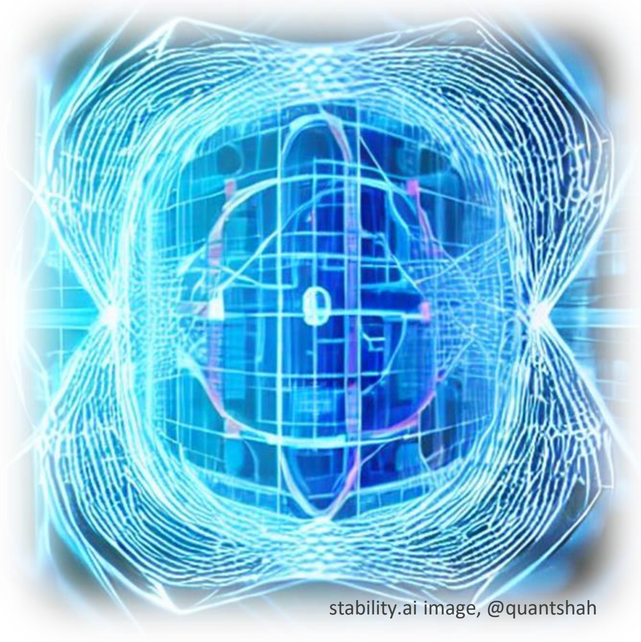
Quantum error correction



stability.ai image, @quantshah

Boulder School 2023:
Non-Equilibrium
Quantum Dynamics

Presented by Victor V. Albert

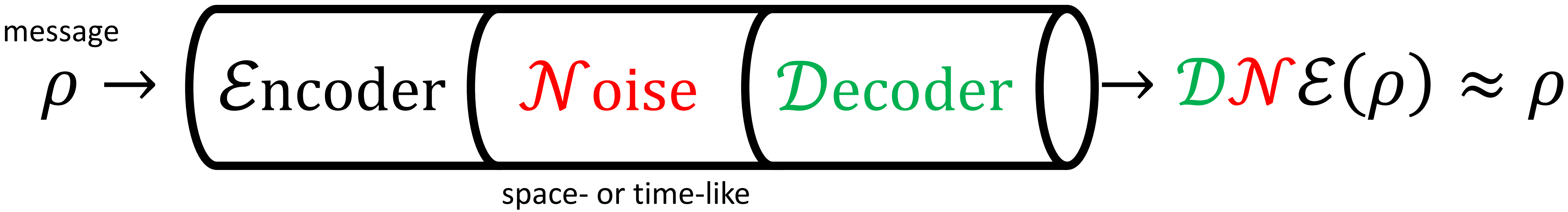
Joint Center for Quantum Information and Computer Science

NIST

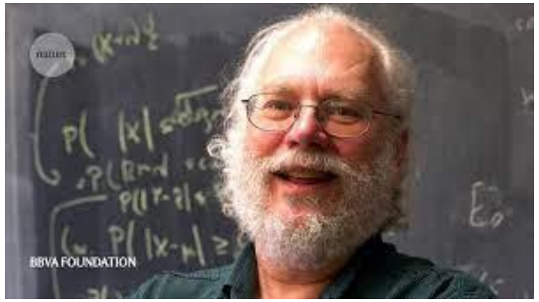
University of Maryland 1856

GOAL OF ERROR CORRECTION

- To preserve messages sent through a noisy transmission channel by encoding the messages in an **error-correcting code**
  - *More precisely:* to make sure the rate of corruption of encoded (i.e., **logical**) information is lower than that of the same information sent without the extra encoding step.



QEC: ORIGINS & GROWTH

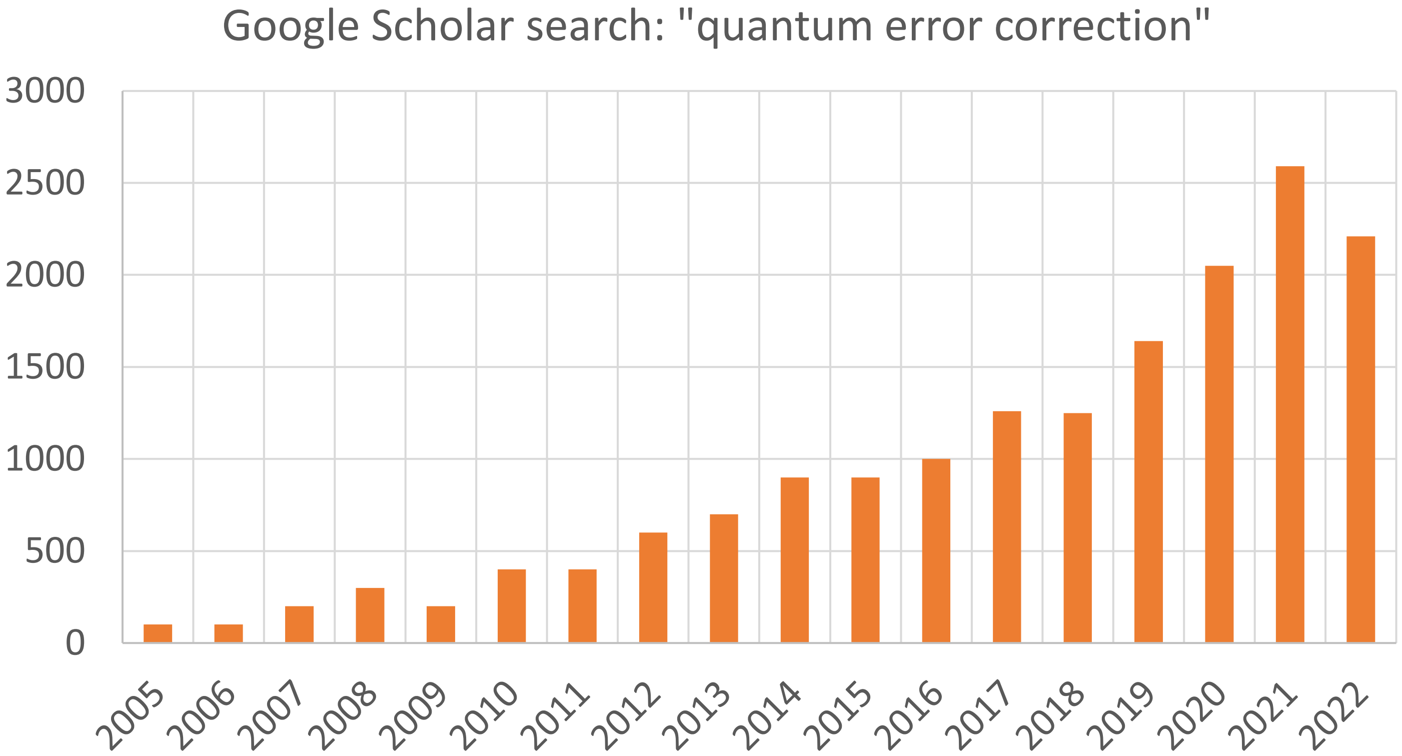


Peter Shor:

- Quantum error-correcting codes (1995)
- Fault-tolerant syndrome measurement (1996)
- Fault-tolerant universal quantum gates (1996)
- Using QEC to prove security of QKD (2000)

Alexei Yu. Kitaev:

- Topological quantum codes (1996-2003)
- Physically protected quantum computing (1997)
- Computing with nonabelian anyons (1997)
- CSS-to-homology dictionary (1998)
- Magic state distillation (1999-2004)
- Majorana modes in quantum wires (2000)



Other pioneers:

- Stabilizer codes (Gottesman + Calderbank, Rains, Shor, Sloane)
- FT error correction (Shor, Steane, Knill)
- QEC conditions (Knill, Laflamme)
- Concatenated threshold theorem (Aharonov, Ben-Or)

Some material from: J. Preskill, QEC 2017 talk

THE MANY TOPICS OF QEC

1. Deterministic or random code constructions that reach **boundary of what is possible**.
    - MDS, perfect, random quantum, generalized homological product, good QLDPC, singleton-bound approaching approximate, covariant, locally testable, triorthogonal
2. Constructing **practical codes** for near-term realization.
    - 2-3D surface, 2-3D color, dynamically generated (Floquet, spacetime circuit), tetron Majorana, single-shot, self-correcting quantum, cluster-state, homological rotor
3. Working with a quantum device to **realize codes**.
    - repetition, small distance block, 2D rotated surface, 2D color, two-component cat, square- and hexagonal-lattice GKP, dual-rail
4. Relating phases of **quantum matter** to error-correcting codes.
    - geometrically local Hamiltonian-based (topological, fracton, ETH, MPS)
5. Relating **gravitational field theories**, among others, to error-correcting codes.
    - holographic (HaPPY), renormalization group cat, matrix model
6. Development of codes for **sensing/metrology**.
    - Error-corrected sensing, metrological

https://errorcorrectionzoo.org/

INTRO TO QEC:
A CODE IS A SUBSPACE

ALPHABETS AND HILBERT SPACES

X	classical states (elements of X)	quantum states (functions on X)
$\mathbb{Z}_2^n = \mathbb{F}_2^n$	bits	qubits
$\mathbb{F}_q^n$	q-ary strings	Galois qudits
$\mathbb{Z}_q^n$	q-ary strings over $\mathbb{Z}_q$	modular qudits
$\mathbb{R}^n$	reals	oscillators
G	finite group	group-valued qudit

