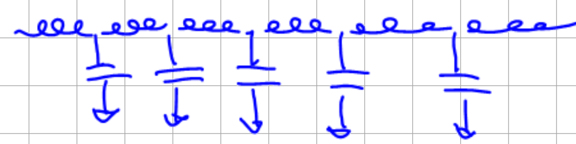
Reflection of microwave light from a resonator

transmission line

equivalent circuit  inductance per unit length ℓ

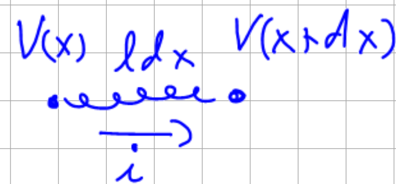
capacitance " " " c

characteristic impedance $Z_c \equiv \sqrt{\ell/c} \sim 50-300\,\Omega$

speed of light $v = \dfrac{1}{\sqrt{\ell c}}$

$$V(x,t) = \frac{q(x,t)}{c} \leftarrow \text{charge density}$$

i = current through inductor



$$\ell \frac{\partial i(x,t)}{\partial t} = -\frac{\partial V}{\partial x}$$

charge conservation $\dfrac{\partial i}{\partial x} = -\dfrac{\partial q}{\partial t} = -c\dfrac{\partial V}{\partial t}$

$$l \frac{\partial}{\partial x}\frac{\partial i}{\partial t} = -\frac{\partial^2 V}{\partial x^2}$$

$$-lc\frac{\partial^2 V}{\partial t^2} = -\frac{\partial^2 V}{\partial x^2}$$

$$\boxed{v^2\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t^2} = 0}$$

Right moving solution $V(x,t) = V_R\left(t - \frac{1}{v}x\right)$ V_R is arbitrary

purely linear dispersion

$$-\frac{\partial V}{\partial x} = +\frac{1}{v}\frac{\partial V_R}{\partial t} = l\frac{\partial i}{\partial t}$$ from charge conservation

from solution

$$\Rightarrow \quad i_R(x,t) = \frac{1}{vl}V_R = \frac{1}{Z_c}V_R\left(t - \frac{1}{v}x\right)$$

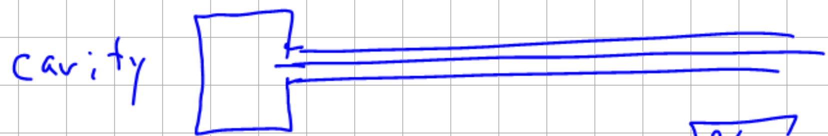
$Z_c = \sqrt{l/c}$

characteristic impedance

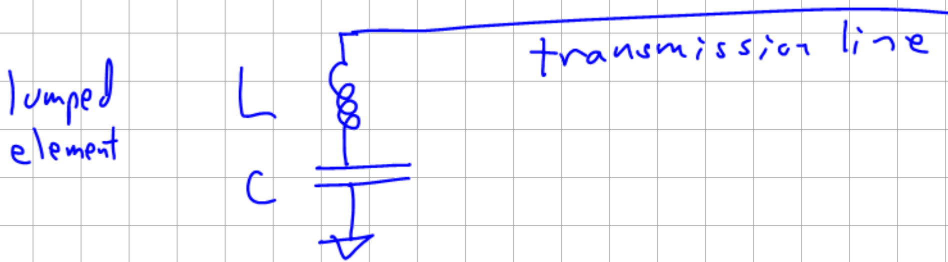
Left moving solution: $V_L\left(t + \frac{1}{v}x\right)$

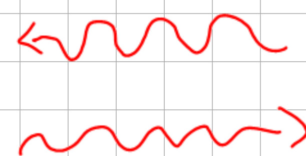
$$i_L = -\frac{1}{Z_c}V_L\left(t + \frac{1}{v}x\right)$$

Reflection of a wave from a resonator



$$Z_c = \sqrt{\ell/c}$$





$$V_{in} = V_L(t + x/v)$$

$$V_{out} = V_R(t - x/v)$$

frequency ω solution $\quad V_{in} = V_0 e^{j\omega(t + x/v)}$

$$V_{out} = V_1 \, e^{j\omega(t - x/v)}$$

$j = -\sqrt{-1}$ for electrical engineers

Assuming no internal losses in the resonator, energy conservation (unitarity) demands

$$V_1 = r(\omega) V_0$$

$\uparrow$ S matrix = reflection coefficient

$$|r(\omega)| = 1 \implies r(\omega) = e^{i\theta(\omega)}$$

Reflection from an impedance mismatch

$$V(x,t) = V_{in} + V_{out}$$

$$V(0,t) = (V_0 + V_1)\, e^{j\omega t}$$

$$I(x,t) = \frac{1}{Z_c}\left(V_{out} - V_{in}\right)$$

right mover left mover

$$I(0,t) = \frac{1}{Z_c}(V_1 - V_0)\, e^{j\omega t}$$

Boundary condition @ $x=0$ $\dfrac{V(0,t)}{I(0,t)} = -Z$ Ohm's Law

I flowing to the right!

