Introduction to Theory of Mesoscopic Systems

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Lecture 4

Beforehand

Anderson Localization, Mesoscopic Fluctuations, Random Matrices, and Quantum Chaos

Today, tomorrow

e-e interactions in disordered conductors

Zero-dimensional Fermi-liquid

Universal Hamiltonian

ORIGINS

E.P. Wigner, Conference on Neutron Physics by Time of Flight, November 1956

P.W. Anderson, *"Absence of Diffusion in Certain Random Lattices"*; Phys.Rev., 1958, v.109, p.1492

L.D. Landau, *"Fermi-Liquid Theory"* Zh. Exp. Teor. Fiz.,1956, v.30, p.1058

J. Bardeen, L.N. Cooper & J. Schriffer, *"Theory of Superconductivity"*; Phys.Rev., 1957, v.108, p.1175.

~~What does it mean - non-Fermi liquid ?~~

Q: What is the difference between **Fermi-liquid** and **non-Fermi liquid** ?

A: The difference is the same as between **bananas** and **non-bananas.**

What does it mean Fermi liquid ?

Fermi Liquid

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

$\Rightarrow$ *Fermi Liquid*

What does it mean?

- Fermi statistics
- Low temperatures
- Not too strong interactions
- Translation invariance

$\Rightarrow$ Fermi Liquid

It means that

1. Excitations are similar to the excitations in a Fermi-gas:
   a) the same quantum numbers – momentum, spin ½ , charge e
   b) decay rate is small as compared with the excitation energy

2. Substantial renormalizations. For example, in a Fermi gas

$$\partial n/\partial\mu, \quad \gamma = c/T, \quad \chi/g\mu_B$$

are all equal to the one-particle density of states ν.
These quantities are different in a Fermi liquid

Signatures of the Fermi - Liquid state ?!

1. Resistivity is proportional to T^2 :

L.D. Landau & I.Ya. Pomeranchuk *"To the properties of metals at very low temperatures"*; Zh.Exp.Teor.Fiz., 1936, v.10, p.649

…The increase of the resistance caused by the interaction between the electrons is proportional to T^2 and at low temperatures exceeds the usual resistance, which is proportional to T^5.

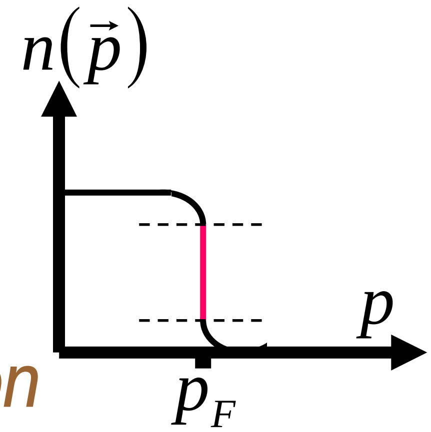
… the sum of the momenta of the interaction electrons can change by an integer number of the periods of the reciprocal lattice. Therefore the momentum increase caused by the electric field can be destroyed by the interaction between the electrons, not only by the thermal oscillations of the lattice.

Signatures of the Fermi - Liquid state ?!

1. Resistivity is proportional to T^2 :

L.D. Landau & I.Ya. Pomeranchuk *"To the properties of metals at very low temperatures"*; Zh.Exp.Teor.Fiz., 1936, v.10, p.649

Umklapp electron – electron scattering dominates the charge transport (?!)



2. Jump in the momentum distribution function at $T=0$.

2a. Pole in the one-particle Green function

$$G(\varepsilon, \vec{p}) = \frac{Z}{i\varepsilon_n - \xi(\vec{p})}$$

Fermi liquid = $0<Z<1$ (?!)

Landau Fermi - Liquid theory

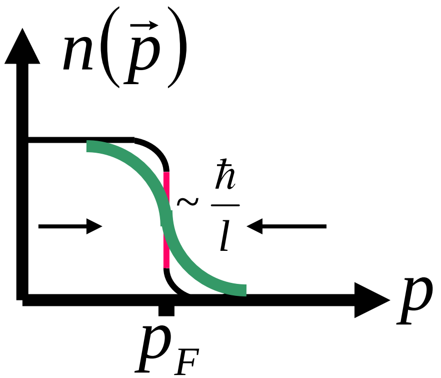
Momentum	$\vec{p}$
Momentum distribution	$n(\vec{p})$
Total energy	$E\{n(\vec{p})\}$
Quasiparticle energy	$\xi(\vec{p}) \equiv \delta E / \delta n(\vec{p})$
Landau f-function	$f(\vec{p}, \vec{p}') \equiv \delta\xi(\vec{p}) / \delta n(\vec{p}')$

Q: ***Can Fermi – liquid survive without the momenta***

Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder ?

Q: *Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder* ?