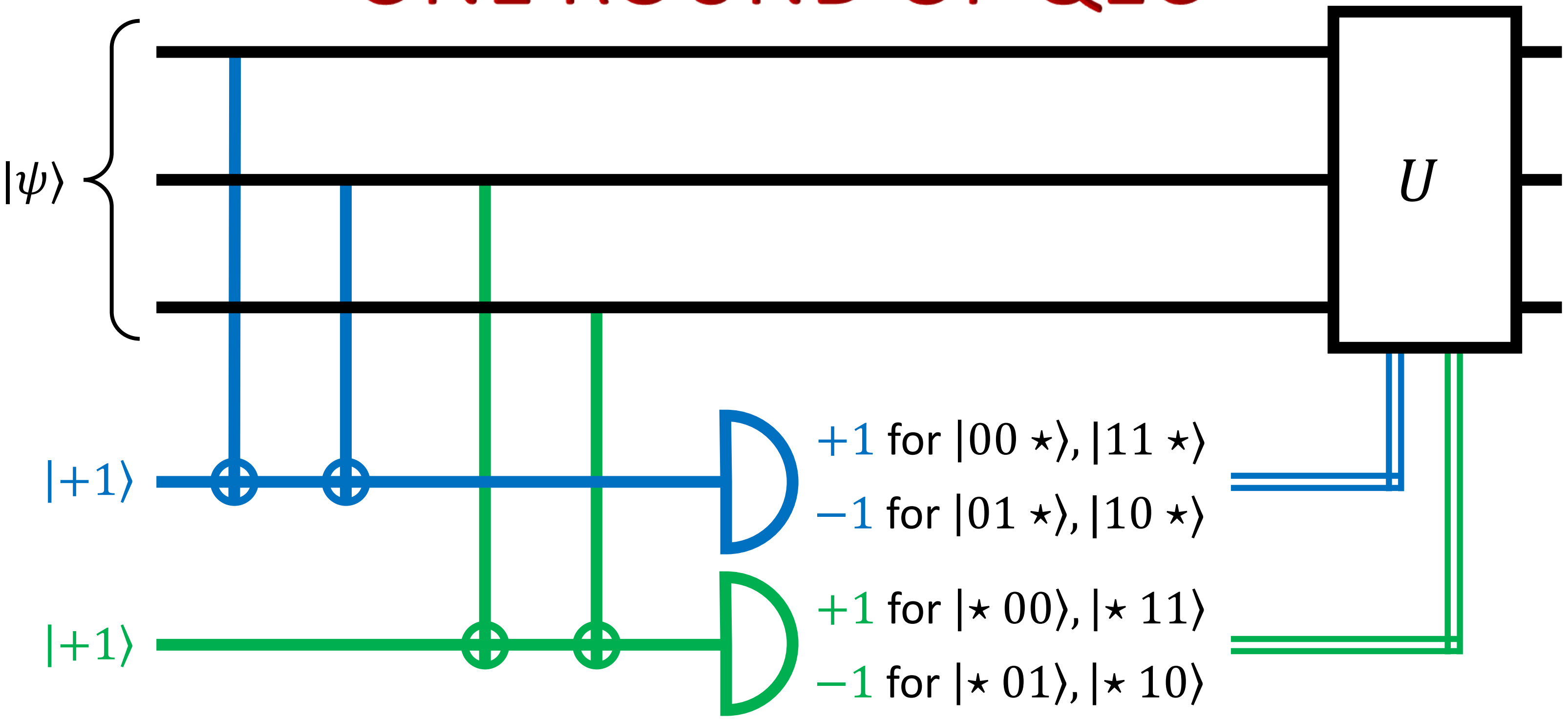
Table 1: Common classical alphabets and their corresponding quantum Hilbert spaces.

$\rightarrow$

$\ell^2\Big($  $\Big)$

ONE ROUND OF QEC

Measure the syndrome, not the data!



Lookup table

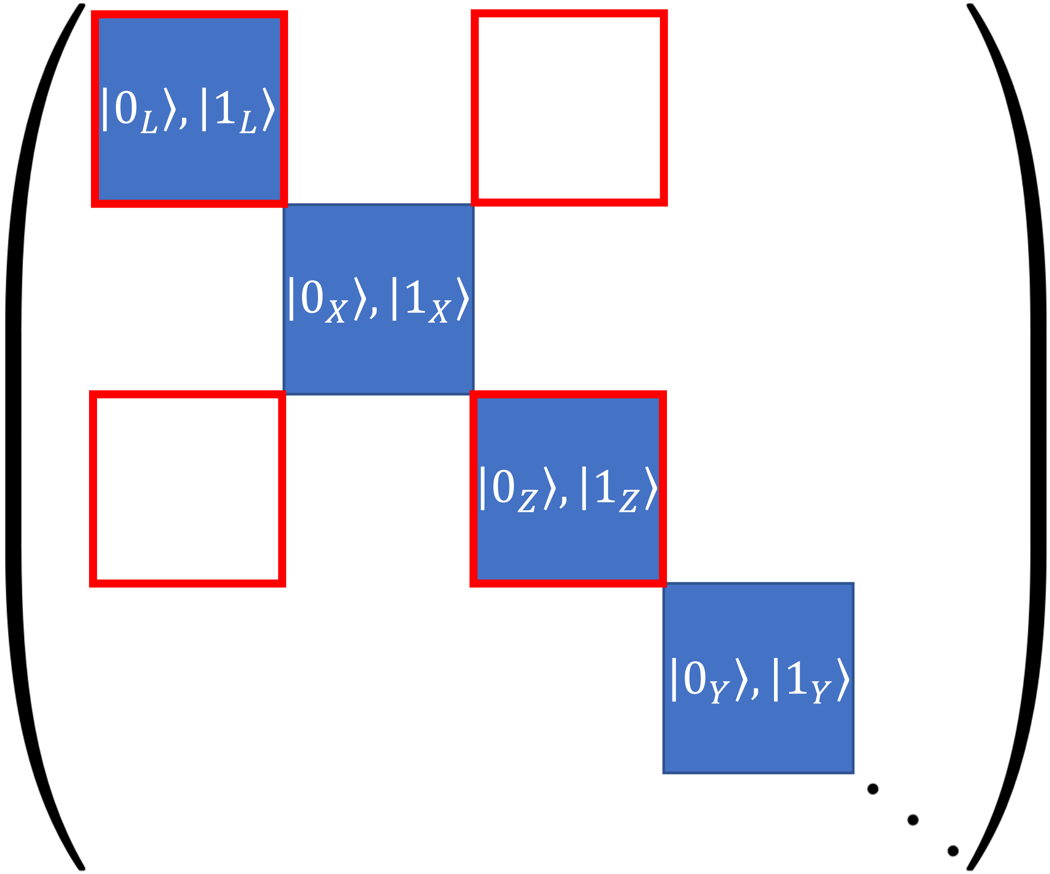
Syndrome	Likely error
$+1, +1$	III
$+1, -1$	IIX
$-1, +1$	XII
$-1, -1$	IXI

- To resolve code and error spaces (**error diagnosis**), measure eigenvalues of commuting set of observables (**check operators**; here, ZZI and IZZ with $Z = \sigma_z$) and apply recovery U conditional on parity-check eigenvalue (**error syndrome**). This is **one round** of correction:
  1. **Diagnose**: measure error syndromes using ancillary qubits.
  2. **Decode**: given a syndrome, determine which recovery U to apply.
- Correction rounds generalize straightforwardly to other types of errors and other codes.

ERROR SPACE STRUCTURE

Noise is continuous, but measured errors are discrete!

1. Check operator measurement collapses system onto codespace or an error space.
2. Paulis are a basis for single-qubit operators
   → General errors are detectable!



Example: Z-axis rotation:

$$R_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \cos\theta I - i\sin\theta Z$$

The result superposition of error space collapses to one error space upon a round of EC:

$$\begin{smallmatrix} I & Z \\ I & I \end{smallmatrix} |\psi_L\rangle = \cos\theta|\psi_L\rangle - i\sin\theta \begin{smallmatrix} I & Z \\ I & I \end{smallmatrix} |\psi_L\rangle$$
$$= \cos\theta\,(c_0|0_L\rangle + c_1|1_L\rangle) - i\sin\theta\,(c_0|0_Z\rangle + c_1|1_Z\rangle)$$

GENERAL QEC CONDITIONS

- Errors E_j are detectable iff they act trivially on the codewords:

1. Environment does not distinguish codewords

$$\langle 0_L|E_j|0_L\rangle = \langle 1_L|E_j|1_L\rangle$$

2. Environment cannot connect distinct codewords:

$$\langle 0_L|E_j|1_L\rangle = \langle 1_L|E_j|0_L\rangle = 0$$

Error-detection conditions

$$PE_jP = c_jP$$

$$P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$$

Example: constant need not be zero: $ZZII|0_L\rangle = |0_L\rangle$ and $ZZII|1_L\rangle = |1_L\rangle$

- Errors $E_{j,k}$ mapping to
…different error spaces are correctable if they are detectable.
…same error space are correctable if detectable + undo each other.

Error-correction conditions

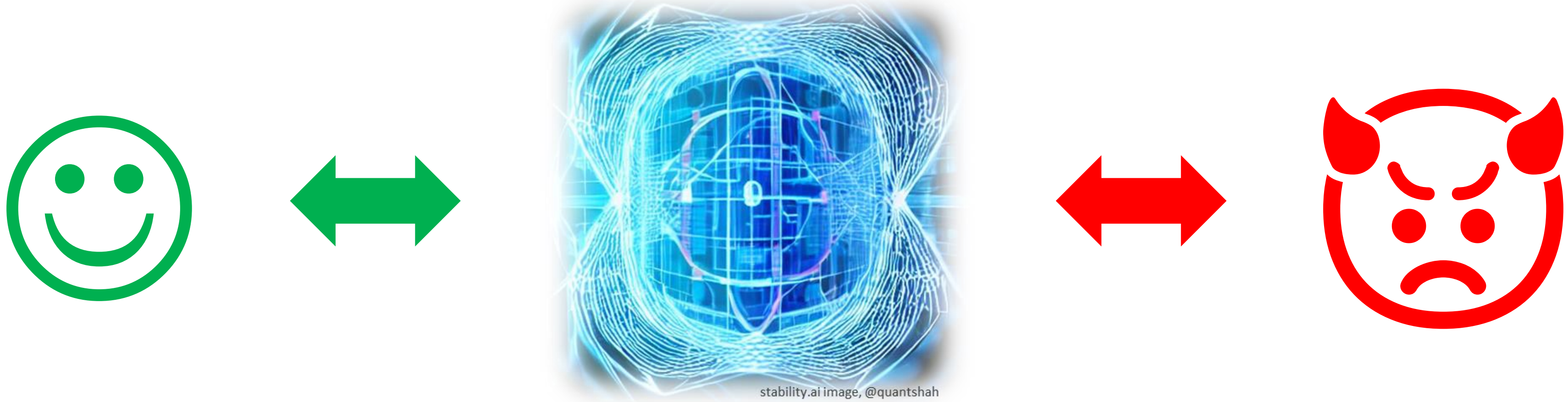
$$PE_j^\dagger E_kP = c_{jk}P$$

Example: single-qubit bit flips are not correctable for four-qubit code b/c they cannot undo each other:

$$XIII = IXII \cdot X_L$$

$$\langle 1_L|E_1^\dagger E_2|0_L\rangle = \langle 1_L|XXII|0_L\rangle = \langle 1_L|X_L|0_L\rangle = 1 \neq 0\,.$$

ENVIRONMENTAL PERSPECTIVE IS USEFUL



$$\mathcal{N}(\rho_L) = \mathrm{tr}_{\mathrm{env}}\left\{U_{\mathcal{N}}\rho_L U_{\mathcal{N}}^\dagger\right\} = \sum_j N_j \rho_L N_j^\dagger$$

$$\widehat{\mathcal{N}}(\rho_L) = \mathrm{tr}_{\mathrm{sys}}\left\{U_{\mathcal{N}}\rho_L U_{\mathcal{N}}^\dagger\right\} = \sum_{j,k} \mathrm{tr}\left\{N_j^\dagger N_k \rho_L\right\} |k\rangle\langle j|$$

Rel-n to data processing inequality:
quant-ph/9604034
[25] = quant-ph/9604022

Finally, we mention that for superoperators $\mathcal{A}$, there is a simple information theoretic characterization of $\mathcal{A}$-correcting codes due to Nielsen and Schumacher [25]. Let $|e\rangle = (1/\sqrt{k})\Sigma_i |i_L\rangle|i_L\rangle$ be the perfectly entangled state of the code from which we can define the density matrices:

$$\overline{\rho} = \frac{1}{k}\sum_{ai} A_a |i_L\rangle\langle i_L| A_a^\dagger \quad \text{and} \quad \rho = \sum_a I\otimes A_a |e\rangle\langle e| A_a^\dagger \otimes I. \tag{29}$$

The entropy of a density matrix σ is denoted by $S(\sigma)$.

