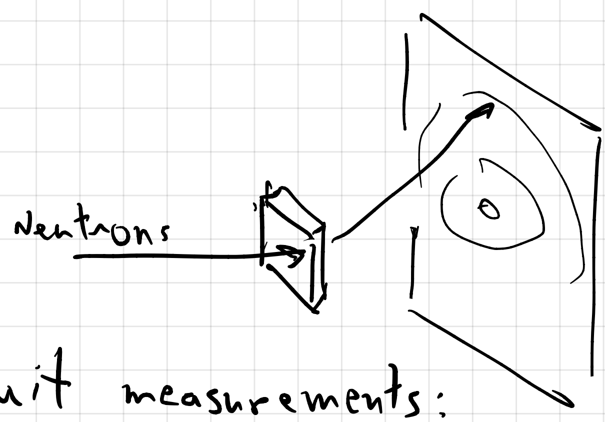
Lectures plan

- Lecture 1 - Measurement induced collective phenomena
  - A quantum information perspective on the non-local effect of measurements.
  - Impact of measurements on quantum critical states.
- Lecture 2 - Observing post-measurement states without post-selection
- Lecture 3 - Decoherence induced transitions and mixed state topological order
  - A new perspective on error thresholds in topological codes.

Measurements play a new kind of role in recent experiments with quantum devices

- Traditional experiments:

Measurements used as a diagnostic of the pristine state

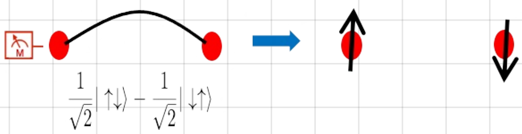


- Monitored systems / mid circuit measurements:

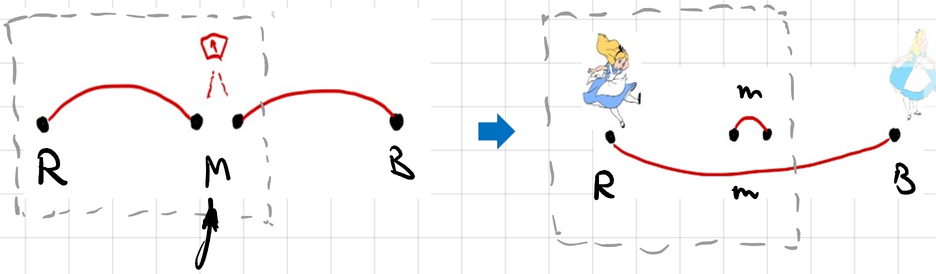
Measurements have non trivial impact on the quantum state (quantum collapse)

Non local effect of measurements

- Quantum Collapse can destroy quantum correlations:



- And can also create new ones:



measure in Bell basin

The local measurement on M created a quantum correlation between A and B that did not exist before the measurement.

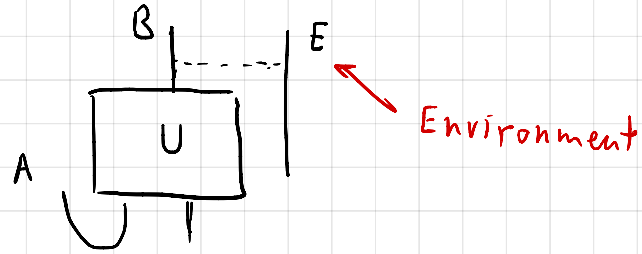
- The nonlocal effect of the measurement was facilitated by the pre existing entanglement

- It allows Alice to teleport her state to Bob

How to quantify the ability to teleport a quantum state = transmit quantum information?

Coherent information

The amount of quantum information transmited through a quantum channel is quantified by the coherent information

$\mathcal{N}(\rho) =$ 

E is the purification of ρ_{AB}

$$I_c(A \rangle B) = S_B - S_{AB} = S_B - S_E$$

Note that classically we must have $S_{AB} \geq S_B$

$\Rightarrow$ positive I_c is inherently quantum, resulting from entanglement between A and B

$$-S_A \leq I_c(A \rangle B) \leq S_A \leftarrow \text{equality if } \rho_{AB} \text{ pure}$$

Digression

Relation to classical conditional information

$$H(A|B) = H_{AB} - H_B \geq 0$$

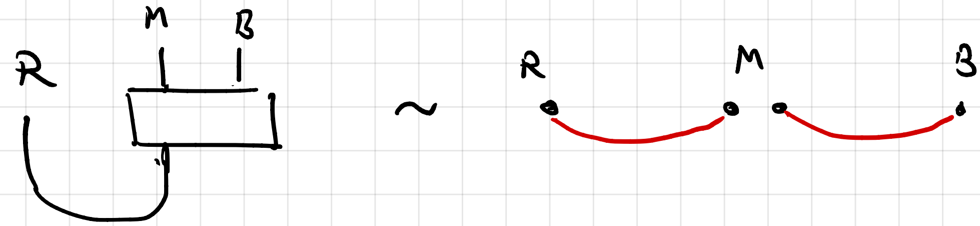
Uncertainty in A remaining after learning the value of B

$$I_c(A \rangle B) = -S(A|B) \leftarrow \text{quantum conditional information}$$

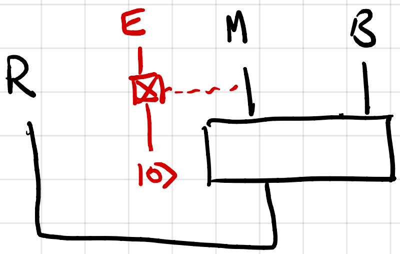
$I_c(A \rangle B) > 0$ is non classical

Effective circuit model for the teleportation protocol:

Before measurement:

$\Psi_{RBM} =$ 

After measurement:



Environment ancilla (E) purifies the quantum channel defined by the measurements.

 = CX

CX: $|m\rangle|0\rangle \to |m\rangle|m\rangle$

$$\Rightarrow \rho_M = \mathrm{diag}(P_m) = \rho_E$$

What is the coherent information we should calculate to quantify the teleportation?

- Is it $I_c(R\rangle B)$?

No! calculate it to show it is zero, as it must be! Now everything is unitary, cannot produce correlation between R and B. B must get the information on classical measurement outcomes.