1. *Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, l. The step in the momentum distribution function is broadened by this uncertainty*



2. *Neither resistivity nor its temperature dependence is determined by the umklapp processes and thus does not behave as* T^2

3. *Sometimes (e.g., for random quenched magnetic field) the disorder averaged one-particle Green function even without interactions does not have a pole as a function of the energy, ε. The residue , Z, makes no sense.*

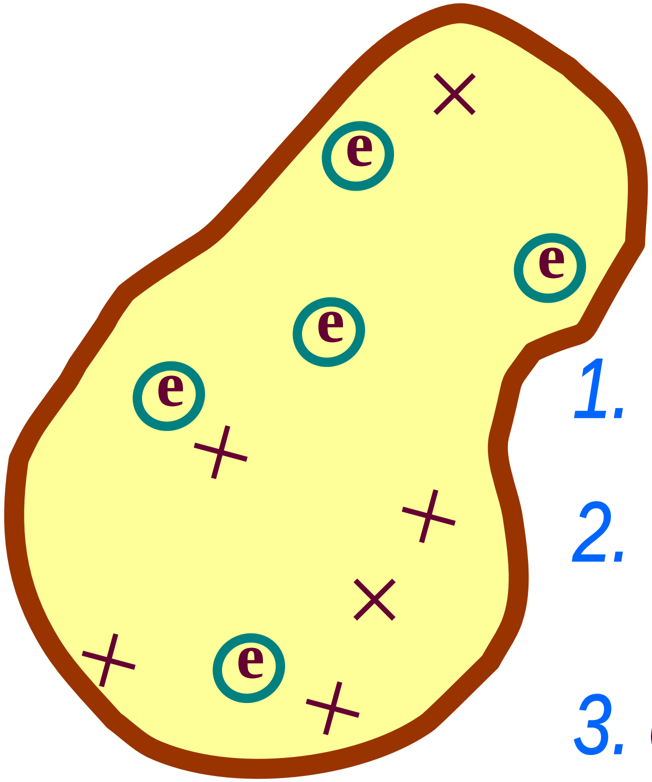
Nevertheless even in the presence of the disorder

I. *Excitations are similar to the excitations in a disordered Fermi-gas.*

II. *Small decay rate*

III. *Substantial renormalizations*

Quantum Dot



1. Disorder (×impurities)
2. Complex geometry

$\Big\}$ chaotic one-particle motion

3. ***e*-*e*** interactions

Realizations:

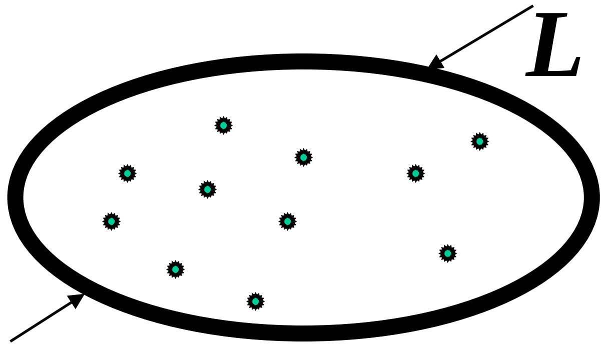
- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes
-
-

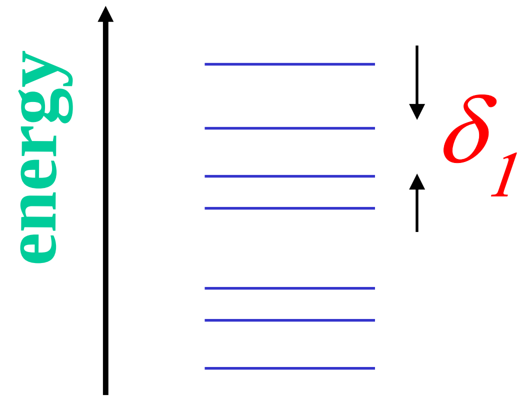
One-particle problem (*Thouless, 1972*)

Energy scales

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T *has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)*

$$g = E_T / \delta_1$$

dimensionless Thouless conductance

$$g = Gh/e^2$$

Zero Dimensional Fermi Liquid

Finite System $\Longrightarrow$ Thouless energy E_T

$$\boxed{\varepsilon << E_T \xrightarrow{def} \mathbf{0D}}$$

At the same time, we want the typical energies, ε, to exceed the mean level spacing, δ_1:

$$\delta_1 << \varepsilon << E_T$$

$$\boxed{g \equiv \frac{E_T}{\delta_1} >> 1}$$

Thouless Conductance and One-particle Quantum Mechanics

0 **1** ***g***

Localized states	***Extended states***
Insulator	***Metal***
Poisson spectral statistics	**Wigner-Dyson spectral statistics**