Theorem III.6. Let $\mathcal{A}$ be a superoperator. Then $\mathcal{C}$ is an $\mathcal{A}$-correcting code if and only if $S(\overline{\rho}) - S(\rho) = \log_2 k$.

EXAMPLE: NON-PAULI CHANNEL

$$\mathcal{N}(\rho_L) = \mathrm{tr}_{\mathrm{env}}\left\{U_{\mathcal{N}}\rho_L U_{\mathcal{N}}^\dagger\right\}$$
$$= \sum_j N_j \rho_L N_j^\dagger$$

$$N_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad N_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

SIMPLE CODES → IMPORTANT CODES

FOUR-QUBIT CODE → CONCATENATED CODE

- Another basis for the codespace yields a pattern:

$m_2 = 2$ copies

$$|\pm_L\rangle = \frac{1}{\sqrt{2}}\left(|0_L\rangle \pm |1_L\rangle\right) = \frac{1}{2}\left(|00\rangle \pm |11\rangle\right)^{\otimes 2}$$

$m_1 = 2$ qubits

- The four-qubit code can be viewed as a bit-phase **concatenated code**:

$$|0\rangle \rightarrow |00\rangle \qquad |1\rangle \rightarrow |11\rangle$$

m_1 bit-flip repetition code

→

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

change to phase basis

→

$$|+\rangle \rightarrow |++\rangle \qquad |-\rangle \rightarrow |--\rangle$$

m_2 phase-flip repetition code

- Concatenating larger codes ($m_1 = m_2 = 3$) yields **Shor nine-qubit code**, the first to correct single-qubit errors:

$$|\pm_L\rangle = \frac{1}{2}\left(|000\rangle \pm |111\rangle\right)^{\otimes 3}$$

- For general m_1, m_2, one obtains **quantum parity / generalized Shor codes**.

EC ZOO

https://errorcorrectionzoo.org/c/quantum_parity

FOUR-QUBIT CODE → STABILIZER CODE

- Recall four-qubit code: $|0_L\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ $\quad |1_L\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle)$
- Recall observation of operator for which the codespace is a +1-eigenvalue eigen-subspace:

$$ZZII|0_L\rangle = |0_L\rangle \quad \text{and} \quad ZZII|1_L\rangle = |1_L\rangle$$

- Operators $IIZZ$ and $XXXX$ satisfy this as well. The three mutually commuting operators generate the code's **stabilizer group** $\mathsf{S}_{\text{four-qubit}}$:

$$\mathsf{S}_{\text{four-qubit}} = \langle ZZII, IIZZ, XXXX\rangle = \langle S_1, S_2, S_3\rangle = \{S_1^a S_2^b S_3^c \mid a, b, c \in \mathbb{Z}_2\}$$

  - **Example**: Four-qubit code is an [[4,1,2]] code. $[[n, k, d]]$

Distance
logical qubits
physical qubits

- Advantages of stabilizer codes (over other codes):
  - ✓ **Efficient presentation** in terms of stabilizer generators.
  - ✓ **Syndromes** obtained for free: group generators are check operators.
  - ✓ **Detectable/undetectable** errors determined simply from check operators
  - ✓ **General idea**: works for bosons, fermions, modular qudits, Galois qudits, molecules.

https://errorcorrectionzoo.org/c/stabilizer

FOUR-QUBIT CODE → CSS CODE

- Re-write stabilizer generators of four-qubit code as binary matrices:

$$XXXX \longleftrightarrow \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} = H_X$$

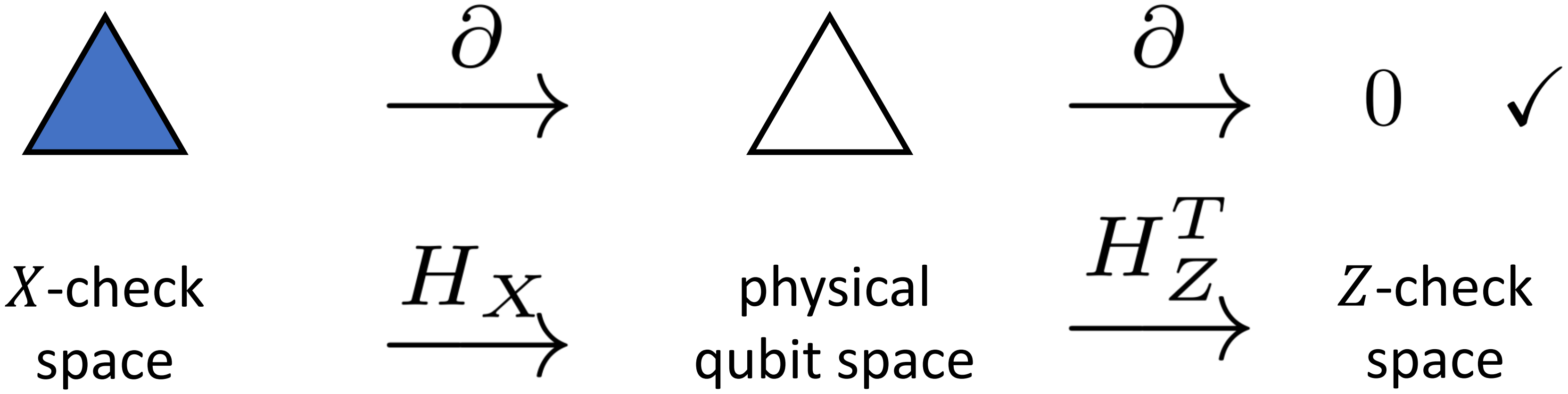
$$\begin{matrix}ZZII \\ IIZZ\end{matrix} \longleftrightarrow \begin{pmatrix}1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1\end{pmatrix} = H_Z$$

- Stabilizer commutation requirement equivalent to following **CSS condition** on matrices:

$$H_X H_Z^T = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1\end{pmatrix} \overset{\text{mod } 2}{=} 0 \quad \checkmark$$

- We just need a particular pair of matrices:
  - Borrow from classical codes → **CSS codes**
  - Embed into chain complex → **CSS-to-homology dictionary**

Calderbank-Shor-Steane: https://errorcorrectionzoo.org/c/css
Kitaev 1998

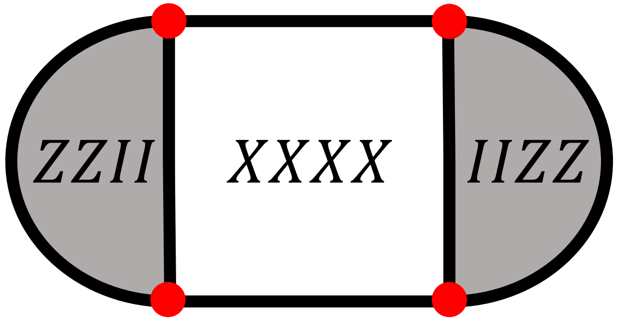


$\partial^2 = 0$
"boundaries don't have boundaries"

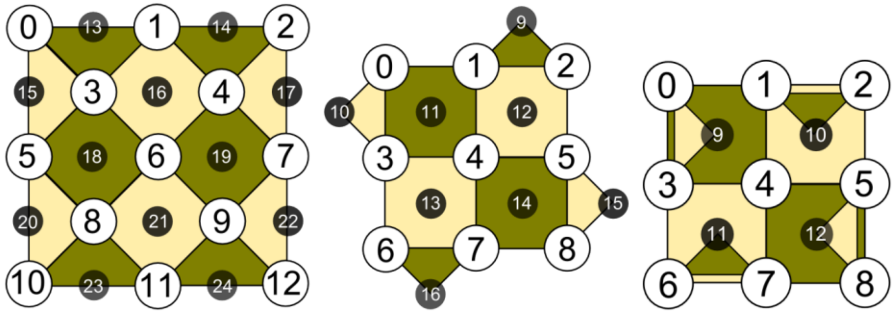
$H_X H_Z^T = 0$
CSS condition

FOUR-QUBIT → SURFACE CODE → TOPOLOGICAL CODES

- Arrange the four qubits (●) into a square and observe geometrical pattern formed by stabilizer generators:

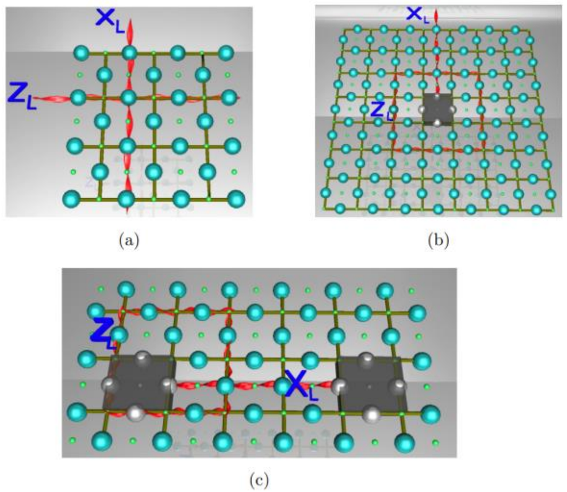


Springboard to other geometries; connection to topological phases.