The correct measure is

$$I_c(R\rangle MB) = S(MB) - S(E) = S(MB) - S(M)$$

Let's see that this gives something sensible.

$$\rho_{BM} = \begin{pmatrix} \boxed{p_{m_1}\rho_{B,m_1}} & & \\ & \boxed{p_{m_2}\rho_{B,m_2}} & \\ & & \ddots \end{pmatrix}$$

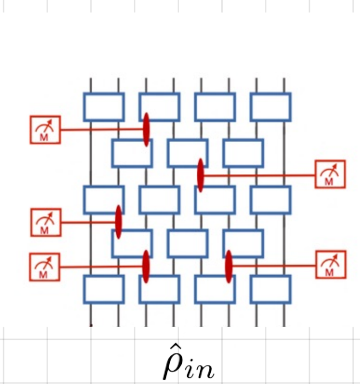
$$\Rightarrow \quad I_c(R\rangle MB) = \sum_m p_m \,\mathrm{tr}\left[\rho_{B,m} \log(p_m \rho_{B,m})\right] - \sum_m p_m \log p_m$$

$$= \sum_m p_m \log p_m \cdot \cancel{\mathrm{tr}\,\rho_{B,m}} + \sum_m p_m \rho_{B,m} \log \rho_{B,m} - \sum_m p_m \log p_m$$

$$= \sum_m p_m \rho_{B,m} \log \rho_{B,m} = \sum_m p_m S_m(B) = \overline{S_m(B)} \quad \checkmark$$

Exercise: If Bob is coupled to an environment E'? Design specific couplings to E' (decoherence channels that would give $I_c = 0$ and $I_c = -\log 2$ respectively.

Application to the MIPT



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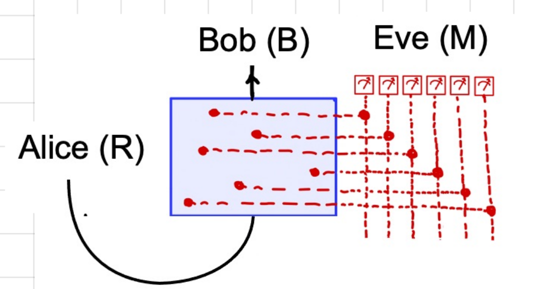
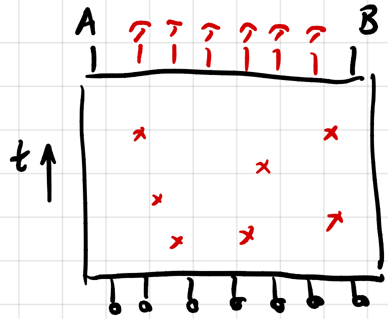


$$I_c(R\rangle BM) = \sum_m p_m S_m(B)$$

coherent information

~ Purification transition

Another setup:

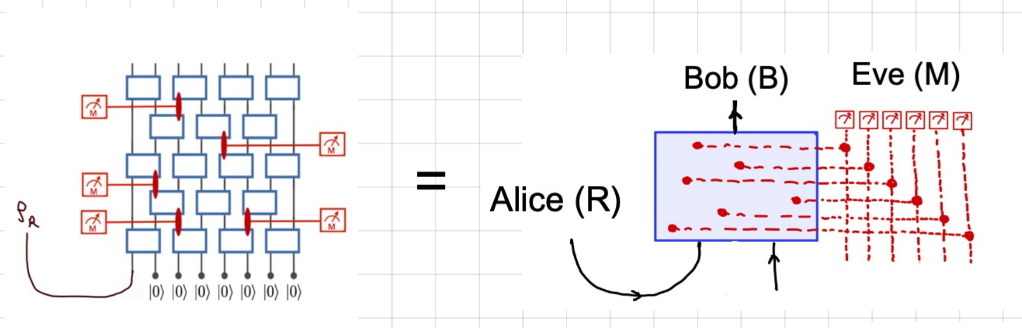


$$I_c(A \rangle B) = \sum_m p_m S_m(B)$$

if pure state $= \overline{S_m(A)} = \frac{1}{2}\overline{I_m(A,B)}$

For more on relation to teleportation see Sarang's lectures.

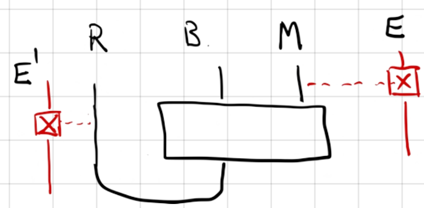
Information transition from Eve's perspective



How much can Eve learn about Alice's state from her measurements?

This is quantified by the **classical** mutual information between Alice and Eve

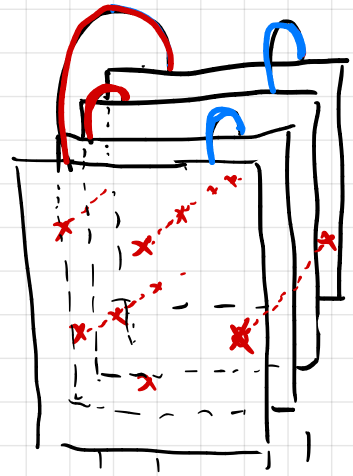
$$I(\tilde{R},\tilde{M}) = S(\tilde{M}) + S(\tilde{R}) - S(\tilde{R}\tilde{M})$$



Do all the different information measures undergo the same transition?
How do they translate to the effective statistical mechanics description ?

Brief review of the stat mech model

(See Romain's talk for details)



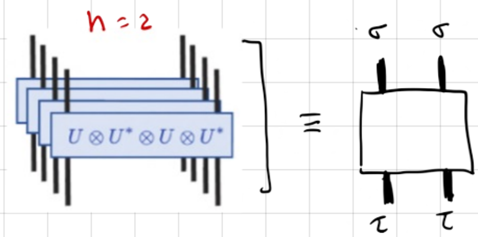
$|\psi_0\rangle\langle\psi_0| \otimes |\psi_0\rangle\langle\psi_0|$

- Need to compute time evolution of a replicated density matrix

Unitary evolution: path integral over $2n$ independent paths $U \otimes U^* \otimes U \otimes U^* \otimes \ldots$

Measurements: glue all paths to the same measurement outcome

$$\mathbb{E}\left[(U \otimes U^*)^n\right] = \mathbb{E}\left[\ \ \right] \equiv$$

