General classification in the bulk - results from K-Theory (Topological Band Theory)

→: "Topological Band Theory":

-: Try to repeat the previous analysis (done for class A) for other symmetry classes.

Due to the presence of time-reversal and charge-conjugation symmetries [which send $\vec{k} \to (-\vec{k})$] the conditions that the simplified Hamiltonians $Q(\vec{k})$ satisfy are more difficult to analyze:

TABLE: List of simplified Hamiltonians

-: The structure of the $Q(\vec{k})$ is most easily revealed by

* either looking at $d=0$ (zero-dim'l space)

* or at special points $\vec{k}=\vec{k}_0$ in the Brillouin zone where $+\vec{k}_0$ and $-\vec{k}_0$ differ by a reciprocal lattice vector.

The list of resulting matrices $Q(\vec{k})$ is displayed in the right most column of $\boxed{\text{TABLE "classifying spaces".}}$

About this TABLE

*: divide 10 classes into

-: 2 "complex" (A, AIII) which have no reality condition that relates H^* to H

-: 8 real (all others) which have at least one reality condition.

*: order the real classes in a particular way

→: Classification of topological properties of $\boxed{Q(\vec{k})}$ in the general case: is precisely solved by K-Theory

Result (Kitaev 2009):

$$\boxed{\pi\left(\overline{T^d}, R_d\right) = \underbrace{\pi_0\left(R_{q-d}\right)} \oplus \underbrace{\left[\bigoplus_{s=0}^{d-1} \binom{d}{s} \cdot \pi_0\left(R_{q-s}\right)\right]}}$$

number of connected components (known!)

$\hat{=}$ strong topol. insulators

weak topol. insultators (= top. order in lower-dimensional planes, ..)

TABLE OF topological insulator + Supercond.

* Schnyder, Ryu, Furusaki, Ludwig: 2008, 2009, 2010
* Kitaev: 2009

Classification by lack of Anderson localization at the boundary

→ The technically simplest way to obtain the TABLE of topol. ins. + supercond.s, is to focus on the boundaries.

General theme:

-: boundaries of topol. ins. + S.C. are anomalous
[Ryu, Moore, Ludwig PRB(2012) for fermionic topol ins. + S.C.]
also when interacting

-: here: [= in the absence of interactions]
* boundaries must always be conducting
* must avoid Anderson localization

Method: We reduce the problem of classifying topol. ins. (S.C.s) in d spatial dim's to a problem of Anderson localization in $\bar{d} = (d-1)$ dim's ("at the boundary").

By studying the lack of Anderson localiz. in $\bar{d} = (d-1)$ dim's, we solve the classification problem of top. ins.(S.C.s) in d dim's

→: Basic underlying principles

—: Theoretical description of problems of Anderson localization [non-interacting Fermions in the presence of static (disorder) potentials] is very systematic and geometrical:

—: On length scales much larger than the mean free path (the microscopic length scale) there is a description in terms of a Non-Linear-Sigma-Model (NLSM) in the spatial dimension. {Here we focusing on the boundary of the Top. Ins. (S.C.)}

* NLSM = generalization of class. Stat. Mech. of Heisenberg Ferromagnet where spin $\vec{S} \in$ 2-dim'l sphere S_2 in 3-dim'l Euclidean Space.

* **Anderson localization:**
Replace $\vec{S}$ by a "Spin" = element of the corresponding generalization of spheres ("Cartan") occuring in the 6^{th} column of Ten-Fold-Way (Cartan) TABLE.

NLSM: ($\bar{d}$ = spatial dimension of boundary)

$$\boxed{S = \frac{1}{g}\int d^{\bar{d}} r \sum_{\mu=1}^{\bar{d}} \mathrm{Tr}\left[\partial_\mu \Phi(r)\, \partial_\mu \Phi(r)\right] + S_{top}\{\Phi(r)\}}$$

$$\Phi(r) \in G/H = \text{"target space"}$$

→: A NLSM (without S_{top}) always has an Anderson-localized phase (where correlation length = finite) at least for large "g". The Anderson-localized phase is not conducting.

→. This cannot happen at the boundary of a Topol. Ins. (Supercond.).

The appearance of a finite correlation length can be avoided if we can add an additional term of topological origin "S_{top}" to the action, which has no adjustable parameter.

Whether such a term exists depends

(i): on the "target space" G/H of the NLSM (i.e. on the symmetry class)

and:

(ii): on the dimension $\bar{d} = (d-1)$ of the boundary

(a): $\mathbb{Z}_2$ topol. term

$$\Leftrightarrow \quad \pi_{\bar{d}}(G/H) = \pi_{d-1}(G/H) = \mathbb{Z}_2$$

(b): Wess-Zumino-Witten term

$$\Leftrightarrow \quad \pi_{\bar{d}+1}(G/H) = \pi_d(G/H) = \mathbb{Z}$$

This information yields precisely the Table of Top. Ins. + Supercond.,
same result as from Topol. Band Theory (K-Theory).