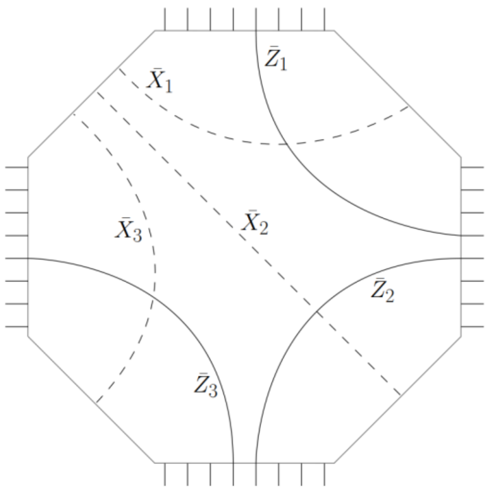


(a) Surface-25 (b) Surface-17 (c) Surface-13

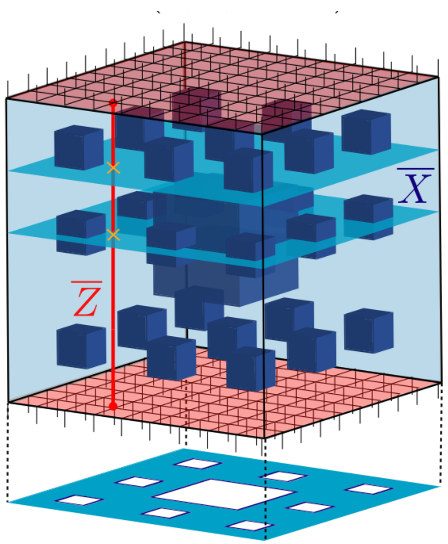
Rotated surface codes: arXiv:1404.3747



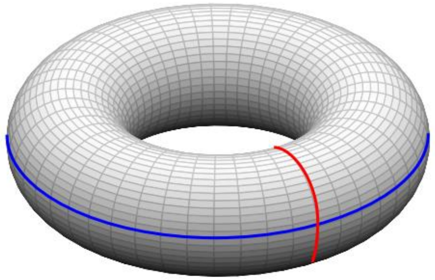
Geometries with holes
arXiv:1111.4022



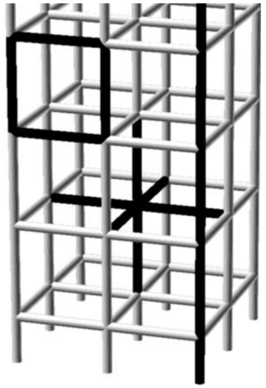
Hyperbolic geometries
arXiv:1506.04029



Fractal geometries
arXiv:2201.03568



Nontrivial boundaries
→ toric code
quant-ph/9707021



3D version
arXiv:1406.4227

?

Exotic manifolds >=4D
arXiv:math/0002124,
arXiv:1310.5555

SURFACE-to-STABILIZER CODE ROSETTA STONE

stabilizer code	surface code
codespace	ground-state subspace
errors	anyonic excitations
QEC conditions	topological quantum order (TQO)
joint +1 Z & X stabilizer eigenspace	flux conservation & gauge/1-form symmetry
logical Paulis	non-contractible Wilson loops
code switching	anyon condensation

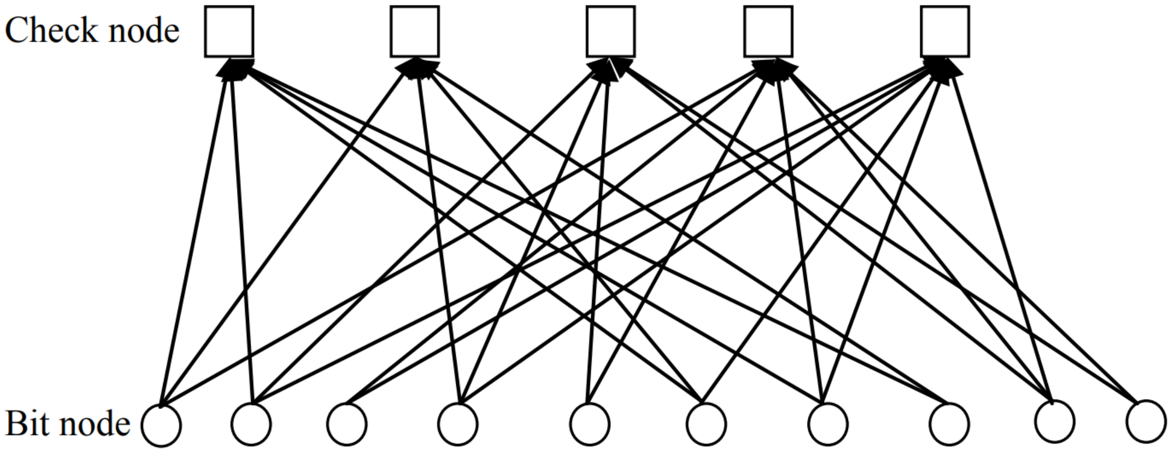
STABILIZER EXPERIMENTS

Parameters	Name	Platforms
[[n,1]]	Repetition	NMR (Waterloo), SC circuits (Google, IBM), silicon (RIKEN, Delft), NV centers (Wratchup, Kosaka, Hanson groups), ions (Blatt group)
[[4,1,2]] variants	Four-qubit	Photonic (Rarity group), ions (IonQ), SC circuits (IBM, Google, Delft, Wallraff, Monz groups)
[[5,1,3]]	Five-qubit perfect	NMR (Waterloo), SC circuits (Pan group), ions (Quantinuum), NV centers (Delft)
[[7,1,3]]	Steane	Ions (Blatt, Monz groups, Quantinuum), Rydberg arrays (Lukin group)
[[9,1,3]]	Shor	Ions (Linke group, IonQ), photonics (Pan group)
[[9,1,3]]	Bacon-Shor subsystem	Ions (IonQ)
[[2m,2m-2,3]], m=2,3	Iceberg	Ions (Quantinuum)
[[m^2,m,3]], m=2,3	Heavy-Hexagon subsystem	SC circuits (IBM)
[[m^2,m,3]], m<=7	Quantum Parity / Shor	Ions (Linke group)
[[9,1,3]]	Surface-17	SC circuits (Wallraff, Pan groups)
n=19 planar, 24 toric	Kitaev surface	Rydberg arrays (Lukin group)
n=9,25, d=3,5 planar	XZZX surface	SC circuits (Google)

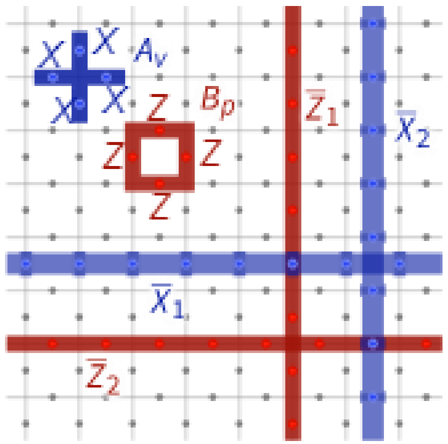
Is our system good enough that adding these extra qubits actually improves logical performance?

For up-to-date references, see:
https://errorcorrectionzoo.org/list/realizations

STABILIZER CODES & LOCALITY



local AKA few-body AKA "CS-local"



geometrically local AKA "physics-local"

$[[2m, 2m-2, 2]]$ error-detecting code [1,2]

edit | "…" cite | ↗

Description

Also known as the *iceberg* code. CSS stabilizer code for $m \geq 2$ with generators $\{XX\cdots X, ZZ\cdots Z\}$ acting on all $2m$ physical qubits. Admits a basis such that each

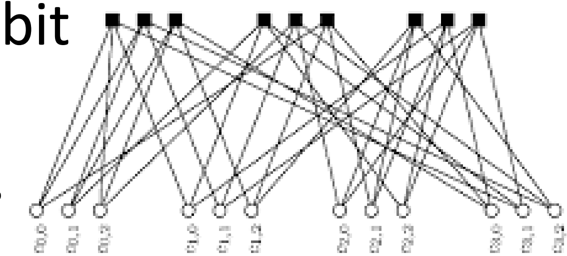
non-local

Rao, Channel Coding Techniques for Wireless Communications

QLDPC EXPLOSION

https://errorcorrectionzoo.org/c/good_qldpc
https://errorcorrectionzoo.org/c/qldpc

- **QLDPC code**: stabilizer code such that the **number of qubits** participating in each stabilizer generator and the **number of stabilizer generators** that each qubit participates in are both independent of n.
  - ‼ Geometric locality **not required** (!) → this is "locality" in the CS sense.



- **Asymptotically good QLDPC**: a family $[[n_i, k_i, d_i]]$ for $i \geq 1$ for which the **rate** k_i/n_i and **relative distance** d_i/n_i remain constant as $i \to \infty$.

arXiv:2206.07750 ▸ Dinur-Hsieh-Lin-Vidick (DHLV) code — DHLV code construction yields asymptotically good QLDPC codes.

arXiv:2111.03654 ▸ Expander lifted-product code — Lifted products of certain classical Tanner codes are the first asymptotically good QLDPC codes.

arXiv:2202.13641 ▸ Quantum Tanner code — Quantum Tanner code construction yields asymptotically good QLDPC codes.

- **Geometric locality** has to be dropped ☹. Codes on lattices in any dimension:
  - ❌ Cannot be good QLDPC codes arXiv:quant-ph/0304161, arXiv:0810.1983, arXiv:2106.00765, arXiv:2109.10982
  - ❌ Admit limited set of transversal gates Bravyi-Koenig arXiv:1206.1609

STRUCTURE of STABILIZER CODES

FOUR-QUBIT CODE AS A STABILIZER CODE

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$

$$S = \left\langle \begin{smallmatrix}X&X\\X&X\end{smallmatrix}, \begin{smallmatrix}Z&I\\Z&I\end{smallmatrix}, \begin{smallmatrix}I&Z\\I&Z\end{smallmatrix} \right\rangle \equiv \langle M_1, M_2, M_3\rangle = \left\{ M_1^a M_2^b M_3^c \,|\, a,b,c \in \mathbb{Z}_2 \right\}$$

Pauli P	Description	$\Pi P \Pi =$	Rel-n to stab. gen's
Stabilizers	Act trivially within codespace	Π	$S = \prod_{j=1}^{n-k} M_j^{p_j}$
Detectable errors	Map codespace into error spaces	0	$EM_j = -M_j E$ for some j
Logical Paulis	Act nontrivially within codespace	L	$LM_j = M_j L$ for all j

ERROR SPACES / PAULI FRAMES

$$|0_L\rangle \propto |\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\rangle + |\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\rangle$$

$$|1_L\rangle \propto |\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\rangle + |\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\rangle$$

E	p	q	r	$\|0_E\rangle$	$\|1_E\rangle$	Previous lecture
$\begin{smallmatrix}I&I\\I&I\end{smallmatrix}$	$+$	$+$	$+$	$\|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\rangle + \|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\rangle + \|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\rangle$	$\|0_L\rangle, \|1_L\rangle$
$\begin{smallmatrix}I&Z\\I&I\end{smallmatrix}$	$-$	$+$	$+$	$\|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\rangle - \|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\rangle - \|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\rangle$	$\|0_Z\rangle, \|1_Z\rangle$
$\begin{smallmatrix}I&I\\X&I\end{smallmatrix}$	$+$	$-$	$+$	$\|\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\rangle + \|\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\rangle + \|\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\rangle$	$\|0_X\rangle, \|1_X\rangle$
$\begin{smallmatrix}I&I\\I&X\end{smallmatrix}$	$+$	$+$	$-$	$\|\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\rangle + \|\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\rangle + \|\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\rangle$	
$\begin{smallmatrix}I&Z\\X&I\end{smallmatrix}$	$-$	$-$	$+$	$\|\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\rangle - \|\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\rangle - \|\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\rangle$	$\|0_Y\rangle, \|1_Y\rangle$
$\begin{smallmatrix}I&Z\\I&X\end{smallmatrix}$	$-$	$+$	$-$	$\|\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\rangle - \|\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\rangle - \|\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\rangle$	
$\begin{smallmatrix}I&I\\X&X\end{smallmatrix}$	$+$	$-$	$-$	$\|\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\rangle + \|\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\rangle + \|\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\rangle$	
$\begin{smallmatrix}I&Z\\X&X\end{smallmatrix}$	$-$	$-$	$-$	$\|\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\rangle - \|\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\rangle$	$\|\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\rangle - \|\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\rangle$	

