\chi(\sigma^{-1}\mu^{-1}\sigma\mu)} = t + \underbrace{\text{sub-leading in } 1/D.}_{} \qquad (39)$$

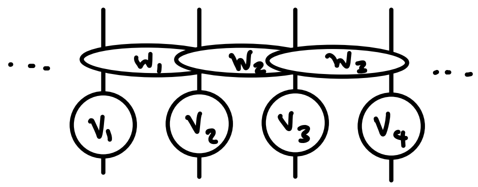
(these are actually exactly zero...)

How does this behavior emerge in a <u>local</u> quantum evolution? Guess...

$$K(t) \sim \begin{cases} t^{L/\xi(t)} & t \leq t_{th} \\ t & t_H \geq t \geq t_{th} \end{cases} \tag{40}$$

for an L-qubit system undergoing local unitary evolution. Here t_{th} is the <u>Thouless time</u>

Consider a $(1+1)$-d <u>Floquet</u> dynamics, from [A. Chan, A. de Luca, J. Chalker, PRL 121 060601 (2018)]



$\} \; \mathcal{U}$ & $\mathcal{U}(t) = \mathcal{U}^t$

Here, V_j is a $q \times q$ Haar-random unitary (single-qudit unitary) while W_j is a two-qudit gate

$$W_j \equiv \sum_{\{\sigma_j, \sigma_{j+1}\}} e^{i\varphi^{(j)}_{\sigma_j, \sigma_{j+1}}} |\sigma_j, \sigma_{j+1}\rangle\langle\sigma_j, \sigma_{j+1}| \tag{41}$$

where $\sigma_j \in \{1, \ldots, q\}$. So $\{W_j\}$ mutually commute and are diagonal in the computational basis. Furthermore $\varphi^{(j)}_{\sigma_j, \sigma_{j+1}}$ is independently chosen for each j, σ_j, σ_{j+1} from a Gaussian distribution with <u>zero mean</u> and variance ε.

$\underset{\substack{V \sim \text{Haar} \\ W \sim \text{Gauss}}}{\mathbb{E}} |\mathrm{Tr}\, \mathcal{U}^t|^2$ can be studied as follows; the Haar average of the $\{V_i\}$ unitaries gives rise to a t-state spin σ_i at each site i, in the large-q limit (labeling one of t cyclic shifts). The $\{W_i\}$ gates will couple these "spins". Note that

$$(W_1 \otimes W_1^*)^{\otimes t}$$

is invariant under permutations of W_1 and independently under permutations of W^*, so the resulting stat. mech. description will have an $S_t \times S_t$ symmetry (t-state Potts model).

It turns out that

$$\mathop{\mathbb{E}}_{\substack{V \sim \text{Haar} \\ W \sim \text{Gauss}}} k(t) = \sum_{\{\sigma\}} e^{-H[\sigma]} \qquad (42)$$

where $H[\sigma] = J \sum_j (1 - \delta_{\sigma_j, \sigma_{j+1}})$ where $J = e^{\varepsilon t}$. This is a ferromagnetic t-state Potts model with a t-dependent interaction. No long-range order in 1d. However, a direct calculation of the partition function gives

$$\mathop{\mathbb{E}}_{\substack{V \sim \text{Haar} \\ W \sim \text{Gauss}}} k(t) = \begin{cases} t^{L/\xi(t)} & t \leq t_{th}(L) \\ t & t \geq t_{th}(L) \end{cases} \qquad (43)$$

where $t_{th}(L) \sim \frac{\log L}{\varepsilon}$ and $\xi(t) \sim \frac{\log t}{t} e^{\varepsilon t}$. Compare with $t_{th}(L) \sim L^2/D$ for an L-site system with a locally-conserved density that relaxes diffusively.

In many-body localized phase (this model appears to exhibit an MBL transition when q is finite, see A. Chan et al. paper), $K(t)$ saturates to asymptotic value in a system-size-independent time, and does not exhibit a "ramp" (Poissonian level statistics)

In higher dimensions, t_{th} is independent of system size and is large when the inter-site coupling is small, in the ergodic phase.

IX. Other "structured" unitary circuits (a *very* brief discussion...)

- Clifford circuit dynamics: uniform distribution over the group of qudit Clifford unitary gates with prime dimension is a unitary 2-design. Efficiently classically-simulable by Gottesmann-Knill Theorem (no operator entanglement is generated in Pauli basis)

- Random unitary evolution w/ conservation laws, e.g. $U(1)$ symmetry reveal diffusive transport of conserved densities and the emergence of hydrodynamics, (see V. Khemani, D. Huse, A. Vishwanath, PRX 8 031057 (2018))

- Dual unitary circuits: can exchange space & time directions, and still obtain unitary evolution. Playground for many interesting exact calculations, (see e.g. P. Kos, B. Bertini, T. Prosen, PRX 11, 011022 (2021)) to study the emergence of ergodicity, entanglement dynamics, operator entanglement, etc. "Maximally" chaotic