Boulder notes by Victor V. Albert. Most of the lecture closely follows his notes; some extra parts are below.

I. ASHWIN VISHWANATH 2

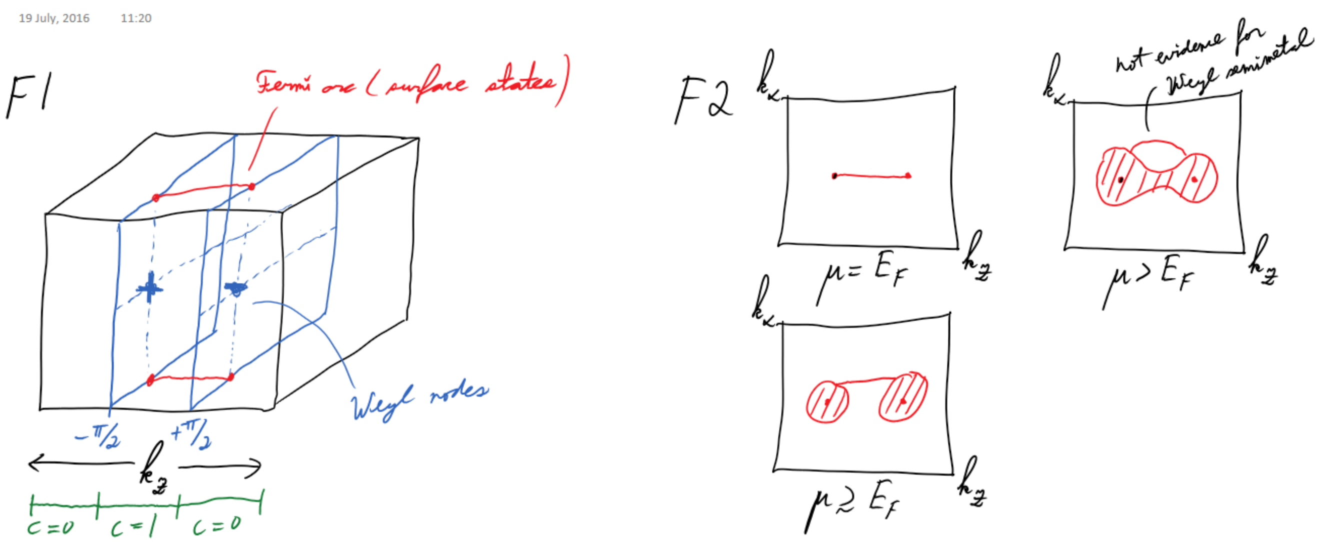
I.1. Weyl semimetals

While the lecture is mostly in his notes, some things are not mentioned. The Hamiltonian in eq. (9) is $H = \vec{d}\cdot\vec{\sigma}$ and we look at surfaces in the $k_x - k_y$ plane (although similar things hold for other cuts):

- $k_z = 0$: The vector $\vec{d}(k_x, k_y)$ looks like a skyrmion when plotted in the 2D $k_x - k_y$ plane. Chern number is 1.
- $k_z = \pi$: $\vec{d} = \langle 0, 0, 3 - \cos k_x - \cos k_y \rangle$. Chern number is 0.

The Fermi surface looks different depending on the chemical potential.

- Let's assume that the chemical potential is at $E_F = 0$, so the Fermi surface is two points at the resective Weyl nodes $\pm k_0$. More generally, H has different Chern numbers for the region $|k_z| < k_0$ inbetween the Weyl nodes and the region outside of them. The BZ used to be a torus, but now it is punctured at the two Weyl nodes since there is a gap closing at each of them. Since the Chern number is 1 in the region $|k_z| < k_0$, there exist surface states there. In the band structure in the $k_x - k_z$ plane, those states are represented by a **Fermi arc** (F1).
- If the chemical potential is not at the Weyl nodes, then the two point Fermi surface becomes two spheres around those points and the Fermi arcs move around (F2).



We $k_0 = \pi$, then the Fermi arcs annihilate and the entire BZ has Chern number 1. Therefore, the system becomes a 3D quantum Hall insulator (a weak TI from Charlie Kane's lectures).