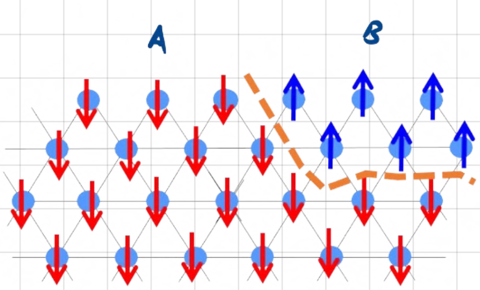


- Haar average projects the local replicated Hilbert space to n! states corresponding to the possible pairings of ket (forward propagating) and bra (backward propagating) copies.

$$|+\rangle \equiv \sum_\alpha |\alpha\rangle|\alpha\rangle \sum_\beta |\beta\rangle|\beta\rangle \equiv \cup\cup = \oplus$$

$$|-\rangle = \sum_{\alpha\beta} |\alpha\rangle|\beta\rangle|\beta\rangle|\alpha\rangle \equiv (\cup) = \ominus$$

- The real time path integral is mapped to the partition function of a classical "spin" model. The boundary condition set by the object we calculated



$$S^{(n)}(A) = F^{(n)}_{DW} - F^{(n)}_0$$

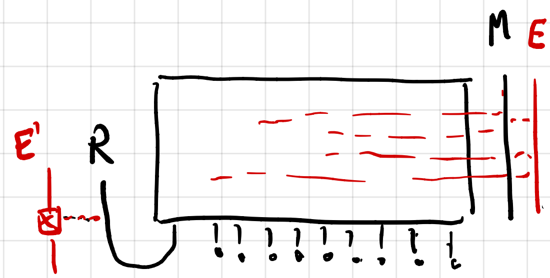
Sub system entanglement entropy $\rightarrow$ domain wall free energy

Volume law $\leftrightarrow$ ferromagnetic phase.

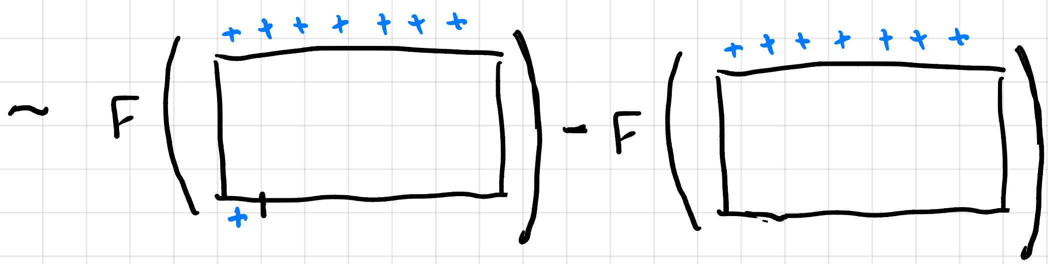
Other information measures map to the same model with different boundary conditions

<u>Exercise:</u> Derive the boundary conditions and show what stat-mech quantity each of the 3 information measures maps to.

<u>Solution for Learnability</u>



$$I(R,M) = S(M) + S(R) - S(RM) = S(M) - \sum_r P_r S_r(M)$$



prob that R is in state $|r\rangle$

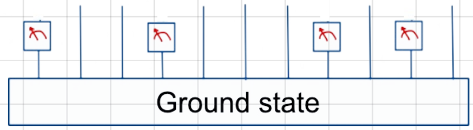
$$\sim \left.\frac{\partial F}{\partial h_R}\right|_{h_R=0} = m_r = \text{magnetization at site } R$$

Behaves like an order parameter

Non-local effects from measuring entangled ground states

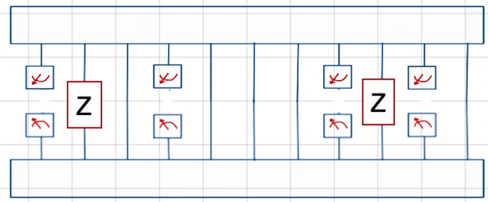
1. Given a quantum state,
   measure a subset of the qubits.

$|\Psi_m\rangle =$



2. Evaluate the correlations in the resulting state
   between unmeasured spins.
   Note: There is no dynamics!

$\langle\Psi_m|Z(x)Z(x')|\Psi_m\rangle =$



Garratt, Weinstein, EA, arXiv:2207.09476
Weinstein, Sajith, EA, Garratt, arXiv:2301.08268
Yang, Mao, Jian, arXiv230108255
Murciano et. al. arXiv:230204325
Sun, Yao and Jian arXiv:230111337

Measurement operators:

$$M_{\pm,j} = \sqrt{p}\,\frac{1 \pm \hat{\sigma}_j}{2} \,, \qquad M_{0,j} = \sqrt{1-p}$$

$$M_m = \prod_{j=1}^{L} M_{m,j}$$

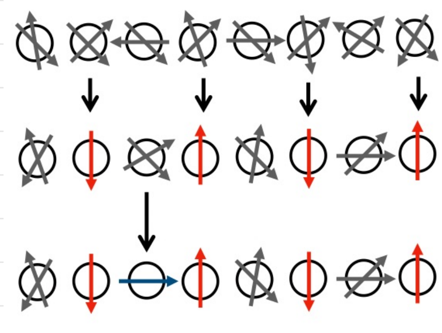
$$|\Psi_m\rangle = \frac{M_m|\Psi_0\rangle}{\sqrt{\langle M_m^2\rangle_0}} \qquad P_m = \langle M_m^2\rangle$$

How to characterize the post-measurement state?

Experimental sequence: $|\psi\rangle$

Measure $\hat{Z}_i$ $|\psi_m\rangle$

Measure probe observable $\hat{O}$:
outcome o_m



Repeat

Can we average the outcome o_m over runs ?

$$\overline{o_m} = \sum_m p_m \langle \psi_m | \hat{O} | \psi_m \rangle = \mathrm{tr}\left[\bar{\rho} \hat{O} \right] \qquad \bar{\rho} = \sum_m M_m \rho M_m$$

Local quantum channel cannot have a non-local effect

Need non-linear averages
(fluctuations between trajectories)

$$\sum_m p_m \left(\langle \hat{O}^2 \rangle_m - \langle O \rangle_m^2 \right)$$

Now focus on critical ground states

Examples: (1) Quantum phase transitions
(2) 1d quantum liquids

Scale invariance:

- entangled on all scales
- $\langle \phi(x) \phi(0) \rangle_0 - \langle \phi(x) \rangle_0 \langle \phi(0) \rangle_0 \sim x^{-2\Delta}$

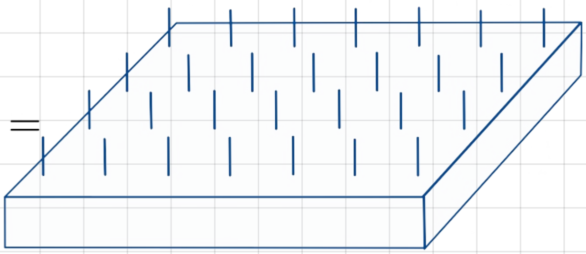
Local measurements can have a non local effect
How do they modify the critical correlations ?