LOGICAL PAULIS

$$|0_L\rangle \propto \left|\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right\rangle$$

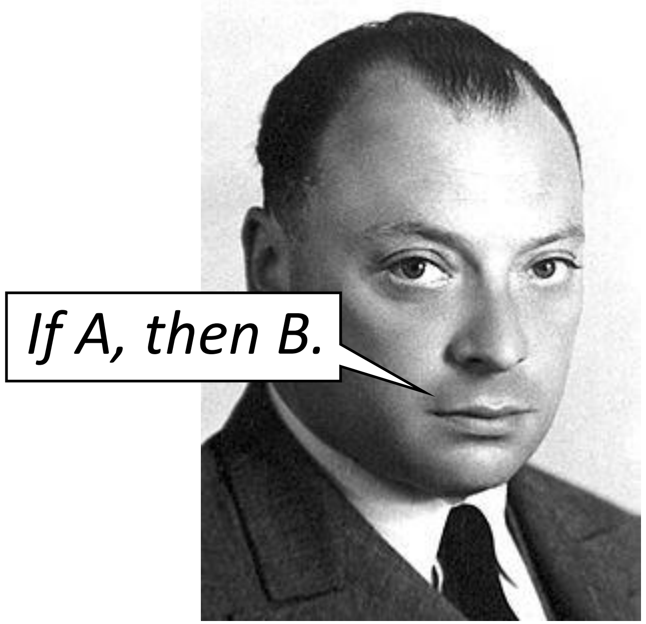
$$L \in \{\overline{I}, \overline{X}, \overline{Z}, \overline{Y}\} = \left\{ \begin{smallmatrix} I & I \\ I & I \end{smallmatrix}, \begin{smallmatrix} X & I \\ X & I \end{smallmatrix}, \begin{smallmatrix} Z & Z \\ I & I \end{smallmatrix}, \begin{smallmatrix} Y & Z \\ X & I \end{smallmatrix} \right\}$$

$$\overline{I} \cong \begin{matrix} I & I \\ I & I \end{matrix} \cong \begin{matrix} X & X \\ X & X \end{matrix} \cong \begin{matrix} Z & I \\ Z & I \end{matrix} \cong \begin{matrix} I & Z \\ I & Z \end{matrix}$$

$$\overline{X} \cong \begin{matrix} X & I \\ X & I \end{matrix} \cong \begin{matrix} I & X \\ I & X \end{matrix} \cong \begin{matrix} Y & I \\ Y & I \end{matrix} \cong \begin{matrix} X & Z \\ X & Z \end{matrix}$$

$$\overline{Y} \cong \begin{matrix} Y & Z \\ X & I \end{matrix} \cong \begin{matrix} Z & Y \\ I & X \end{matrix} \cong \begin{matrix} X & Z \\ Y & I \end{matrix} \cong \begin{matrix} Y & I \\ X & Z \end{matrix}$$

$$\overline{Z} \cong \begin{matrix} Z & Z \\ I & I \end{matrix} \cong \begin{matrix} Y & Y \\ X & X \end{matrix} \cong \begin{matrix} I & Z \\ Z & I \end{matrix} \cong \begin{matrix} Z & I \\ I & Z \end{matrix}$$



MOVING LOGICALS & CLEANING LEMMA

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$

$$L \cdot S \to L \cdot S \cdot S'$$

- Either you "can clean $\mathtt{M}$": all logicals can be chosen to act outside of $\mathtt{M}$
- Or you "cannot clean $\mathtt{M}$": $\exists$ a logical acting entirely within $\mathtt{M}$

$$g(\mathtt{M}) + g(\mathtt{M}^{\perp}) = 2k$$

arxiv:0810.1983

HARDNESS OF DECODING

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$

- General stabilizer recovery consists of three parts:

$$E \cdot L \cdot S$$

- S: Stabilizer group equivalence
- L: Residual logical operations
- E: Map back to codespace

- ML decoding:

$$(L \cdot S)_\star = \arg\max_{L,S} \Pr(L, S \mid E)$$

- (Degenerate) ML decoding:

$$L_\star = \arg\max_{L} \sum_{S} \Pr(L, S \mid E)$$

RIP David Poulin
arXiv:1310.3235

STABILIZER CODES: SUMMARY

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$

Code-preserving Paulis make up the **normalizer**

$$\mathsf{N}(\mathsf{S}) \equiv \mathsf{N}_{\mathsf{P}_n}(\mathsf{S}) = \text{everything in } \mathsf{P}_n \text{ that commutes with everything in } \mathsf{S} \qquad (53$$

while logical Pauli representatives L make up the quotient group $\mathsf{N}(\mathsf{S})/\langle i, \mathsf{S}\rangle$:

	[[4, 1, 2]] code	Size	Stabilizer	Size
stabilizer group	$\langle \begin{smallmatrix}X&X\\X&X\end{smallmatrix}, \begin{smallmatrix}Z&I\\Z&I\end{smallmatrix}, \begin{smallmatrix}I&Z\\I&Z\end{smallmatrix} \rangle$	$2^3 = 8$	S	2^{n-k}
code-preserving Paulis $L \cdot S$	$\langle i, \begin{smallmatrix}X&I\\X&I\end{smallmatrix}, \begin{smallmatrix}I&X\\I&X\end{smallmatrix}, \begin{smallmatrix}Z&Z\\I&I\end{smallmatrix}, \begin{smallmatrix}I&I\\Z&Z\end{smallmatrix}, \begin{smallmatrix}Z&I\\I&Z\end{smallmatrix} \rangle$	$4 \cdot 2^5 = 128$	$\mathsf{N}(\mathsf{S})$	$4 \cdot 2^{n+k}$
logical Paulis			$\mathsf{N}(\mathsf{S}) - \langle i, \mathsf{S}\rangle$	
logical Paulis modulo $\langle i, \mathsf{S}\rangle$	$\{ \begin{smallmatrix}I&I\\I&I\end{smallmatrix}, \begin{smallmatrix}X&I\\X&I\end{smallmatrix}, \begin{smallmatrix}Y&Z\\X&I\end{smallmatrix}, \begin{smallmatrix}Z&Z\\I&I\end{smallmatrix} \}$	$\frac{4\cdot 2^5}{4\cdot 2^3} = 4$	$\mathsf{N}(\mathsf{S})/\langle i, \mathsf{S}\rangle$	4^k

SUBSYSTEM CODES:
A CIRCUIT-CENTRIC APPROACH

From STABILIZER to SUBSYSTEM codes

- The four-qubit code can be extended to the [[4,2,2]] stabilizer code:

$$\mathsf{S} = \left\langle \begin{smallmatrix} X & X \\ X & X \end{smallmatrix}, \begin{smallmatrix} Z & Z \\ Z & Z \end{smallmatrix} \right\rangle$$

$$|00_L\rangle \propto \left| \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right\rangle \qquad |01_L\rangle \propto \left| \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right\rangle$$

$$|10_L\rangle \propto \left| \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \right\rangle \qquad |11_L\rangle \propto \left| \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right\rangle$$

- Logical Pauli representatives are

$$\Big\{ \underbrace{\begin{smallmatrix} X & I \\ X & I \end{smallmatrix}, \begin{smallmatrix} Z & Z \\ I & I \end{smallmatrix}}_{\text{qubit I}}, \underbrace{\begin{smallmatrix} X & X \\ I & I \end{smallmatrix}, \begin{smallmatrix} Z & I \\ Z & I \end{smallmatrix}}_{\text{qubit II}} \Big\}$$

- Let us **not use the second qubit** for storage, but as a tunable knob or “gauge” degree of freedom that we can set as we please.

SUBSYSTEM CODES

- We need another (this time, non-Abelian) group **G** to keep track of which “gauge” qubits we have picked.

$$\langle i, \mathsf{S} \rangle \subseteq \mathsf{G} \subseteq \mathsf{N}(\mathsf{S})$$

1. $\mathsf{G} = \langle i, \mathsf{S} \rangle$: no qubits are gauge –> stabilizer code
2. $\mathsf{G} = \mathsf{N}(\mathsf{S})$: all qubits are gauge –> no logical subspace

Similarity to gauging in electromagnetism only conceptual. Electric and magnetic potentials can be changed via gauge transformations without affecting the physically observable fields. Similarly, the gauge qubits can be manipulated without affect the logical information, **but** such effects are observable.

SUBSYSTEM CODES: ADVANTAGE 1

3. gauge fixing can allow you to switch between different codes, and many gadgets can be understood as subsystem codes

(a) Gauge fixing second qubit to $|0_L\rangle$ (and forgetting it) gets back to $[[4, 1, 2]]$ code with stabilizer group $\langle \begin{smallmatrix} X & X \\ X & X \end{smallmatrix}, \begin{smallmatrix} Z & I \\ Z & I \end{smallmatrix}, \begin{smallmatrix} I & Z \\ I & Z \end{smallmatrix} \rangle$.

(b) Gauge fixing second qubit to $|+_L\rangle$ (and forgetting it) yields "dual" $[[4, 1, 2]]$ code with stabilizer group $\langle \begin{smallmatrix} Z & Z \\ Z & Z \end{smallmatrix}, \begin{smallmatrix} X & X \\ I & I \end{smallmatrix}, \begin{smallmatrix} I & I \\ X & X \end{smallmatrix} \rangle$.

(c) Gauge fixing the second qubit to the maximally mixed state yields a way to do computation with mixed states.

$$|00_L\rangle \propto |\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\rangle + |\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\rangle \qquad |01_L\rangle \propto |\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\rangle + |\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\rangle$$

$$|10_L\rangle \propto |\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\rangle + |\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\rangle \qquad |11_L\rangle \propto |\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\rangle + |\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\rangle$$

SUBSYSTEM CODES: ADVANTAGE 2

Measuring lower-weight operators can yield error syndromes.