I.2. Weyl semimetals in a magnetic field

Introduce a magentic field $\vec{B} = \vec{\nabla} \times \vec{A} = B\hat{z}$ and put it into Ham:

$$H = \vec{p}\cdot\vec{\sigma} \longrightarrow \left(\vec{p} - \vec{A}\right)\cdot\vec{\sigma}\,.$$

You will get Landau levels of the Dirac type (see Pablo Jarillo-Herrero) with energies

$$E_n \propto \pm\sqrt{Bn + p_z^2}\,.$$

I.3. Graphene as a topological semimetal

Consider graphene and look at lines in the BZ to see how they change upon crossing nodes. In contrast to Charlie Kane's lectures and motivation, we need to make sure that the gap in graphene is **closed** as opposed to trying to open it. Therefore, we cannot look at any 2D invariants since the system is gapped. Since we are gapped, we need to preserve the Hamiltonian under time-reversal, inversion, and $su(2)$ spin rotation. The combined $\tilde{T} = TI$ with $\tilde{T}^2 = +1$, which determines that our symmetry class is AI. But which dimension do we pick? In 1D, class AI is trivial. Teo and Kane looked at defects in otherwise translationally invariant Hamiltonians and saw that defect real-space coordinates contribute a negative dimension:

$$H\left(\underbrace{\vec{k}}_{d_k}\underbrace{\vec{r}}_{d_r}\right) \qquad\qquad d = d_k - d_r\,.$$

So instead of looking at $d = 1$, we look at $d = 7$, which does have a $\mathbb{Z}_2$ index.

II. ASHWIN VISHWANATH 3

Topological entanglement entropy (TEE) is the constant offset γ in the entropy of the reduced density matrix of a region A of space:

$$S_A = -\mathrm{Tr}\left\{\rho_A \log \rho_A\right\} = \alpha L - \gamma,$$

where L is the perimeter of A and α is a constant. If $\gamma > 0$ in the ground state of a Hamiltonian on a 2D plane, the phase is **long-range entangled** (LRE). If $\gamma = 0$, the phase is short-range entangled. For the toric code, $\gamma = \log 2$.

III. ASHWIN VISHWANATH 4

III.1. 1D cluster state model

Consider the frustration-free model in $d = 1$ with $\mathbb{Z} \times \mathbb{Z}$ symmetry:

$$H = -J\sum_i Z_i X_{i+1} Z_{i+2} \equiv -J\sum_i \tilde{X}_i\,.$$

The two symmetries are $g_\mu = \prod_i X_{2i+\mu}$. The frustration-free nature can be checked by ordering the terms by site:

$$\begin{pmatrix} Z & X & Z & & \\ & Z & X & Z & \\ & & Z & X & Z \end{pmatrix}.$$

There is a unique ground state in PBC, but with open BC, we have a symmetry protected degeneracy. Possible commuting edge operators are

$$\Sigma_L^Z = Z_0 \qquad\qquad \Sigma_L^X = X_0 Z_1 \qquad\qquad \Sigma_L^Y = Y_0 Z_1.$$

These edge operators commute with H and anti-commute with each other, implying a two-fold degeneracy on each edge and a total **four-fold** degeneracy. The symmetry operators are now

$$g_1 = Z_0 Z_{2N} = \Sigma_L^Z \Sigma_R^Z \qquad\qquad g_1 = \Sigma_L^X \Sigma_R^X\,.$$

These satisfy the group multiplication rules when multiplied:

$$g_1 g_2 = g_2 g_1\,.$$

However, the one-edge components of $g_{1,2}$ form a **projective representation** (the group multiplication rules are changed by extra factors):

$$g_1^R = \Sigma_R^Z \qquad\qquad g_1^R = \Sigma_R^X \qquad\qquad g_1^R g_2^R = -g_2^R g_1^R\,.$$

Classification of the projective representations (and therefore the phases) which add $U(1)$ phases to group multiplication rules is related to the second cohomology gorup $H^2(G, U(1))$. Here

$$H^2\left(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)\right) = \mathbb{Z}_2\,.$$