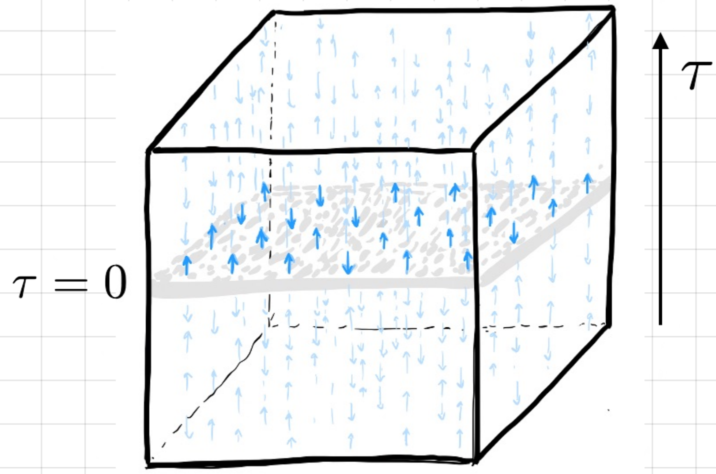
Example: critical transverse field Ising model

Ground state correlations in the d-dimensional quantum Ising model ⟷ Correlations on the τ=0 slice of the d+1 dimensional classical Ising model

$$H = -\sum_{\langle rr' \rangle} Z_r Z_{r'} - \sum_r X_r$$

$|\psi_0\rangle =$

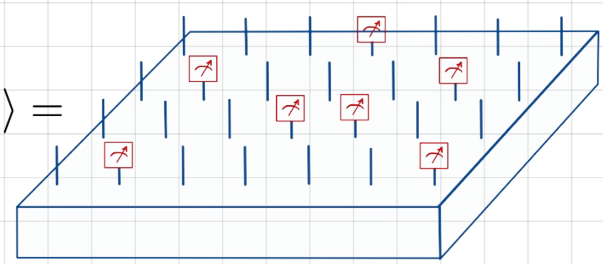


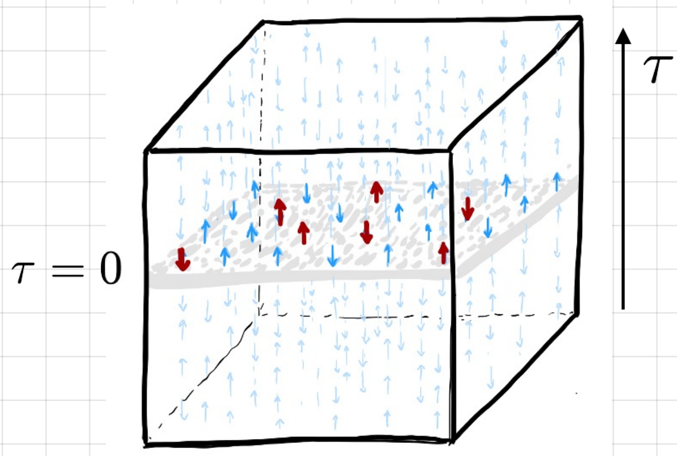


$$S_0[\phi] = \int d^dx \, d\tau \left[(\partial_\tau \phi)^2 + (\nabla \phi)^2 + r\phi^2 + u\phi^4 \right]$$

Post measurement state ⟷ Freeze the measured spins on the τ=0 surface.

$|\psi_m\rangle =$





$$S_m[\phi] = S_0[\phi] - \lambda \int d^dx \; m(x) \, \psi(x)$$

Post measurement correlations mapped to surface criticality with a new kind of disorder on the surface we may ask for example:

How partial measurement of $\hat{Z}_i$ (order parameter) impacts the post measurement connected correlation?

$$C_m(r) = \langle Z_r Z_0 \rangle_m - \langle Z_r \rangle_m \langle Z_0 \rangle_m$$

Simple to solve for the 3 dimensional transverse-field Ising mode → Gaussian fixed point

$$S_0 = \frac{1}{2} \int d^d x \, d\tau \left[(\partial_\tau \phi)^2 + (\nabla \phi)^2 \right]$$

partition function withe the measurement defect

$$Z_m = \int \mathcal{D}\phi \, e^{-S[\phi] + \mu \int d^d x \, m(x) \phi(x)}$$

scaling dimension of ϕ at the Gaussian fixed point

$$d + 1 - 2 + 2[\phi] = 0 \qquad [\phi] = \frac{1-d}{2}$$

$$\Rightarrow \quad \langle \phi(x) \phi(0) \rangle_0 - \langle \phi(x) \rangle_0 \langle \phi \rangle_0 \sim x^{1-d} \sim 1/x^2 \quad \text{for } d = 3$$

scaling of μ (heuristic without replicas):

$$[\mu] = \frac{d}{2} + [\phi] = \frac{1}{2}$$

due to random field $m(x)$

$\Rightarrow$ measurements are relevant

$\Rightarrow$ decouple the half spaces $\tau > 0$ and $\tau < 0$
pin the order parameter at $\tau = 0$

Now we need o.p. near a boundary where the field ϕ is pined.

$$Z_i^{bdry} \sim (\partial_\tau \phi)_{\tau=0}$$

$$[\partial_\tau \phi] = \frac{1-d}{2} - 1 = -\frac{d+1}{2}$$

$$\langle Z_r Z_0 \rangle_m - \langle Z_r \rangle_m \langle Z_0 \rangle_m \sim \langle \partial_\tau \phi(r,0) \partial_\tau \phi(0,0) \rangle - \langle \partial_\tau \phi(r,0) \rangle \langle \partial_\tau \phi(r,0) \rangle$$
$$\sim x^{-(d+1)} \sim x^{-4}$$