$N \times N$

Random Matrices

Quantum Dots with dimensionless conductance g

$N \to \infty$

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

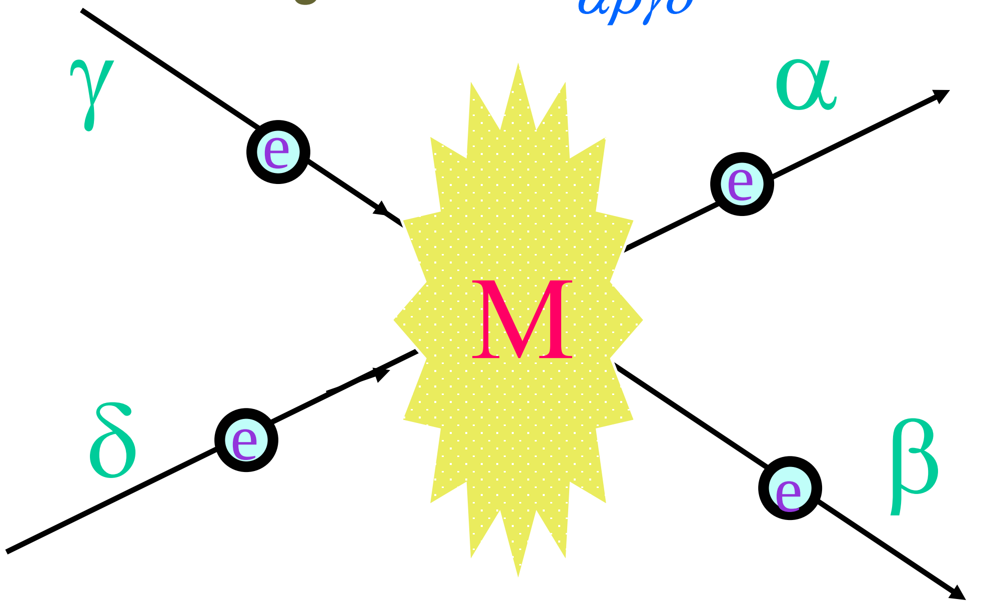
$g \to \infty$

Two-Body Interactions

$|\alpha,\sigma>$ *Set of one particle states. σ and α label correspondingly spin and orbit.*

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha a^{+}_{\alpha,\sigma} a_{\alpha,\sigma} \qquad \hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^{+}_{\alpha,\sigma} a^{+}_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

ε_α *-one-particle orbital energies* $\quad$ $\boldsymbol{M}_{\alpha\beta\gamma\delta}$ *-interaction matrix elements*



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ε_α -one-particle orbital energies $\quad$ $M_{\alpha\beta\gamma\delta}$ -interaction matrix elements

Nuclear Physics	ε_α	are taken from the shell model
	$M_{\alpha\beta\gamma\delta}$	are assumed to be random
Quantum Dots	ε_α	RANDOM; Wigner-Dyson statistics
	$M_{\alpha\beta\gamma\delta}$	? ? ? ? ? ? ? ?

Matrix Elements

$$\hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^{+}_{\alpha,\sigma} a^{+}_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

$$M_{\alpha\beta\gamma\delta}$$

Diagonal - $\alpha,\beta,\gamma,\delta$ *are equal pairwise*

$\alpha=\gamma$ *and* $\beta=\delta$ *or* $\alpha=\delta$ *and* $\beta=\gamma$ *or* $\alpha=\beta$ *and* $\gamma=\delta$

Offdiagonal - *otherwise*

It turns out that

in the limit $g \to \infty$

- *Diagonal matrix elements are much bigger than the offdiagonal ones*

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- *Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging*

Toy model:

*Short range **e-e** interactions*

$$U(\vec{r}) = \frac{\lambda}{\nu}\delta(\vec{r})$$

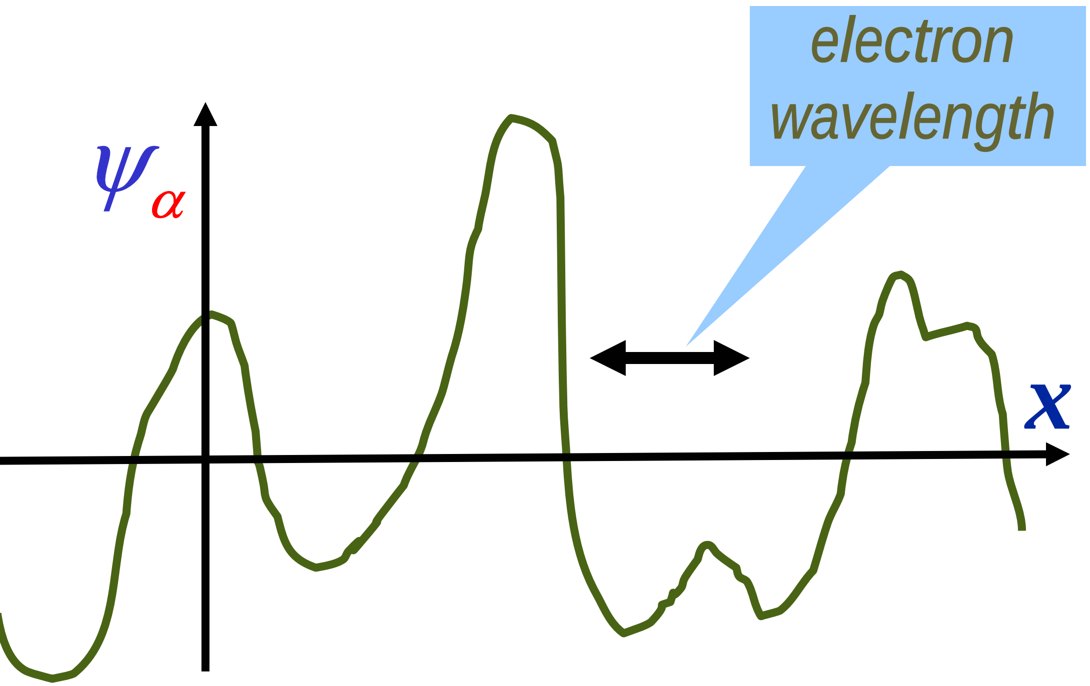
λ *is dimensionless coupling constant*

ν *is the electron density of states*

$$M_{\alpha\beta\gamma\delta} = \frac{\lambda}{\nu}\int d\vec{r}\,\psi^*_\alpha(\vec{r})\psi^*_\beta(\vec{r})\psi_\gamma(\vec{r})\psi_\delta(\vec{r})$$