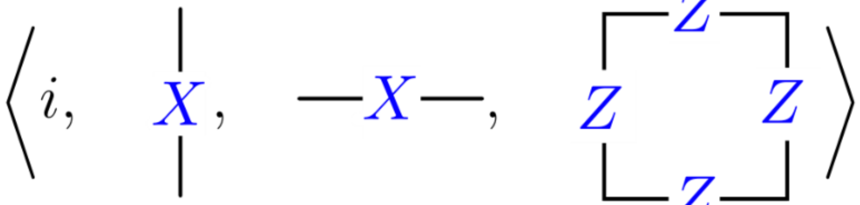
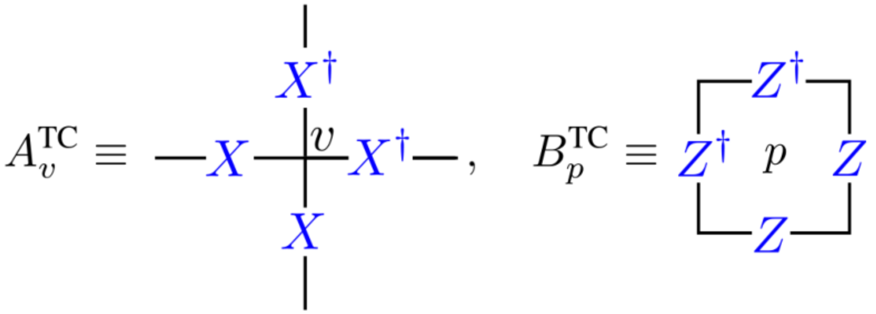
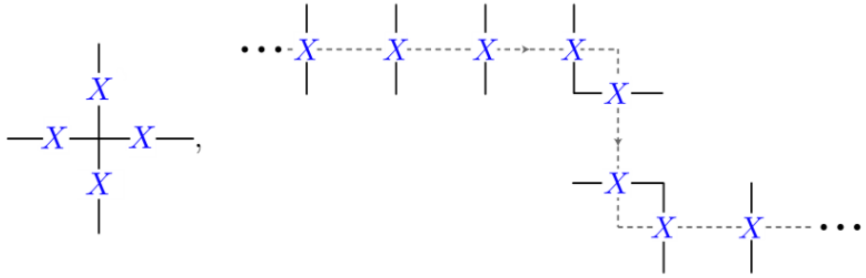
$$\boxed{\begin{matrix} X & X \\ I & I \end{matrix} = -1} \quad + \quad \boxed{\begin{matrix} I & I \\ X & X \end{matrix} = +1} \quad = \quad \boxed{\begin{matrix} X & X \\ X & X \end{matrix} = -1}$$

$$\mathsf{S}_{[[9,1,4,3]]} = \left\langle \begin{matrix} X & X & I \\ X & X & I \\ X & X & I \end{matrix}, \begin{matrix} I & X & X \\ I & X & X \\ I & X & X \end{matrix}, \begin{matrix} Z & Z & Z \\ Z & Z & Z \\ I & I & I \end{matrix}, \begin{matrix} I & I & I \\ Z & Z & Z \\ Z & Z & Z \end{matrix} \right\rangle$$

$$\mathsf{G}_{[[9,1,4,3]]} = \left\langle i, \mathsf{S}, \begin{matrix} X & X & I \\ I & I & I \\ I & I & I \end{matrix}, \begin{matrix} I & I & I \\ X & X & I \\ I & I & I \end{matrix}, \begin{matrix} I & X & X \\ I & I & I \\ I & I & I \end{matrix}, \begin{matrix} I & I & I \\ I & X & X \\ I & I & I \end{matrix}, \begin{matrix} Z & I & I \\ Z & I & I \\ I & I & I \end{matrix}, \begin{matrix} I & Z & I \\ I & Z & I \\ I & I & I \end{matrix}, \begin{matrix} I & I & I \\ Z & I & I \\ Z & I & I \end{matrix}, \begin{matrix} I & I & I \\ I & Z & I \\ I & Z & I \end{matrix} \right\rangle$$

SUBSYSTEM CODES: ADVANTAGE 3

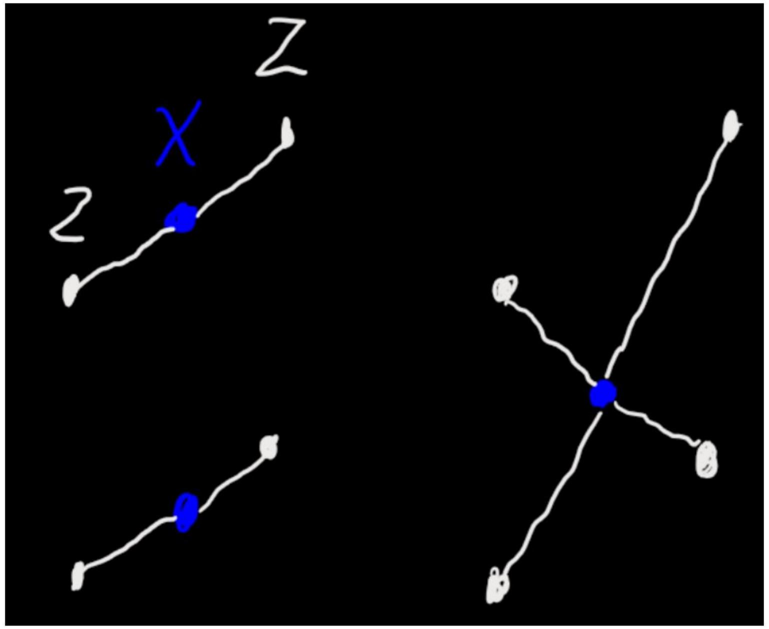
2. Subsystem codes realize more phases of matter than stabilizer codes, e.g.,

	planar surface code [2112.11394]	$\mathbb{Z}_2^{(0)}$ subsystem code [2211.03798]
G		
S		
Hamiltonian	$-\sum_{S \in \mathsf{S}} S$	$-\sum_{G \in \mathsf{G}} J_G G \quad \text{s.t.} \quad J_G \in \mathbb{R}$
Realizes	$\mathbb{Z}_2$ topological order	$\mathbb{Z}_2$ gauge theory (w/ $XXXX$ constraint)

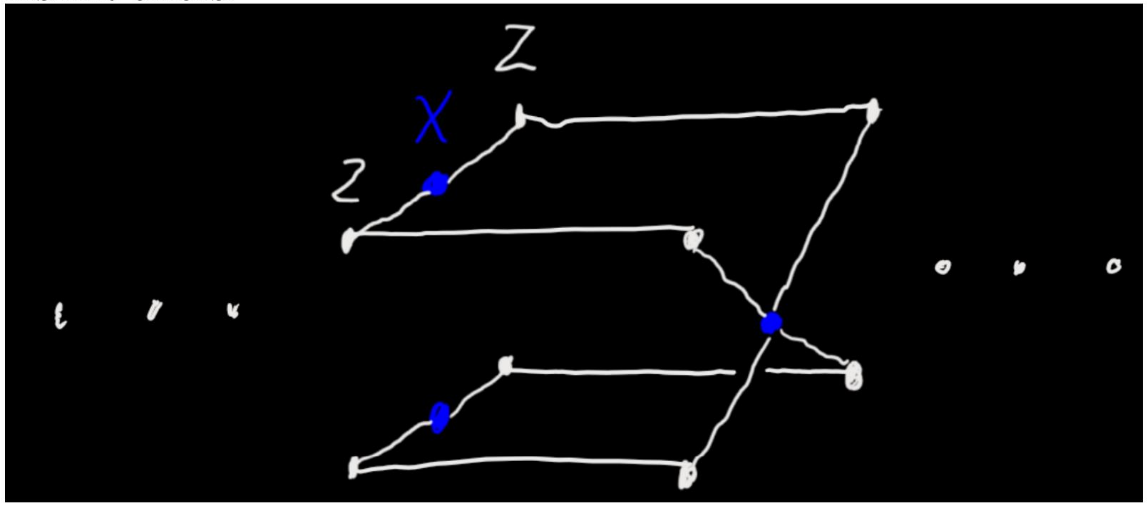
MBQC AS A SUBSYSTEM CODE

$$|0_L\rangle \propto \left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right\rangle$$



clusterization



foliation

CODE SWITCHING

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$

- Code switching can be done by starting with a code state of a stabilizer group **S** and measuring check operators in a new stabilizer group **F**. The new stabilizer group consist of everything in both **S** and **F** that commute with everything in **F**.

$$\boxed{\mathsf{S} \rightarrow \mathsf{N}_{\langle \mathsf{S},\mathsf{F}\rangle}\left(\mathsf{F}\right)\ .}$$

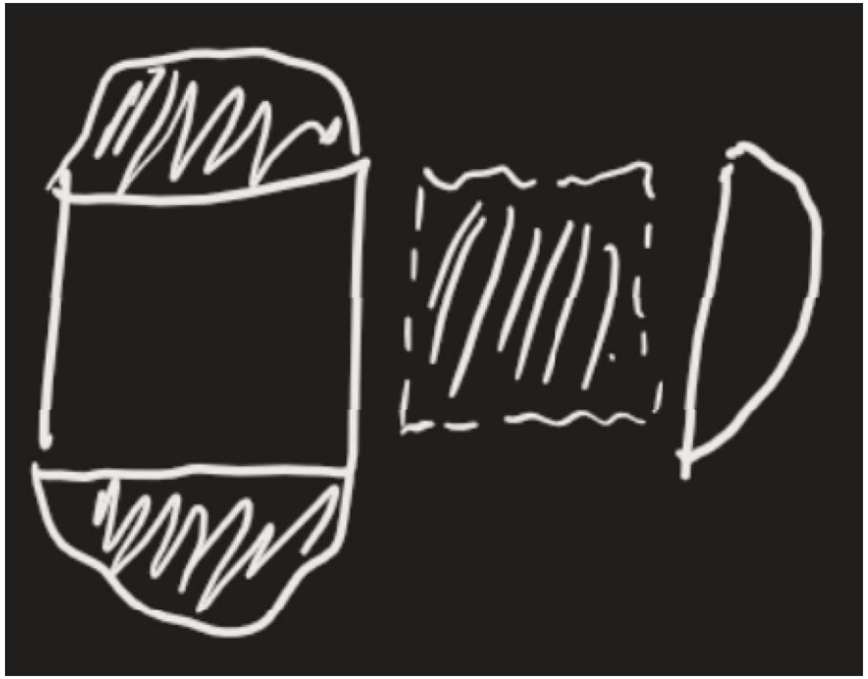
LATTICE SURGERY IS CODE SWITCHING

$$\mathsf{S} \to \mathsf{N}_{\langle \mathsf{S},\mathsf{F}\rangle}(\mathsf{F})\ .$$

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$

- **Lattice surgery** combining [[4,1,2]] and [[2,1,1]] codes into [[6,1,2]].