In more general cases we do need replicas

<u>Replica field theory</u>

$$\overline{\langle \hat{O}_r \rangle_m \langle \hat{O}_0 \rangle_m} = \sum_m P_m \langle \hat{O}_r \rangle_m \langle \hat{O}_0 \rangle_m = \sum_m \frac{\langle \hat{O}_r M_m^2 \rangle_0 \langle \hat{O}_0 M_m^2 \rangle_0}{\langle M_m^2 \rangle_0}$$

where $\hat{O}_r$ is the probe observable at site r

Note that we used the fact that r an 0 are unmeasured sites therefore M_m comute with $\hat{O}_0$ and with $\hat{O}_0$

Non trivial denominator! solve with replicas

$$\overline{\langle \hat{O}_r \rangle_m \langle \hat{O}_0 \rangle_m} = \lim_{n\to 1} \frac{\sum_m P_m^n \langle \hat{O}_r \rangle_m \langle \hat{O}_0 \rangle_m}{\sum_m P_m^n} = \lim_{n \to 1} \frac{\sum_m \langle M_m^2 \rangle_0^{n-2} \langle \hat{O}_r M_m^2 \rangle_0 \langle \hat{O}_0 M_m^2 \rangle_0}{\sum_m \langle M_m^2 \rangle_0^n}$$

$$= \lim_{n\to 1} \frac{\langle \psi_0^{\otimes n} | O_r^{(1)} O_r^{(2)} \sum_m (M_m^2)^{\otimes n} | \psi_0^{\otimes n} \rangle}{\langle \psi_0^{\otimes n} | \sum_m (M_m^2)^{\otimes n} | \psi_0^{\otimes n} \rangle}$$

The denominator is a partition function with measurements inserted at $\tau = 0$ leading to interactions that tend to lock the replicas to the same value of the measured operator.

$$\sum_m (M_m^2)^{\otimes n} = \prod_{j=1}^{L} \left\{ (1-p)^n + p^n \frac{1}{2} \sum_{m_j = \pm 1} (1 + m_j \hat{\sigma}_j)^{\otimes n} \right\}$$

$$\sim \prod_{j=1}^{L} \left\{ 1 + \lambda \sum_{r=1}^{n/2} \sum_{1 < \alpha_1 < \dots < \alpha_{2r} \leq n} \hat{\sigma}_j^{\alpha_1} \cdots \sigma_j^{\alpha_{2r}} \right\}$$

$$\sim e^{-\lambda \sum_{j=1}^{L} \sum_{1 < \alpha_1 < \dots < \alpha_{2r} \leq n} \hat{\sigma}_j^{\alpha_1} \cdots \sigma_j^{\alpha_{2r}}}$$

Now taking the expectation value on the replicated

This leads to the field theory:

$$S = \sum_{\alpha=1}^{n} S_0[\phi^\alpha] - \lambda \sum_{\alpha < \alpha'} \int d^d x \, \psi^\alpha(x, \tau = 0) \, \psi^{\alpha'}(x, \tau = 0)$$

For the gaussian fixed point

$$[\lambda] = d + 2[\phi] = d + 1 - d = 1$$

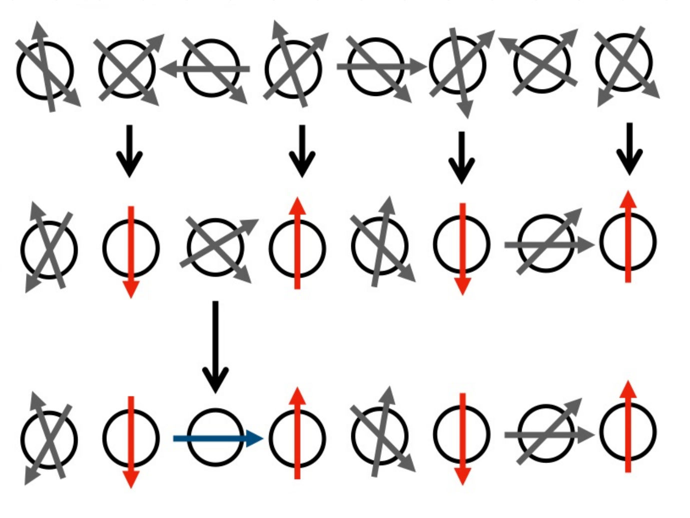
The post-selection problem

$|\psi\rangle$

Measure $\hat{Z}_i$ on many sites and create a state $|\psi_m\rangle$

$|\psi_m\rangle$

To characterize $|\psi_m\rangle$ measure a probe observable $\hat{X}$: outcome x_m



To probe $\langle \hat{X} \rangle_m = \langle \psi_m | \hat{X} | \psi_m \rangle$ need to obtain the same state $|\psi_m\rangle$ many times.
But the probability for this is vanishingly small: $p_m \sim 2^{-M}$

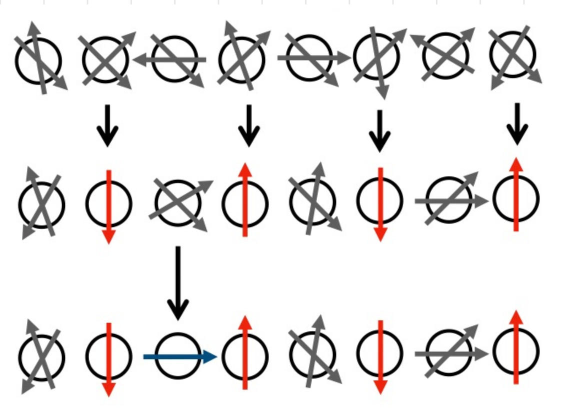
Averaging observables over measurements = dephasing

$|\psi\rangle$

Measure $\hat{Z}_i$

$|\psi_m\rangle$

Measure probe observable X:
outcome x_m



Repeat

$$\overline{x_m} = \sum_m p_m \langle \psi_m | \hat{X} | \psi_m \rangle = \mathrm{tr}\left[\bar{\rho}\hat{X}\right]$$

$$\bar{\rho} = \sum_m \hat{P}_m \rho \hat{P}_m$$

A local quantum channel. cannot have non-local effect!

Need non-linear averages: $\sum_m p_m (\langle X^2 \rangle_m - \langle X \rangle_m{}^2)$ but these are unobservable!

How to ubserve unobservables?

