Comment (ii): Superconducting Systems

→: Go back to p. Ten-5.

We assumed non-superconducting systems [= particle-number $\hat{Q}$ conservation].

→: We can think about Bogoliubov-de Gennes (BdG) Hamiltonians of Fermionic excitations inside superconductors in the same way, except that the Hamiltonian (= the BdG Hamiltonian) now has an "automatic" charge-conjugation symmetry.

We see this as follows:

→: Instead of

$$\boxed{\hat{\psi} = \begin{pmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_N \end{pmatrix}} \quad , \qquad \text{(Ten - 29.1)}$$

use the "Nambu spinor"

$$\boxed{\hat{\chi} = \begin{pmatrix} \hat{\chi}_1 \\ \vdots \\ \hat{\chi}_N \\ \hline \hat{\chi}_{N+1} \\ \vdots \\ \hat{\chi}_{2N} \end{pmatrix} = \begin{pmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_N \\ \hline \hat{\psi}^+_1 \\ \vdots \\ \hat{\psi}^+_N \end{pmatrix} = \begin{bmatrix} \hat{\psi} \\ \hline (\hat{\psi}^+)^t \end{bmatrix}} \quad , \qquad \text{(Ten-29.2)}$$

→: In this notation, the 2nd quantized BdG Hamiltonian of a superconductor reads:

$$\boxed{\hat{H} = \frac{1}{2} \sum_{A,B=1}^{2N} \hat{\chi}^{\dagger}_{A} H_{AB} \hat{\chi}_{B} = \frac{1}{2} \hat{\chi}^{\dagger} H \hat{\chi} = \frac{1}{2} \left(\hat{\psi}^{\dagger}, \hat{\psi}^{t} \right) H \begin{pmatrix} \hat{\psi} \\ (\hat{\psi}^{\dagger})^{t} \end{pmatrix}} \qquad (\text{Ten-30.1})$$

In view of the last equation, the 1st quantized BdG Hamiltonian (a $2N \times 2N$-matrix of numbers) has the familiar form

$$\boxed{H = \begin{array}{cc} \begin{array}{cc} \leftarrow N \rightarrow & \leftarrow N \rightarrow \end{array} & \\ \begin{pmatrix} \Xi & \Delta \\ \Delta^{*} & -\Xi^{t} \end{pmatrix} & \begin{array}{c} \updownarrow N \\ \updownarrow N \end{array} \end{array}} \qquad (\text{Ten-30.2})$$

where $\Xi = \Xi^{\dagger}$ (Hermiticity) and $\Delta = -\Delta^{t}$ (Fermi statistics).

→: Writing out the last of Eq. (Ten-30.1), we find the familiar expression:

$$\boxed{\begin{aligned}\hat{H} = \sum_{a,b=1}^{N} \hat{\psi}_a^{+}\, \Xi_{ab}\, \hat{\psi}_b \; + \\ + \frac{1}{2}\sum_{a,b=1}^{N}\left(\hat{\psi}_a^{+}\, \Delta_{ab}\, \hat{\psi}_b^{+} + \hat{\psi}_a\, \Delta_{ab}^{*}\, \hat{\psi}_b\right)\end{aligned}} \qquad \text{(Ten-31.1)}$$

→: Note that while in the non-superconducting case the vectors of operators

$$\boxed{\hat{\psi} = \begin{pmatrix}\hat{\psi}_1\\ \vdots \\ \hat{\psi}_N\end{pmatrix} \quad \text{and} \quad \left(\hat{\psi}^{+}\right)^{t} = \begin{pmatrix}\hat{\psi}_1^{+}\\ \vdots \\ \hat{\psi}_N^{+}\end{pmatrix}}$$

are linearly independent (clearly!), this is not the case for the vectors $\hat{\chi}$ and $\left(\hat{\chi}^{+}\right)^{t}$: In view of Eq. (Ten-29.2),