III.2. 2D boson-vortex duality (Peskin, Dasgupta/Halperin)

Consider bosons hopping on a lattice with creation operators $e^{i\phi_i}$ with conjugate number operators n_i such that $[n_i, \phi_j] = \delta_{ij}$:

$$H = -t \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j) + U \sum_i n_i^2 .$$

This model is an insulator at $t \ll U$ and a superfluid at $t \gg U$. We look for a description of this model in terms of another set of operators — vortices:

1. The vortices are determined by locations, so they can be thought of as particles $\psi_V^\dagger(r)$.
2. Vortices have long-range interactions, so in 2D $v(|r - r'|) = -\log(|r - r'|)$. This is the same behavior as for electric charges.
3. A boson sees a vortex through 2π flux and visa versa. This turns out to be non-trivial (despite the phase being trivial). There is also then a vector potential $\vec{a}$ and electric/magnetic fields $\vec{e}, b$ causing this flux.

Boson	Vortex
density n	flux $\frac{1}{2\pi}\vec{\nabla} \times \vec{a} = \frac{1}{2\pi} b$
$\frac{1}{2\pi}\vec{\nabla} \times \left(\vec{\nabla}\phi\right)$	vortex density n_V
$\frac{1}{2\pi}\hat{z} \times \left(\vec{\nabla}\phi\right)$	electric field e
Time reversal	Particle-hole symmetry

We can model the vortex interactions and coupling to the EM fields with the Lagrangian

$$L = |(\partial_\mu - ia_\mu)\psi_V|^2 + m^2 |\psi_V|^2 + |\psi_V|^4 + \left(e^2 + b^2\right) + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda .$$

The background field A (last term) keeps track of the boson charges.

In the insulator phase, the bosons are gapped but in the dual picture, the vortices are "condensed".

In the superfluid phase, there is a phononic Goldstone mode and the bosons are condensed. In the dual picture, the vortices are gapped. In the Lagrangian, this corresponds to the photon of the EM photon (when we ignore anything with a ψ_V in it).

III.3. 2+1D SPT using two layers of bosons

Take two layers A and B with background fields A_μ^A and A_μ^B and consider the Lagrangian of a bound state of a vortex in A and a charge in B, $\tilde{\psi}_V = \psi_V^A \Phi_B$:

$$L_{BS} = \left|\left(\partial_\mu - ia_\mu - iA_\mu^B\right)\tilde{\psi}_\nu\right|^2 + \left(e^2 + b^2\right) + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda .$$

If we condense ($\left\langle \tilde{\psi}_V \right\rangle \neq 0$) and use the Meissner effect $a + A^B = 0 \to a = -A^B$, we can ignore almost everything:

$$L_{BS} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu^A \partial_\nu A_\lambda^B .$$

This is a multi Chern-Simons theory with $K = X$ (see Chetan Nayak's talk). Since $|\det K| = 1$, this is a trivial topological phase (while for the toric code $|\det K| = 4$). Taking a look at the response,

$$j_a = \sigma_{ab} E_b \qquad\qquad j_\mu = \frac{\delta L}{\delta A_\mu} = \frac{1}{\pi}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda .$$

To see how this is related to Hall conductance, let's take a look at

$$j_1 = \frac{1}{\pi}\left(\partial_2 A_0 - \partial_0 A_2\right) = \frac{1}{\pi} E_2 \longrightarrow \sigma_{xy} = \frac{1}{\pi} .$$

This indeed implies a quantized conductance if we plug in bosons of charge q and $\hbar = 1$, meaning that $2\pi \to h$ and so $\sigma_{xy} = 2\frac{q^2}{h}$. So for bosons, the Hall conductance is quantized and in multiples of two. If we consider Cooper pairs, $q = 2e$ and conductance is in multiples of 8. We can relate the quantum Hall charge transfer to heat transfer via the Wiedeman-Franz law for free fermions, Cooper pairs on the other hand violate Wiedeman-Franz due to interactions.