$$|\overline{00}\rangle \propto \left|\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&0\\1&1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}\right\rangle$$

$$|\overline{01}\rangle \propto \left|\begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&0\\1&1&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&1\\0&0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&1\\1&1&0\end{smallmatrix}\right\rangle$$

$$|\overline{10}\rangle \propto \left|\begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&0\\1&1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&1\\1&1&1\end{smallmatrix}\right\rangle$$

$$|\overline{11}\rangle \propto \left|\begin{smallmatrix}1&1&0\\0&0&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&0\\1&1&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&1\\1&1&0\end{smallmatrix}\right\rangle$$

$$F = \left\langle \begin{smallmatrix}I&Z&Z\\I&Z&Z\end{smallmatrix}\right\rangle$$

arxiv:1810.10037

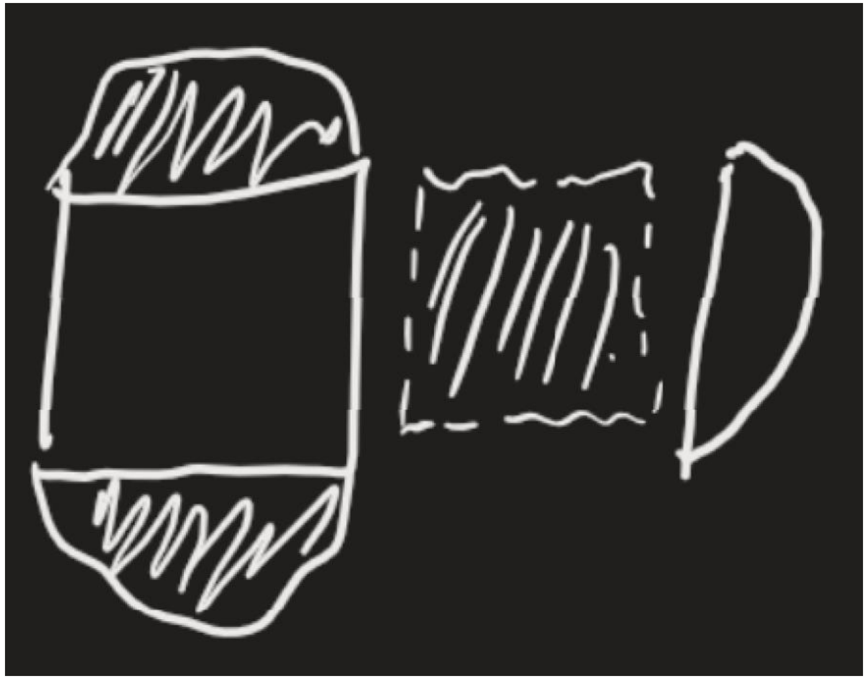
LATTICE SURGERY IS CODE SWITCHING

$$\text{S} \rightarrow \text{N}_{\langle \text{S},\text{F} \rangle}\,(\text{F})\ .$$

$$|0_L\rangle \propto \left|\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right\rangle$$

- **Lattice surgery** combining [[4,1,2]] and [[2,1,1]] codes into [[6,1,2]].



$$F = \left\langle \begin{smallmatrix} I & Z & Z \\ I & Z & Z \end{smallmatrix} \right\rangle$$

Measurement result is +1, so we project:

$$\left(I + \begin{smallmatrix} I & Z & Z \\ I & Z & Z \end{smallmatrix}\right)|\overline{00}\rangle \propto \left|\begin{smallmatrix} 0&0&0 \\ 0&0&0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1&1&0 \\ 1&1&0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 0&0&1 \\ 0&0&1 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1&1&1 \\ 1&1&1 \end{smallmatrix}\right\rangle$$

$$\left(I + \begin{smallmatrix} I & Z & Z \\ I & Z & Z \end{smallmatrix}\right)|\overline{01}\rangle = 0$$

$$\left(I + \begin{smallmatrix} I & Z & Z \\ I & Z & Z \end{smallmatrix}\right)|\overline{10}\rangle = 0$$

$$\left(I + \begin{smallmatrix} I & Z & Z \\ I & Z & Z \end{smallmatrix}\right)|\overline{11}\rangle \propto \left|\begin{smallmatrix} 1&1&0 \\ 0&0&1 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 0&0&0 \\ 1&1&1 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1&1&1 \\ 0&0&0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 0&0&1 \\ 1&1&0 \end{smallmatrix}\right\rangle$$

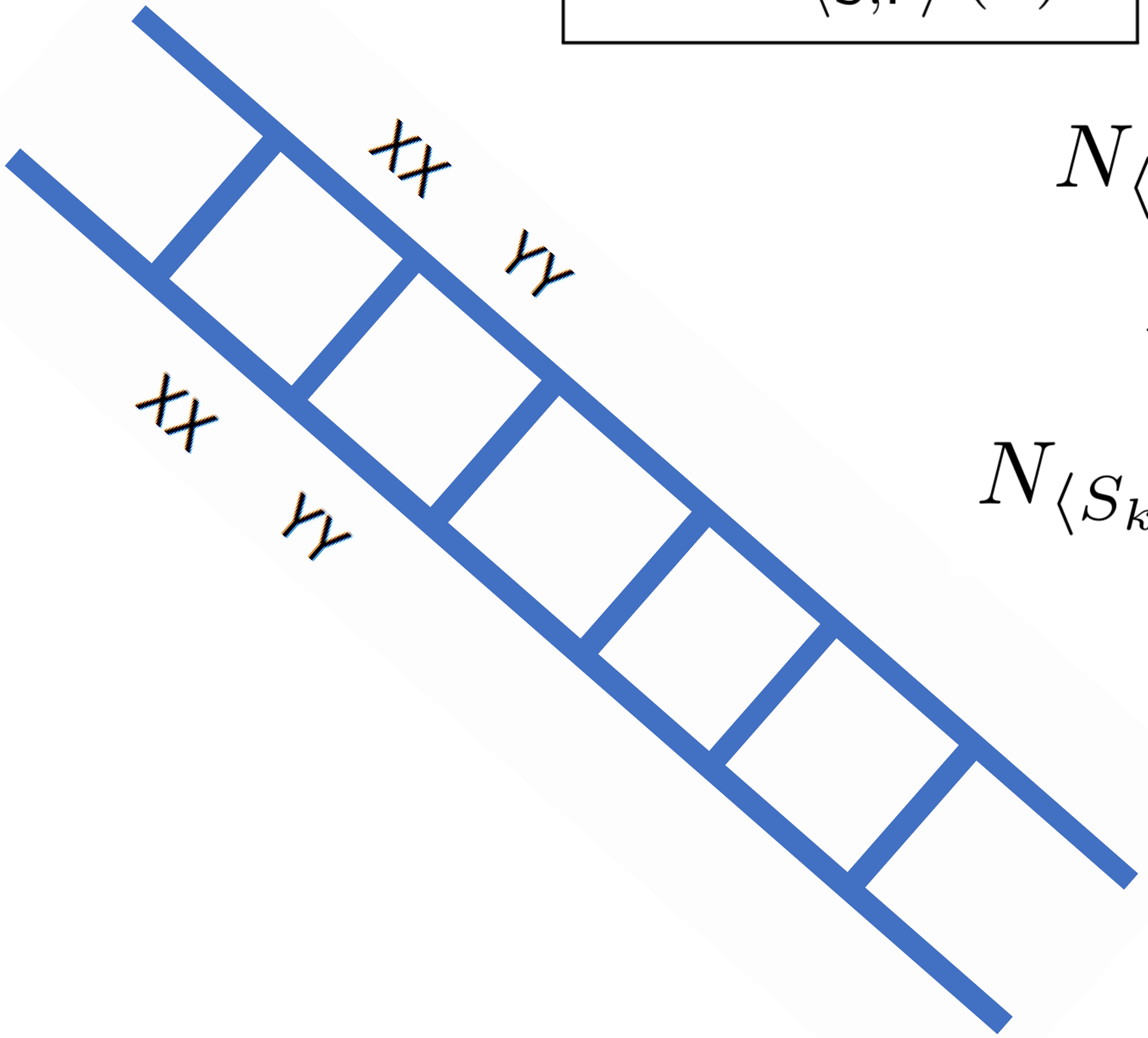
arxiv:1810.10037

FLOQUET QEC is CODE SWITCHING

$$\mathsf{S} \rightarrow \mathsf{N}_{\langle \mathsf{S},\mathsf{F}\rangle}\left(\mathsf{F}\right).$$

$$|0_L\rangle \propto \left|\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\rangle$$

$$|1_L\rangle \propto \left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right\rangle$$



$$\begin{array}{ccc} N_{\langle S_k,F_4\rangle}(F_4) & \xrightarrow{F_1} & N_{\langle S_{k+1},F_1\rangle}(F_1) \\ F_4\uparrow & & \downarrow F_2 \\ N_{\langle S_{k+3},F_3\rangle}(F_3) & \xleftarrow{F_3} & N_{\langle S_{k+2},F_2\rangle}(F_2) \end{array}$$