$\psi_\alpha(\vec{r})$

one-particle eigenfunctions



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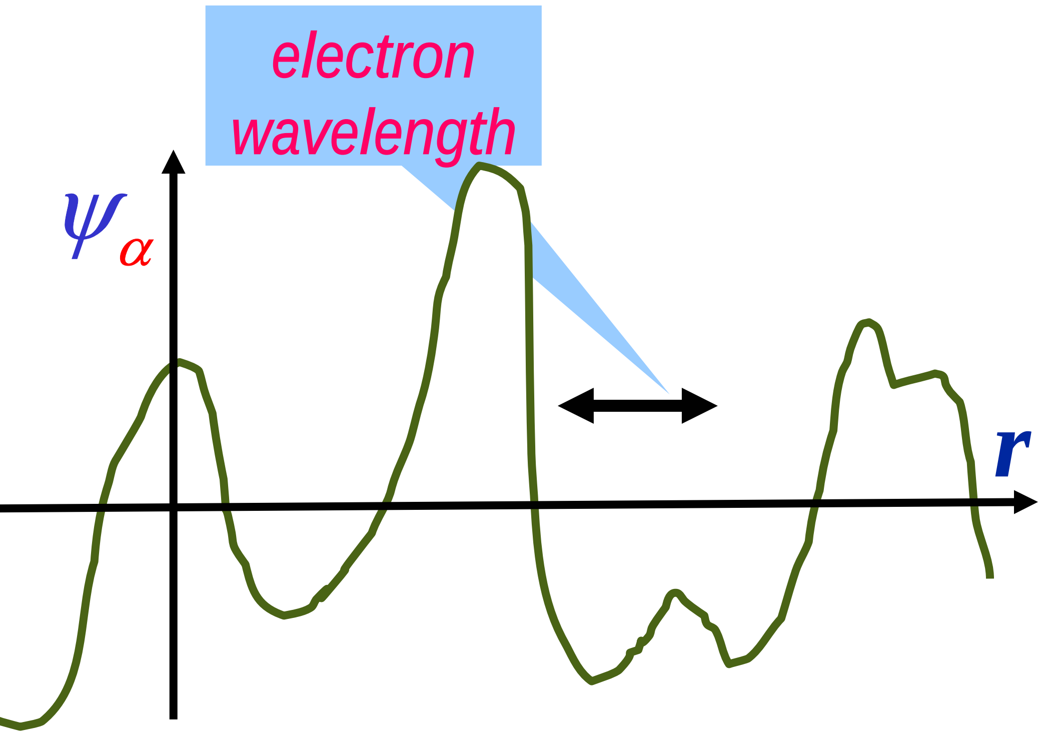
$\psi_\alpha(\vec r)$ one-particle eigenfunctions



$\psi_\alpha(\vec r)$ *is a random function that rapidly oscillates*

$$\left|\psi_\alpha(\vec r)\right|^2 \geq 0$$

$$\psi_\alpha(\vec r)^2 \geq 0$$

*as long as **T**-invariance is preserved*

In the limit

$g \to \infty$

- Diagonal matrix elements are much bigger than the offdiagonal ones

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging

$$\left.\begin{array}{l} M_{\alpha\beta\alpha\beta} = \dfrac{\lambda}{\nu}\displaystyle\int d\vec{r}\left|\psi_\alpha(\vec{r})\right|^2\left|\psi_\beta(\vec{r})\right|^2 \\ \left|\psi_\alpha(\vec{r})\right|^2 \Rightarrow \dfrac{1}{\text{volume}} \end{array}\right\} \Rightarrow \boxed{M_{\alpha\beta\alpha\beta} = \lambda\delta_1}$$

More general: finite range interaction potential $U(\vec{r})$

$$\boxed{M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu}\int\left|\psi_\alpha(\vec{r}_1)\right|^2\left|\psi_\beta(\vec{r}_2)\right|^2 U(\vec{r}_1 - \vec{r}_2)\,d\vec{r}_1 d\vec{r}_2}$$

The same conclusion

Random Matrices:

E_α - spectrum

$\psi_\alpha(i)$ – ***i-th*** component of α***-th*** eigenvector

$$\boxed{\left\langle \psi_\alpha{}^*(i)\psi_\gamma(j)\right\rangle = \frac{1}{N}\delta_{\alpha\gamma}\delta_{ij}}$$

$$\boxed{\left\langle \psi_\alpha(i)\psi_\gamma(j)\right\rangle = \frac{2-\beta}{N}\delta_{\alpha\gamma}\delta_{ij}}$$

in the limit $N \to \infty$

Components of the different eigenvectors as well as different components of the same eigenvector are not correlated

Berry Conjecture:

Exact wavefunctions at energy $\approx \varepsilon_F$ in chaotic systems behave as sums of plane waves with $|\vec{k}| \approx k_F$ and random coefficients:

$$\left\langle \psi_\alpha^*(\vec{r}_1)\psi_\gamma(\vec{r}_2)\right\rangle = \frac{\delta_{\alpha\gamma}}{V} f\left(\frac{2\pi|\vec{r}_1 - \vec{r}_2|}{\lambda}\right)$$

$$\left\langle \psi_\alpha(\vec{r}_1)\psi_\gamma(\vec{r}_2)\right\rangle = (2-\beta)\frac{\delta_{\alpha\gamma}}{V} f\left(\frac{2\pi|\vec{r}_1 - \vec{r}_2|}{\lambda}\right)$$

$$f(x) = \Gamma\left(\frac{d}{2}\right) x^{1-d/2} J_{d/2-1}(x)$$

d *is # of dimensions,*
$J_\mu(x)$ *is Bessel function*

Important:

!