$$\boxed{\left(\hat{\chi}^{+}\right)^{t}} = \begin{bmatrix}\left(\hat{\psi}^{+}\right)^{t} \\ \hline \hat{\psi}\end{bmatrix} = \tau_1 \cdot \begin{bmatrix}\hat{\psi} \\ \left(\hat{\psi}^{+}\right)^{t}\end{bmatrix} \boxed{= \tau_1 \cdot \hat{\chi}} \;, \qquad \text{(Ten-31.2)}$$

where

$$\tau_1 = \left[\begin{array}{c|c} 0_N & \mathbb{1}_N \\ \hline \mathbb{1}_N & 0_N \end{array}\right] \;, \qquad \text{(Ten-31.3)}$$

→: From Eq. (Ten-31.2) ⇒

(i): $\boxed{\hat{\chi} = \tau_1 (\hat{\chi}^{+})^{t}}$ ← (Ten-32.1a)

~~equation~~: and

(ii): $\boxed{(\hat{\chi}^{+})^{t} = \tau_1 \cdot \hat{\chi}}$,

.. which implies

$$\boxed{\hat{\chi}^{+} = \hat{\chi}^{t} \cdot \tau_1} \quad \leftarrow \quad \text{(Ten-32.1b)}$$

→: We can use this to show the invariance of the 1st quantized BdG Hamiltonian [Eq. (Ten-30.1)] under a charge-conjugation symmetry:

(Ten-30.1) (Ten-32.1a,b)

$$\hat{H} = \frac{1}{2}\hat{\chi}^{+} H \hat{\chi} = \frac{1}{2}\hat{\chi}^{t}\tau_1 H \tau_1 (\hat{\chi}^{+})^{t} = \quad \leftarrow \text{write out}$$

$$= \frac{1}{2}\sum_{A,B=1}^{2N} \hat{\chi}_A (\tau_1 H \tau_1)_{AB} \hat{\chi}^{+}_B =$$

$$= \frac{1}{2}\sum_{A,B=1}^{2N} (\tau_1 H \tau_1)_{AB} \cdot \left(-\hat{\chi}^{+}_B \hat{\chi}_A + \delta_{A,B}\right) =$$

$$= \frac{1}{2}\hat{\chi}^{+}(-\tau_1 H \tau_1)^{t}\hat{\chi} + \frac{1}{2}\underbrace{\mathrm{Tr}(\tau_1 H \tau_1)}_{= \mathrm{Tr}(H) = 0}$$

$= \mathrm{Tr}(H) = 0$ due to Eq. (Ten-30.2)

In conclusion, we have:

$$\boxed{\frac{1}{2}\,\vec{\chi}^{\dagger}\,\underline{H}\,\vec{\chi} \;=\; \frac{1}{2}\,\vec{\chi}^{\dagger}\underline{\left(-\tau_1 H \tau_1\right)^{t}}\,\vec{\chi}}$$

which implies

$$\boxed{\underline{H} \;=\; \underline{-\,\tau_1 H^{t} \tau_1}}$$

or:

$$\boxed{\tau_1 H^{*} \tau_1} \underset{\substack{\uparrow \\ (H \text{ is Hermitean})}}{=} \tau_1 H^{t} \tau_1 = \boxed{-\,H}, \qquad \text{(Ten-33.1)}$$

This means that every BdG Hamiltonian H [which is always of the form of Eq. (Ten-30.2)] is $\overbrace{\;}^{\text{automatically}}$ invariant under charge-conjugation $\overbrace{\;}^{\text{[Eq. (++)/p. (Ten-15)]}}$ with $\boxed{U_c = \tau_1}$.