Nonlinear in ρ_m ➡ unobservable $$\overline{\langle X\rangle^2_m} = \sum_m p_m \langle\psi_m|X|\psi_m\rangle^2$$

Alternative linear quantity: $$\overline{w_m x_m} = \sum_m p_m w_m \langle\psi_m|X|\psi_m\rangle$$

The weight w_m utilizes knowledge of the measurement outcomes.

a natural choice for w_m is an estimate of the conditional expectation value obtained by a classical computation $$w_m = \langle\psi_m|X|\psi_m\rangle^{\mathrm{C}}$$

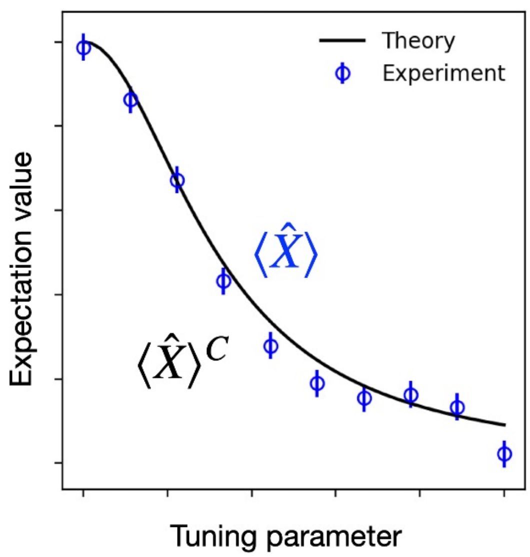
Observe measurement induced phenomena through the ”quantum classical correlation”:

Cross correlation between model and experiment $$\overline{\langle X\rangle^C_m x_m} = \sum_m p_m \langle X\rangle^C_m \langle X\rangle_m$$

Compare to the fully theoretical result: $$\overline{\langle X\rangle^C_m \langle X\rangle^C_m}$$

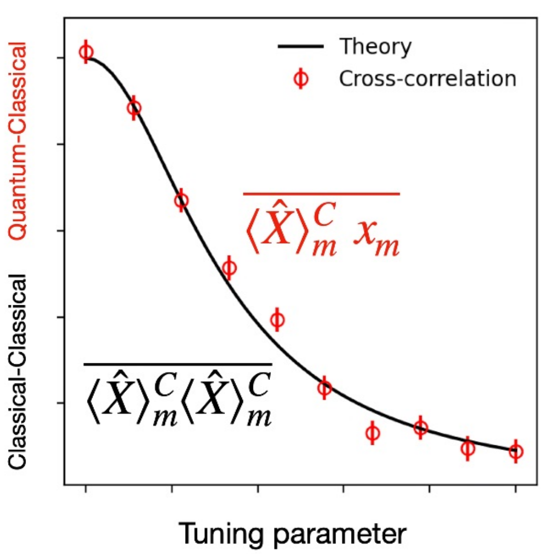
A new paradigm for comparing experiment to theory

Traditional experiments



Post-measurements experiments

Cannot determine conditional expectation values



Problem: In this paradigm, the experimental result is sensitive to our theory (classical model).

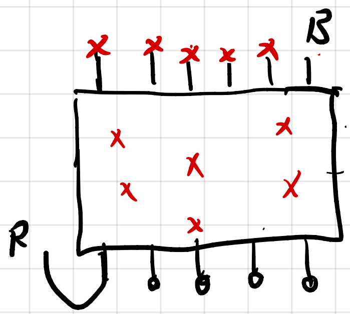
Can we learn about theory independent quantities?

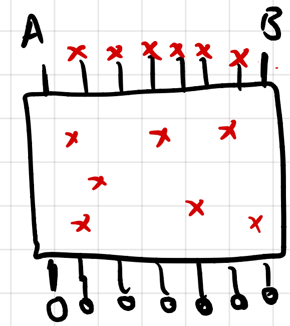
Quantum classical correlators: single shot shadows

want an information quantity

e.g. $I_c(R\rangle BM) = \overline{S_m(B)}$

$I_c(A\rangle BM) = \overline{S_m(B)}$





(Maps to spin-spin rorelations in the stat mech model

we want $\overline{S_m} = \overline{tr[\rho_m \log \rho_m]}$

replace with single-shot measurement outcome ?

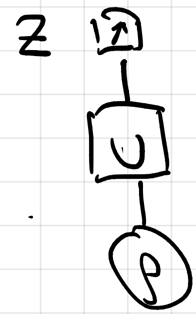
replace by $\log \rho_m^c$ (classically calculated matrix)

what is a matrix valued object we can extract from individual shots that converges to ρ_m upon averaging over shots ?

Solution : a single shot classical shaddow step back what is a classical shaddow?

[Huang, Keung & Preskill Nature Phys 2020]

Suppose we want to learn a state ρ which we can create repeatedly. The shaddow approach is



1) create ρ
2) rotate with random U
3) Measure $\hat{Z}$ outcome $r = \pm 1$
4) repeat

There is an object $\hat{\sigma}(r, U)$ such that $E_r[\sigma] = \rho$

more explicitly

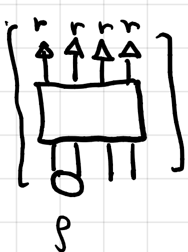
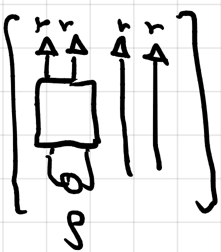
$$\sum_U \sum_r P(U)\, P(r|U)\, \hat{\sigma}(r,U) = \rho$$

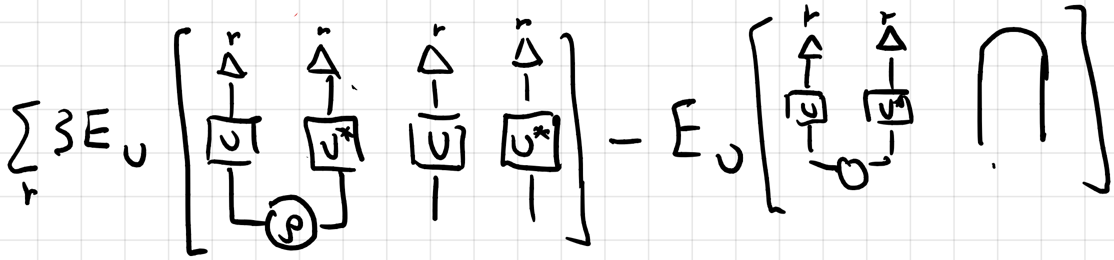
$P(r|U) = \langle r | U \rho U^\dagger | r \rangle$ Born probability