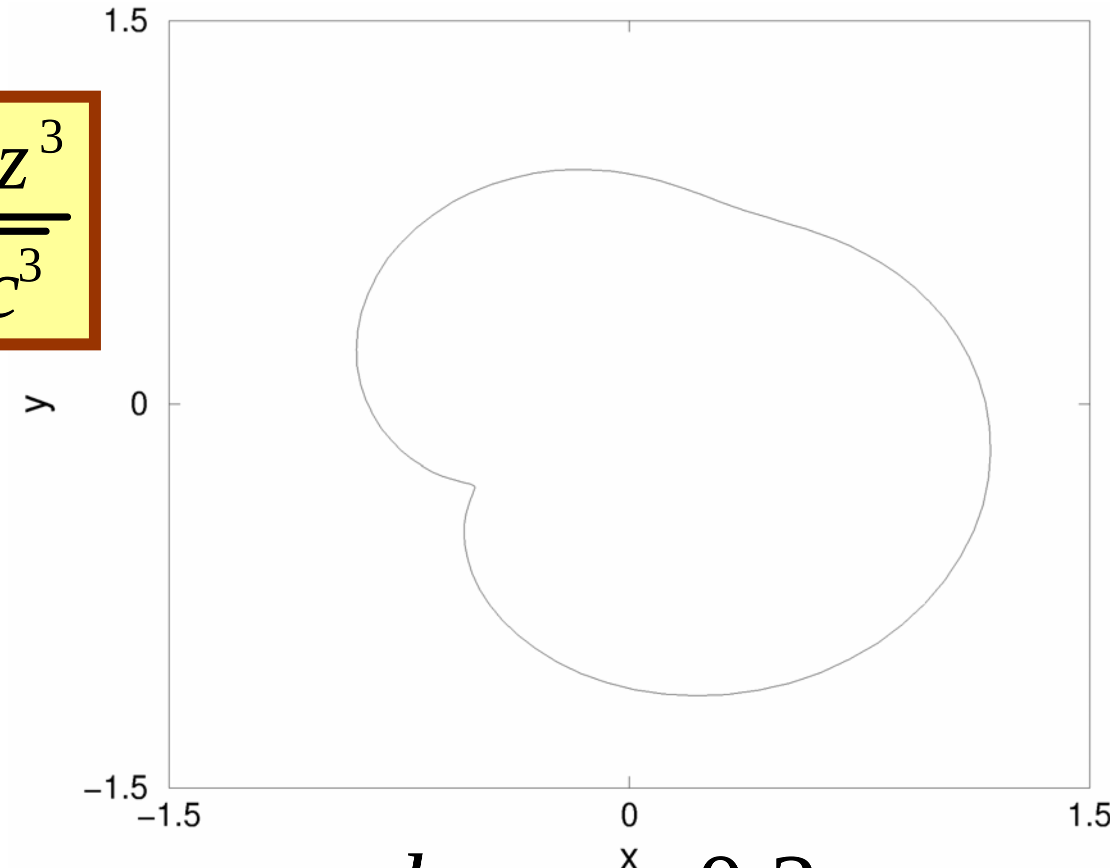
when x *increases* $f(x)$ *decays quickly enough for the integral* $\int_0^\infty f(x) x^{d-1} dx$ *to converge*