$$Z_c \frac{V_0 + V_1}{(V_1 - V_0)} = -Z \qquad V_0 Z_c + V_1 Z_c = -V_0 Z + V_1 Z$$

$$V_1 (Z_c + Z) = -V_0 (Z_c - Z)$$

$$V_1 = V_0 \left(\frac{Z - Z_c}{Z + Z_c} \right) \qquad r(\omega) = \left[\frac{Z(\omega) - Z_c}{Z(\omega) + Z_c} \right]$$

sanity check: $r = 0$ if $Z = Z_c$ ✓ impedance matching condition

For the resonator $Z(\omega) = j\omega L + \frac{1}{j\omega C}$

$$Z = \frac{1}{j\omega C} \left[1 - \frac{\omega^2}{\omega_R^2} \right] = \frac{\omega_R^2 - \omega^2}{j\omega C \omega_R^2} = \frac{(\omega_R + \omega)}{j\omega C \omega_R^2} (\omega_R - \omega)$$

vanishes at resonance

Near resonance $\omega \sim \omega_R$

$$Z \approx \frac{2}{j\omega_R^2 C} (\omega_R - \omega)$$

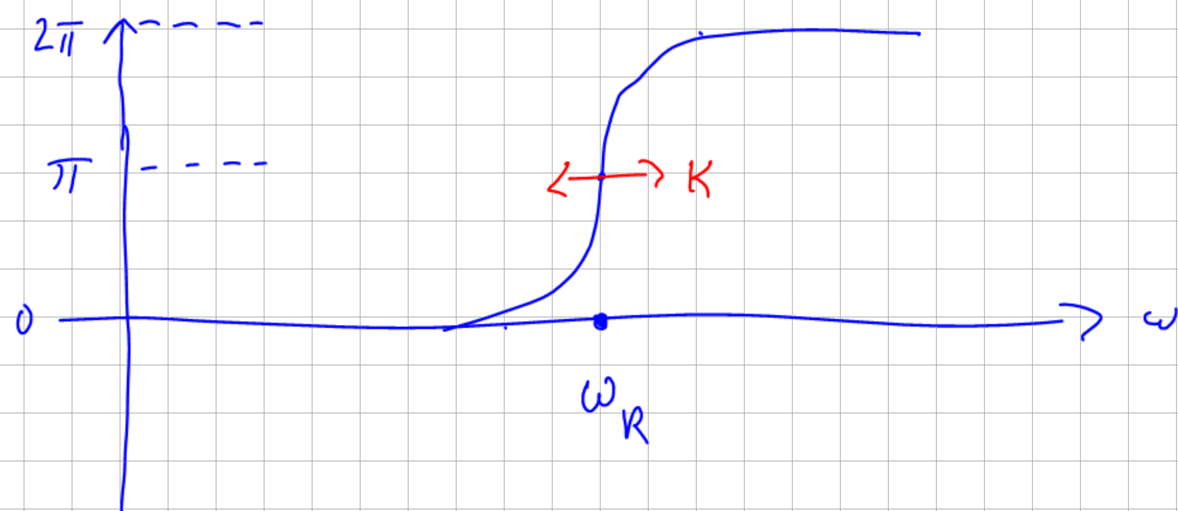
$$r = \frac{z - z_c}{z + z_c} \approx \frac{\omega_R - \omega - j\dfrac{\omega_R^2 c Z_c}{2}}{\omega_R - \omega + j\dfrac{\omega_R^2 c Z_c}{2}} \approx \frac{\omega_R - \omega - jK/2}{\omega_R - \omega + jK/2} = \frac{e^{i\theta/2}}{e^{-i\theta/2}}$$

$$K \equiv \frac{Z_c}{L} = \text{cavity damping rate} = \omega_R^2\, c Z_c = \omega_R^2\, \tau$$

Notice $|r| = 1$ ✓ unitary S matrix Because $Z(\omega)$ purely imaginary (reactive)

$$r = e^{i\Theta(\omega)} = \frac{e^{i\theta/2}}{e^{-i\theta/2}} \qquad \tan\left[\frac{\Theta(\omega)}{2}\right] = -\frac{K}{2(\omega_R - \omega)} = \frac{K}{2(\omega - \omega_R)}$$

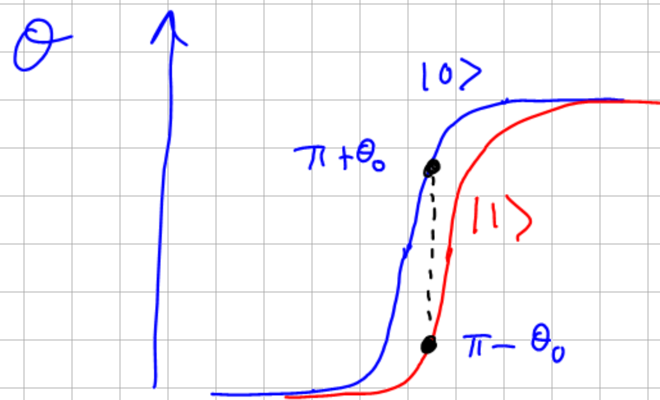
$$\Theta(\omega_R) = \pi$$



Reflection phase shift is a very sensitive function of resonance frequency

Resonance freq. is dispersively shifted by the state of the qubit

$$\tilde{\omega}_R = \omega_R + \chi\,\sigma^z$$



coherent state $e^{\alpha(a^{\dagger} - a)}|0\rangle$

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \otimes |\alpha\rangle_{Left}$$ (product state)

π phase shift @ resonance

$$|\Psi_{out}\rangle = \frac{1}{\sqrt{2}}\, |0\rangle |-e^{i\theta_0}\alpha\rangle_{right} + |1\rangle |-e^{-i\theta_0}\alpha\rangle_{right}$$

output state is <u>entangled</u> between qubit and microwave field

Measurement of phase of reflected microwaves collapses state of qubit to $|0\rangle$ or $|1\rangle$ and measures qubit.