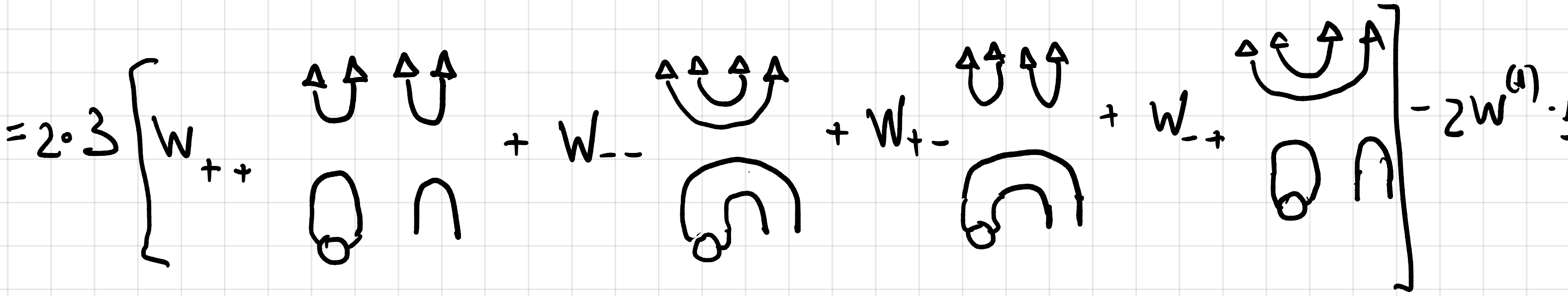
The correct object is: $\hat{\sigma} = 3 U^\dagger |r\rangle\langle r| U - \mathbb{1}$

if U is chosen from Haar ensemble or more precisely it's enough if it forms a 2-design.

$$E_U \left[\sum_r \langle r | U \rho U^\dagger | r \rangle \left(3 U^\dagger |r\rangle\langle r| U - \mathbb{1} \right) \right]$$

$= \sum_r \Big\{ 3 E_U$  $- E_U$  $\Big\} = \rho$





$$= 6\left(\tfrac{1}{3}\mathbb{1} + \tfrac{1}{3}\rho - \tfrac{1}{6}\rho - \tfrac{1}{6}\mathbb{1}\right) - \mathbb{1} = \rho$$

Can we still use this approach if we can create each state ρ_m only once?

i.e. every run we create a different ρ_m and produce a single shot shaddow

$$\sigma_m = 3U^\dagger|r_m\rangle\langle r_m|U - \mathbb{1}$$

The answer is yes because of linearity
We can take perform the average over m last.

$$E_r\left[\mathrm{tr}(\sigma_{m_r} W^c_{m_r})\right] = \sum_m p_m \sum_U \sum_{r_m} P(U)\, p(r_m|U)\, \mathrm{tr}\left(\sigma_m(U)\, W^c_m\right)$$

$$= \sum_m p_m \mathrm{tr}\left[W_m \sum_U \sum_{r_m} P(U) \langle r_m|U \rho_m U^\dagger|r_m\rangle \left(3U^\dagger|r_m\rangle\langle r_m|U - \mathbb{1}\right)\right]$$

$$= \sum_m p_m \mathrm{tr}\left[\rho_m W^c_m\right]$$

Bounding the true entanglement entropy

Upper bound:

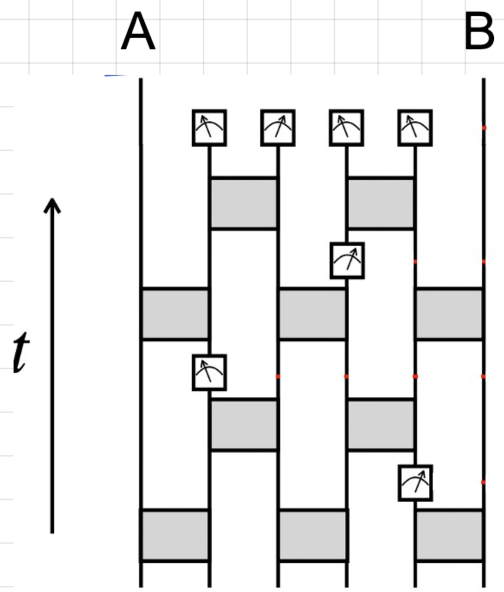
$$S_m^{QC} - S_m = -\left(\mathrm{tr}\left[\rho_m \log \rho_m^C\right] - \mathrm{tr}\left[\rho_m \log \rho_m\right]\right)$$

$$= D\left(\rho_m | \rho_m^C\right) \geq 0$$

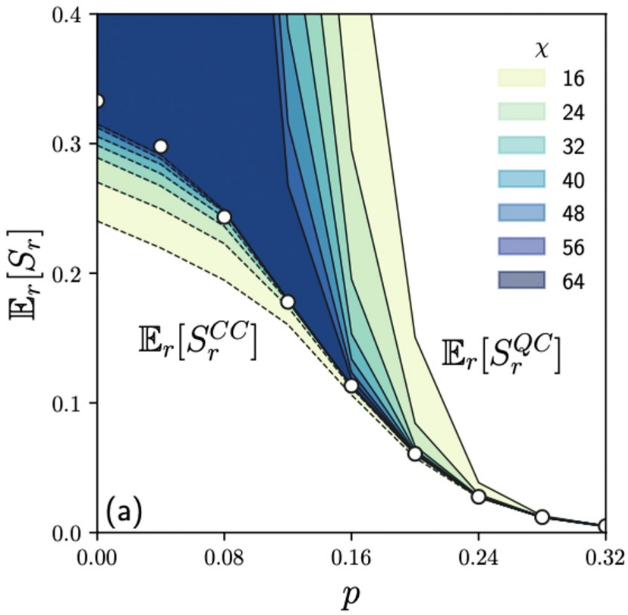
$$S_A^{QC} \geq \overline{S_A}$$

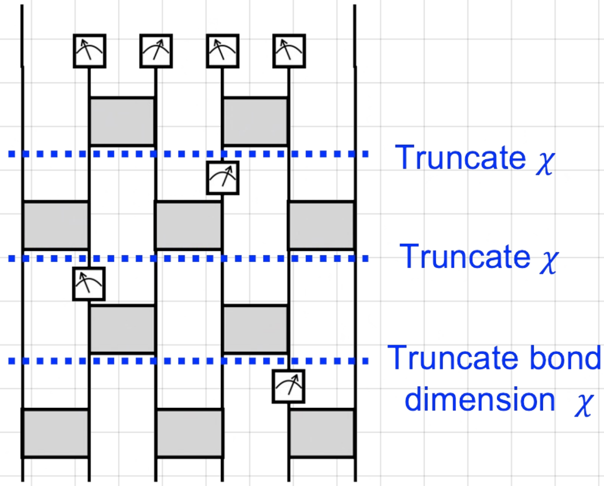
Using the monotonicity of *D* (i.e. $D_A \leq D_{AB}$)
we can get also a **lower bound:**