Only local correlations

AFRICA BILLIARD - *a conformal image of a unit circle*

Sang-Hyeon Ahn

$$\omega(z) = R\frac{z + bz^2 + ce^{i\delta}z^3}{\sqrt{1+2b^2+3c^3}}$$



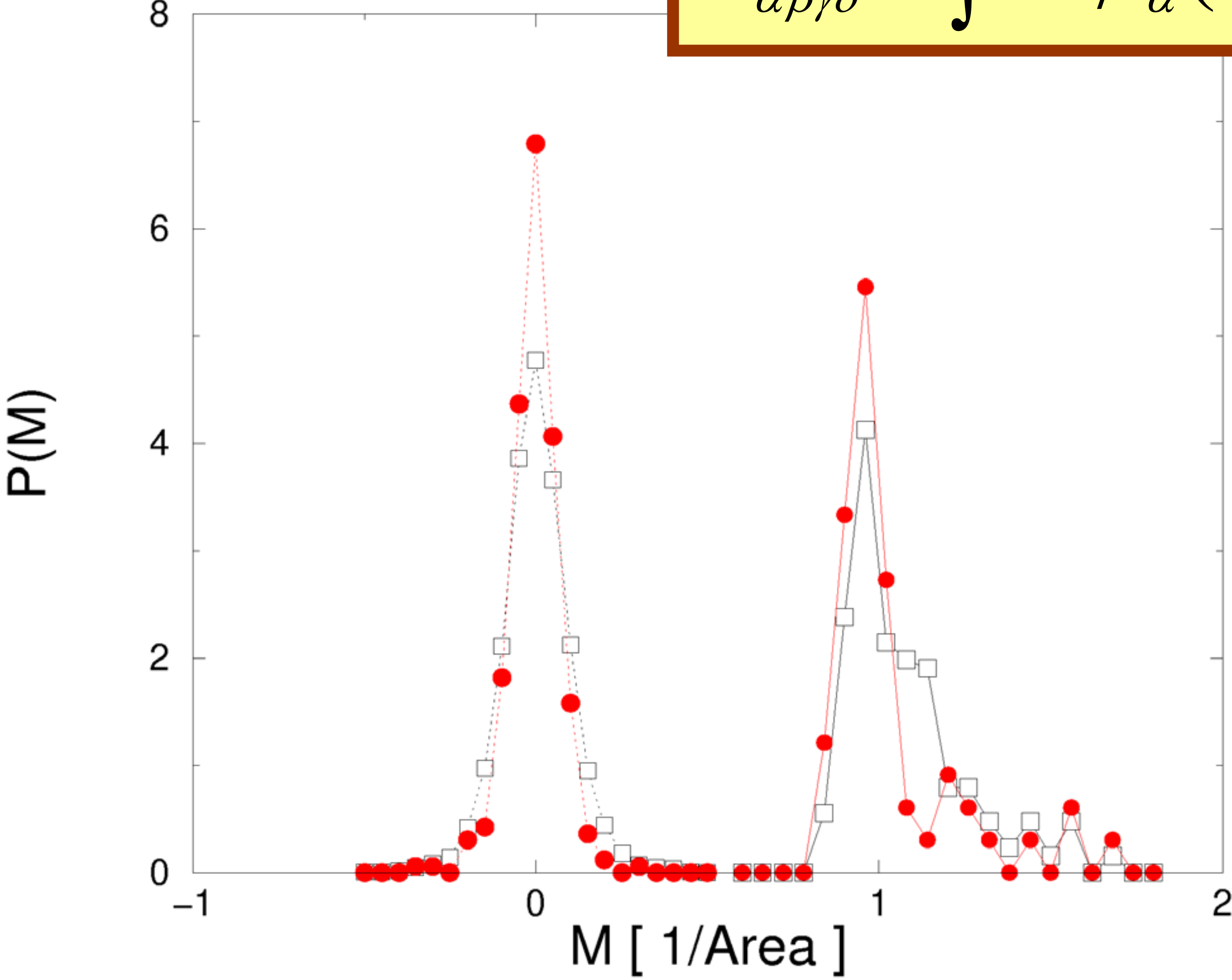
$$b = c = 0.2;$$

$$\delta = 1.5; R = 1$$

Distribution of the matrix elements

Sang-Hyeon Ahn

$$M_{\alpha\beta\gamma\delta} = \int d\vec{r}\, \psi_{\alpha}(\vec{r})\psi_{\beta}(\vec{r})\psi^{*}_{\alpha}(\vec{r})\psi^{*}_{\beta}(\vec{r})$$