arxiv:2107.02194; acknowledge discussions w/ Arpit Dua

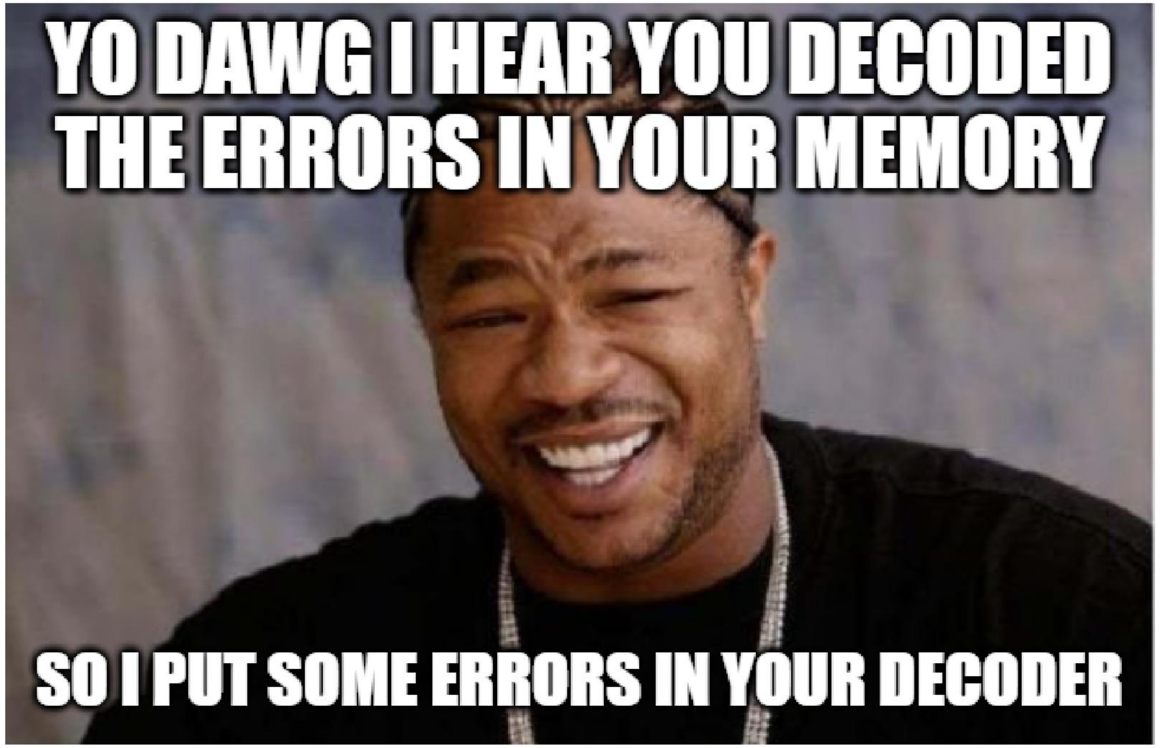
DYNAMICAL PROCEDURES: SUMMARY

	gauging out	gauge fixing	code switching
from/to using	stab. $\to$ gauge $\mathsf{F} \subseteq \mathsf{N}(\mathsf{S})$	gauge $\to$ gauge stab. group $\mathsf{F} \subset \mathsf{G}$	stab. $\to$ stab. stab. group $\mathsf{F} \subset \mathsf{P}_n$
S transforms as G transforms as	$\mathsf{S} \to \mathsf{Z}(\langle i, \mathsf{S}, \mathsf{F} \rangle)$ $\langle i, \mathsf{S} \rangle \to \langle i, \mathsf{S}, \mathsf{F} \rangle$	$\mathsf{S} \to \langle \mathsf{S}, \mathsf{F} \rangle$ $\mathsf{G} \to \mathsf{N}_{\mathsf{G}}(\mathsf{F})$	$\mathsf{S} \to \mathsf{N}_{\langle \mathsf{S}, \mathsf{F} \rangle}(\mathsf{F})$

DYNAMICAL PROCEDURES: SUMMARY

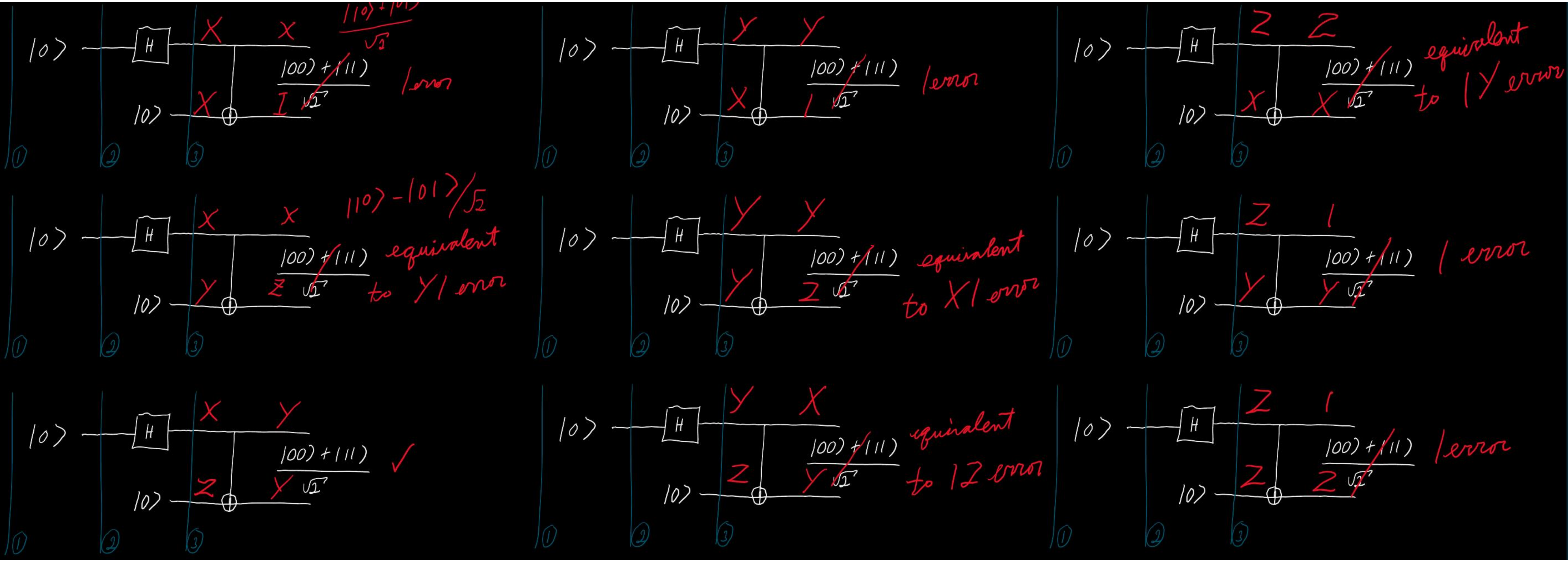
	gauging out	gauge fixing	code switching
from/to using	stab. $\to$ gauge $\mathsf{F} \subseteq \mathsf{N}(\mathsf{S})$	gauge $\to$ gauge stab. group $\mathsf{F} \subset \mathsf{G}$	stab. $\to$ stab. stab. group $\mathsf{F} \subset \mathsf{P}_n$
S transforms as G transforms as	$\mathsf{S} \to \mathsf{Z}(\langle i, \mathsf{S}, \mathsf{F} \rangle)$ $\langle i, \mathsf{S} \rangle \to \langle i, \mathsf{S}, \mathsf{F} \rangle$	$\mathsf{S} \to \langle \mathsf{S}, \mathsf{F} \rangle$ $\mathsf{G} \to \mathsf{N}_{\mathsf{G}}(\mathsf{F})$	$\mathsf{S} \to \mathsf{N}_{\langle \mathsf{S}, \mathsf{F} \rangle}(\mathsf{F})$
MBQC		✓ [1607.02579]	✓ [2212.06775]
lattice surgery		✓ [1810.10037]	✓ [1810.10037]
Floquet codes		× [2107.02194]	✓ [2107.02194]
anyon condensation			✓ [2212.00042]
chilral abelian top. phases	✓ [2211.03798]		

FAULT TOLERANCE



MULTI-QUBIT GATE FAULTS

Two-qubit gate errors occur on two qubits, so we have to take those into account by considering weight-two Paulis at any two-qubit gate locations.



FAULT TOLERANCE

The four-qubit code has a transversal implementation of the CZ-gate on its encoded subspace, $\overline{CZ} \simeq \sqrt{Z} \otimes \sqrt{Z}^\dagger \otimes \sqrt{Z}^\dagger \otimes \sqrt{Z}$, where $\sqrt{Z} = \text{diag}(1, i)$. We can measure this operator as follows. We note that conjugating S^X with the unitary rotation $\tilde{T} = T \otimes T^\dagger \otimes T^\dagger \otimes T$, where $T = \text{diag}(1, \sqrt{i})$, gives the hermitian operator:

$$\overline{W} \equiv \tilde{T} S^X \tilde{T}^\dagger \propto \overline{CZ} S^X. \quad (1)$$

Given that we prepare the code with $S^X = +1$, measuring $\overline{W}$ effectively gives a reading of $\overline{CZ}$.

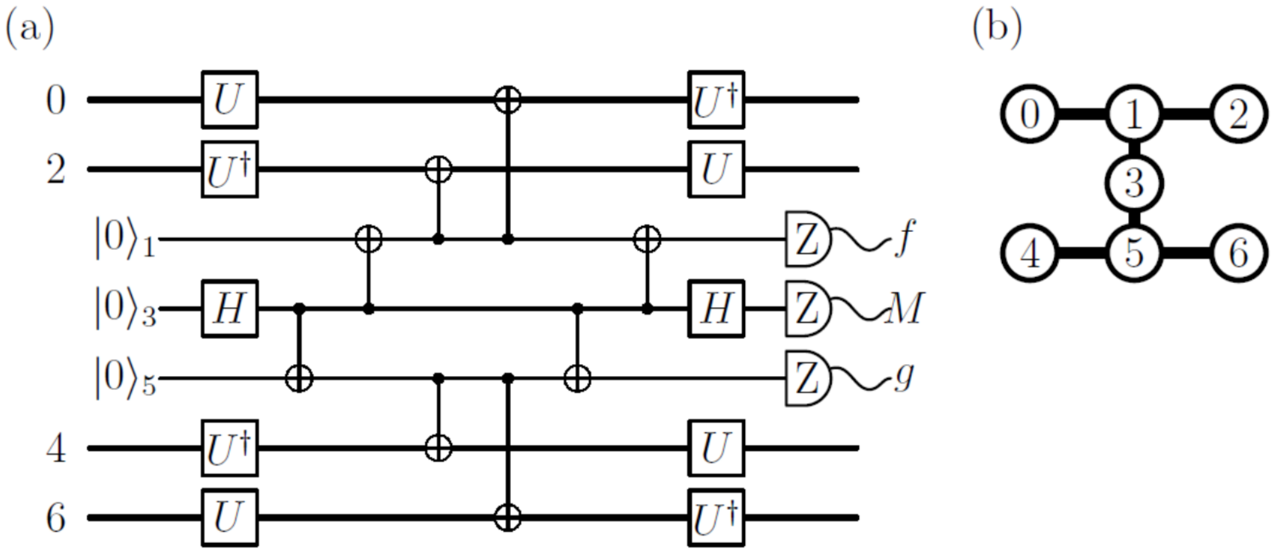


FIG. 1: A fault-tolerant circuit (a) to measure S^X, S^Z and $\overline{W}$ using flag qubits on the heavy-hexagonal lattice architecture (b). The four-qubit code is encoded on qubits with even indices and the other qubits are used to make the fault-tolerant parity measurement. The circuit measures S^X (S^Z) by setting $U = \mathbb{1}$ (H), where H is the Hadamard gate. As explained in the main text, the circuit measures $\overline{W}$ if we set $U = T$. Measurement outcome M gives the reading of the parity measurement, and outcomes f and g flag that the circuit may have introduced a logical error to the data qubits.

SUMMARY

1. Classical states are elements of a space ***X***; quantum states are functions on ***X***.
2. Error-correction paradigm works for spatio-temporal channels & classical/quantum info [Shannon].
3. Quantum codes have to protect against both bit- and phase-flip errors; there is a tradeoff.
4. QEC requires space-time overhead, which can be "Wick-rotated" (e.g., MBQC).
5. Degeneracy makes decoding harder; yields connections to statistical mechanics.
6. Geometric locality is physically relevant, but handicaps code parameters (QLDPC).
7. Circuit-centric approach emerging that requires less overhead for same robustness (e.g., Floquet).
8. Fault tolerance is the art of using QEC to make sure errors are not amplified during performance of desired task.
9. QEC has many non-computational applications (e.g., sensing, holography, topological order).

Thank you!