$$\boxed{S_A^{QC} \geq \overline{S_A} \geq S_A^{QC} - S_{AB}^{QC}}$$

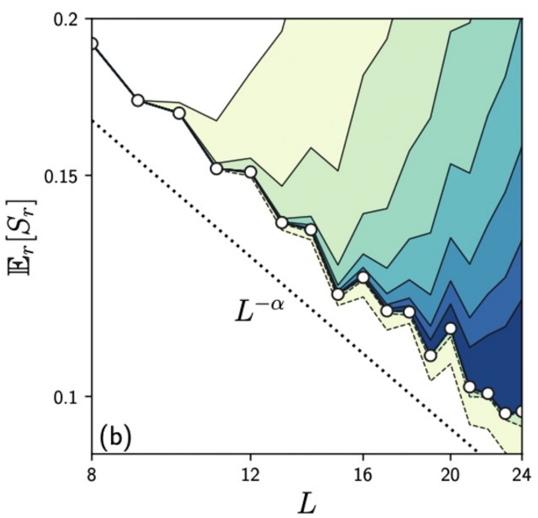


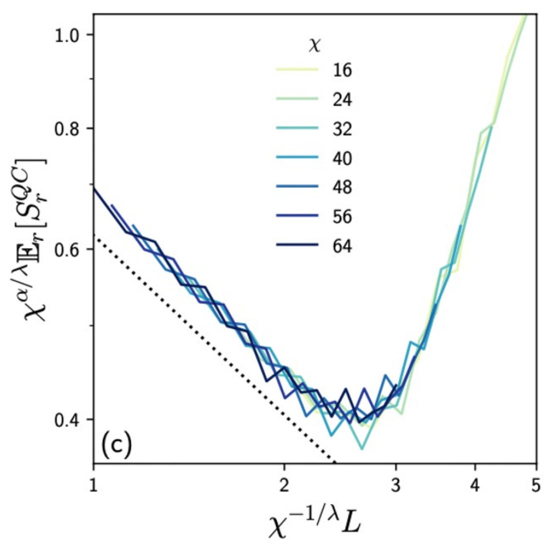
Example: Using MPS as the classical model for ρ_m^C

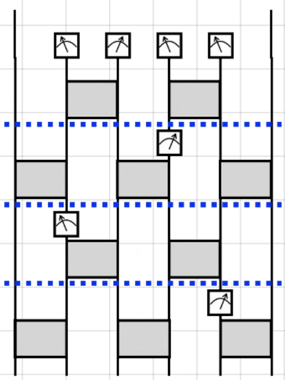




Entanglement estimator vs L at the critical point:



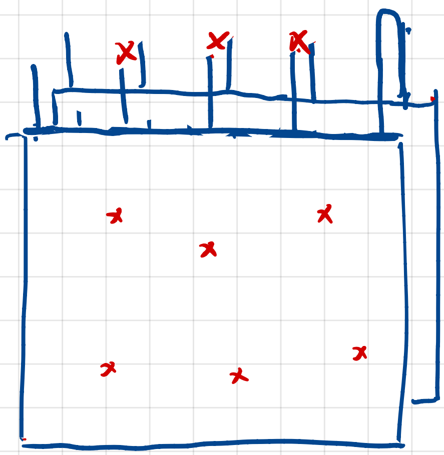




Requires only polynomial resources with system size: $\chi_\star \sim L^\lambda$ $(\lambda \approx 2)$

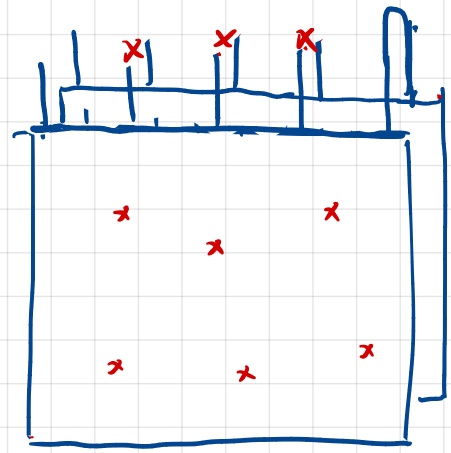
Connection to decoding

experiment

$\rho_m =$ 

$\xrightarrow{m}$

calculation

$\rho_m^c =$ 

$\mathrm{tr}(\rho_m^2) = \frac{1}{2}(1+\gamma_m^2)$

$\mathrm{tr}\left[(\rho_m^c)^2\right] = \frac{1}{2}\left(1+(\gamma_m^{cc})^2\right)$

Decoding:

1. Find unitary rotation U_c s.t. $U_m^c \rho_m^c U_m^{c\dagger} = \frac{1}{2}(1+\gamma^{cc}\hat{Z})$ is parallel to $\hat{Z}$

2. Apply U_c to the experiment: $U_m^c \rho_m U_m^{c\dagger}$

3. measure Z on output

$$P_{+1} = \langle 1| U_m^c \rho_m U_m^{c\dagger} |1\rangle = \frac{1}{2}(1+\gamma_m^{QC}) \qquad \gamma_m^{QC} < \gamma_m$$

4. Average over runs $\quad \mathbb{E}_r(\gamma_{m_r}^{QC}) = \gamma^{QC} \leq \bar{\gamma}_m$

We can extract the same property through classical post-processing alone using the

$$E_r\left[\sigma_{m_r} W_{m_r}^c\right] = \gamma^{QC} \qquad W_m = 2(U_m^c)^\dagger |1\rangle\langle 1| U_m^c - \mathbb{1}$$