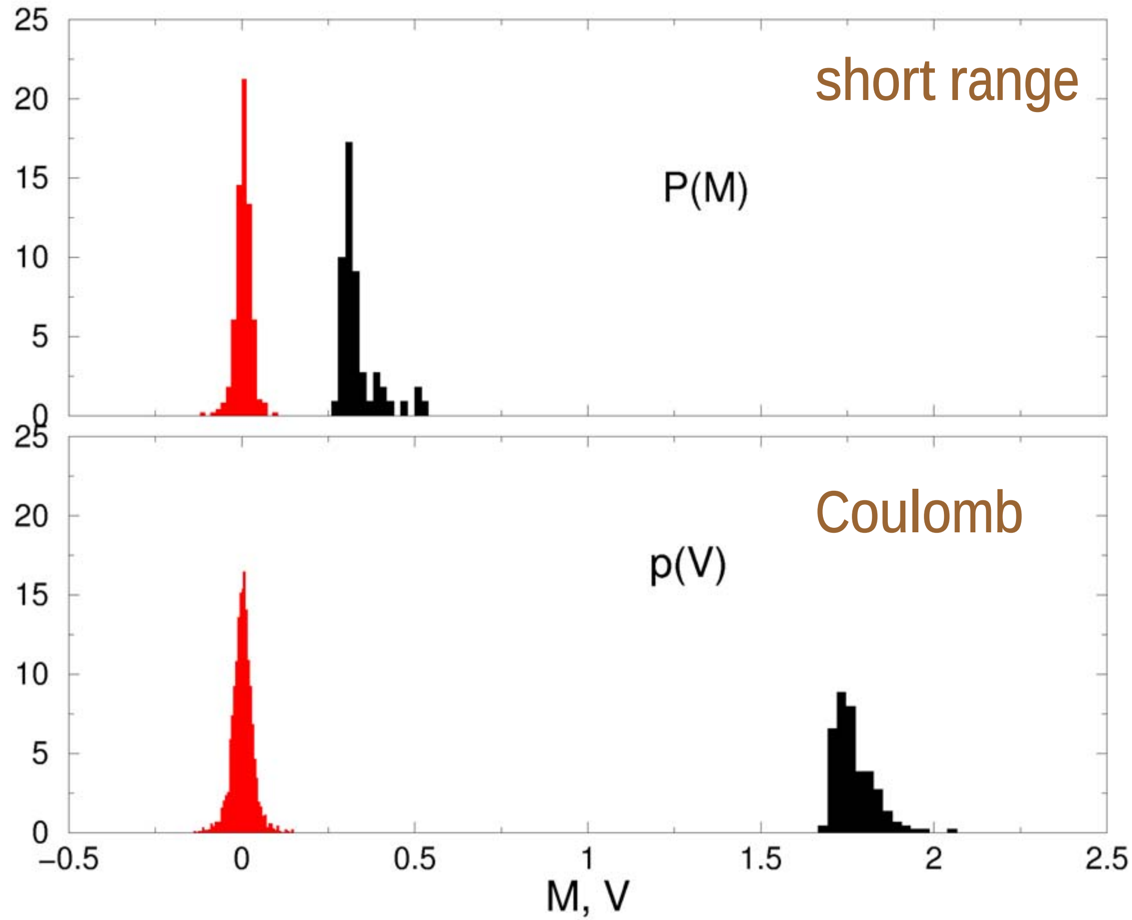
Open boxes -
levels from 20 to 30

Closed circuits –
levels from 30 to 40

$$M_{\alpha\beta\gamma\delta} = \frac{1}{\pi}\int d\vec{r}\, \psi_\alpha(\vec{r})\psi_\beta(\vec{r})\psi^*_\alpha(\vec{r})\psi^*_\beta(\vec{r})$$

$$V_{\alpha\beta\gamma\delta} \propto \int \frac{d\vec{r}_1 d\vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}\psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2)\psi^*_\alpha(\vec{r}_1)\psi^*_\beta(\vec{r}_2)$$

Distribution function of diagonal and offdiagonal matrix elements



Sang-Hyeon Ahn

Universal (Random Matrix) limit - Random Matrix symmetry of the correlation functions:

All correlation functions are invariant under arbitrary orthogonal transformation:

$$\boxed{\tilde{\psi}_{\mu}(\vec{r}) = \sum_{\nu} \int d\vec{r}_1 O_{\mu}^{\nu}(\vec{r}, \vec{r}_1) \psi_{\nu}(\vec{r}_1)}$$

$$\boxed{\int d\vec{r}_1 O_{\mu}^{\nu}(\vec{r}, \vec{r}_1) O_{\nu}^{\eta}(\vec{r}_1, \vec{r}\,') = \delta_{\mu\eta} \delta(\vec{r} - \vec{r}\,')}$$

There are only three operators, which are quadratic in the fermion operators a^+, a, and invariant under RM transformations:

$$\hat{n} = \sum_{\alpha,\sigma} a^+_{\alpha,\sigma} a_{\alpha,\sigma}$$

total number of particles

$$\hat{S} = \sum_{\alpha,\sigma_1,\sigma_2} a^+_{\alpha,\sigma_1} \vec{\sigma}_{\sigma_1,\sigma_2} a_{\alpha,\sigma_2}$$

total spin

$$\hat{T}^+ = \sum_{\alpha} a^+_{\alpha,\uparrow} a^+_{\alpha,\downarrow}$$

????

Charge conservation (gauge invariance) - no $\hat{T}$ or $\hat{T}^{+}$ only $\hat{T}\,\hat{T}^{+}$

Invariance under rotations in spin space - no $\hat{S}$ only $\hat{S}^{2}$

Therefore, in a very general case

$$\boxed{\hat{H}_{\mathrm{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^{+}\hat{T}.}$$

Only three coupling constants describe all of the effects of e-e interactions

In a very general case only three coupling constants describe all effects of electron-electron interactions:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \hat{H}_{\text{int}}$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{T}^+ \hat{T}.$$

I.L. Kurland, I.L.Aleiner & B.A., 2000

See also

P.W.Brouwer, Y.Oreg & B.I.Halperin, 1999

H.Baranger & L.I.Glazman, 1999

H-Y Kee, I.L.Aleiner & B.A., 1998

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For a short range interaction with a coupling constant λ

$$E_c = \frac{\lambda\delta_1}{2} \qquad J = -2\lambda\delta_1 \qquad \lambda_{BCS} = \lambda\delta_1 (2-\beta)$$

where δ_1 is the one-particle mean level spacing

$$\boxed{\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}}$$

$$\boxed{\hat{H}_0 = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha}}$$

$$\boxed{\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^{+}\hat{T}.}$$

Only one-particle part of the Hamiltonian, $\hat{H}_0$ ***, contains randomness*** **!**

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^+\hat{T}.$$

E_c determines the charging energy (Coulomb blockade)

J describes the spin exchange interaction

λ_{BCS} determines effect of superconducting-like pairing

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}}$$

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{\mathrm{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^{+}\hat{T}.$$

I. Excitations are similar to the excitations in a disordered Fermi-gas.

II. Small decay rate

III. Substantial renormalizations

Isn't it a Fermi liquid ?

Fermi liquid behavior follows from the fact that different wave functions are almost uncorrelated

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha}$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^{+}\hat{T}.$$

I. *Excitations are similar to the excitations in a disordered Fermi-gas.*

II. *Small decay rate*

III. *Substantial renormalizations*

Small decay rate

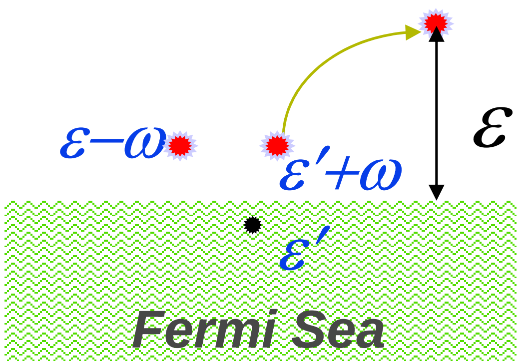
Q:

- Why is it small
- What is it equal to
- What is the connection between the decay rate of the quasiparticles and the dephasing rate

?

Quasiparticle decay rate at $T = 0$ *in a clean Fermi Liquid.*

I. d=3



$$\frac{\hbar}{\tau_{e-e}(\varepsilon)} \propto \left(\begin{matrix}\text{coupling}\\ \text{constant}\end{matrix}\right)^2 \frac{\varepsilon^2}{\varepsilon_F} \qquad d = 3$$

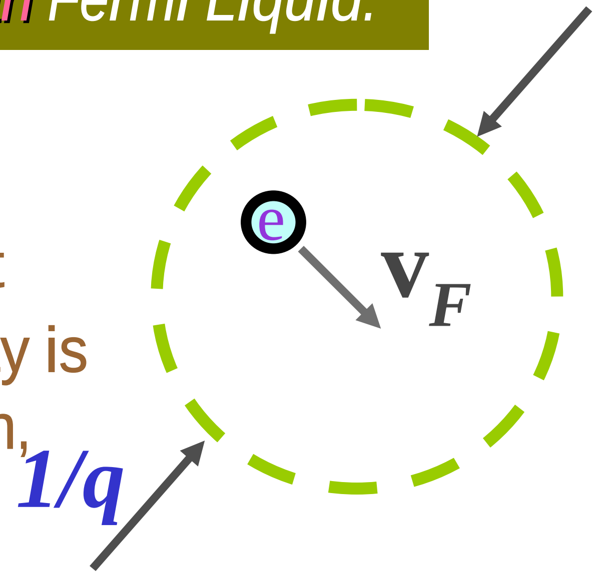
Reasons:

- At small ε the energy transfer, ω, is small and the integration over ε' and ω gives the factor ε^2.

- The momentum transfer, q, is large and thus the scattering probability at given ε' and ω does not depend on ε', ω or ε

Quasiparticle decay rate at $T = 0$ in a clean Fermi Liquid.

II. Low dimensions



Small momenta transfer, q , become important at low dimensions because the scattering probability is proportional to the squared time of the interaction, $(q\mathbf{v}_F)^{-2}$

$$\frac{\hbar}{\tau_{e-e}(\varepsilon)} \propto \begin{cases} \varepsilon^2/\varepsilon_F & d=3 \\ \left(\varepsilon^2/\varepsilon_F\right)\log\left(\varepsilon_F/\varepsilon\right) & d=2 \\ \varepsilon & d=1 \end{cases}$$

Quasiparticle decay rate at $T = 0$ in a clean Fermi Liquid.

III. Applicability

$$\frac{\hbar}{\tau_{e-e}(\varepsilon)} \propto \begin{cases} \varepsilon^2/\varepsilon_F & d=3 \\ \left(\varepsilon^2/\varepsilon_F\right)\log\left(\varepsilon_F/\varepsilon\right) & d=2 \\ \varepsilon & d=1 \end{cases}$$

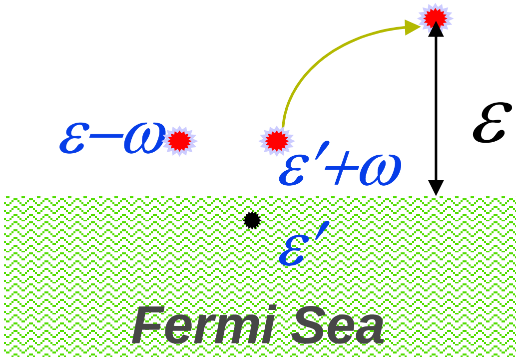
Conclusions:

1. For d=3,2 from $\varepsilon \ll \varepsilon_F$ it follows that $\varepsilon\tau_{e-e} \gg h$, i.e., that the qusiparticles are well determined and the Fermi-liquid approach is applicable.

2. For d=1 $\varepsilon\tau_{e-e}$ is of the order of h, i.e., that the Fermi-liquid approach is not valid for 1d systems of interacting fermions.

Luttinger liquids

Quasiparticle decay rate at $T = 0$ *in a 0D Fermi Liquid.*



Electronic spectrum is discrete

Need offdiagonal matrix elements

Quasiparticle decay is beyond the "universal Hamiltonian"

Quasiparticle decay rate is small as g^{-1}

$$\tau_{ee}(\varepsilon) \geq g \frac{\hbar}{\varepsilon}$$

CONCLUSIONS

One-particle chaos + moderate interaction of the electrons $\mapsto$ to a rather simple Hamiltonian of the system, which can be called Zero-dimensional Fermi liquid.

The main parameter that justifies this description is the Thouless conductance, which is supposed to be large

Excitations are characterized by their one-particle energy, charge and spin, but not by their momentum.

These excitations have the lifetime, which is proportional to the Thouless conductance, i.e., is long.

This approach allows to describe Coulomb blockade (renormalization of the compressibility), as well as the substantial renormalization of the magnetic susceptibility and effects of superconducting pairing