→: Therefore, we can treat both systems, non-superconducting and superconducting, within the same notation,

if we denote by

$$\boxed{\hat{\underline{\Psi}} = \begin{cases} \hat{\psi} = \begin{bmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_N \end{bmatrix} & \text{non-superconducting} \\ \frac{1}{\sqrt{2}}\hat{\chi} = \begin{bmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_N \\ \hat{\psi}_1^+ \\ \vdots \\ \hat{\psi}_N^+ \end{bmatrix} \ \left(\text{"Nambu-Spinor"}\right) & \text{superconducting} \end{cases}}$$

and in both cases, the 2nd quantized Hamiltonian reads

$$\boxed{\hat{H} = \hat{\underline{\Psi}}^+ \, \underset{\text{1st quantized Hamiltonian}}{H} \, \hat{\underline{\Psi}}}$$

The Ten-Fold Way

→: Now return to the problem of classifying all 1^{st} quantized Hamiltonians H which are invariant under some group G_0 of symmetries that are unitarily realized on the single-particle Hilbert space.
[Note that includes also the case where $G_0 = \{1\}$ = trivial.]

⇒: As discussed, these Hamiltonians are characterized by symmetry-less block Hamiltonians $H^{(\lambda)}$.

Since those quantum mechanical symmetries which are not yet accounted for by invariance under G_0 are the anti-unitarily realized symmetries (on the single-particle Hilbert space), it must be the case that these blocks $H^{(\lambda)}$ can be classified by their behavior under these anti-unitary symmetries.

As discussed, there can only be two such anti-unitary symmetries: $\boxed{T \text{ and } C.}$
As discussed, we also need to consider $\boxed{S = T \cdot C.}$

→: Therefore, the problem of classifying all Bloch Hamiltonians $H^{(2)}$, which for brevity we simply denote by H, has been reduced to the problem of classifying all possible ways in which H can respond to T, C and S.

This is a complete classification scheme of all 1st quantized Hamiltonians H, because after systematically eliminating all unitarily realized symmetries, the only possible remaining quantum mechanical symmetries are T, C and S.

→: <u>Classification:</u>

–: It would at first appear that there are

$$\boxed{3 \times 3 = 9}$$

ways a Hamiltonian can respond to T $(T^2 = 0, \pm 1)$ and C $(C^2 = 0, \pm 1)$.

–: This is not quite, but almost, true:

For 8 of the 9 choices the behavior under S is uniquely fixed by the behavior under T and C.

There is one choice,

$$\boxed{T = 0 \quad \text{and} \quad C = 0}$$

where this is not the case: Here, S can be $S = 0$ or $S = 1$.

–: So we obtain

$$(3 \times 3 - 1) + 2 = 8 + 2 = 10$$ possible behaviors

–: Each of these 10 possibilities is called a symmetry class. The time evolution operators $U(t) = \exp\{itH\}$ for each class is listed in the TABLE.

Ten Fold Way [CARTAN CLASSES]

-: Examples (TABLE)

-: First row "A": $T = C = S = 0$ (no symmetries)

=> No constraint on the Hamiltonian =>

=> H = general Hermitean $N \times N$-matrix
[apart from locality]

=> time-evolution operator = general unitary $N \times N$-matrix

=> "$U(N)$"

-: 2^{nd} row "AI": H invariant under T with $T^2 = +1$

* We know that for such a Hamiltonian we can choose a basis in which H = symmetric $= H_S$.

[exercise: $U_T H^* U_T^+ = H$ ← p. Ten-11

Since $T^2 = +1$ we know that $(U_T)^t = U_T$; (symmetric).
Can show there exists a basis where $U_T = \mathbb{1}$.]

* Now consider an arbitrary hermitian Hamiltonian H and decompose it into symmetric and antisymmetric pieces:

$$\boxed{\begin{aligned} H &= \underbrace{\tfrac{1}{2}(H + H^t)} + \underbrace{\tfrac{1}{2}(H - H^t)} = \\ &= \quad H_s \quad + \quad H_a \end{aligned}}$$

* So, in the special basis where the $T^2 = +1$ Hamiltonian is symmetric $= H_s$, we can write it as

$$\boxed{H_s = H - H_a}$$

upon exponentiation, the corresponding time-evolution operator is $\boxed{e^{itH_s}}$.

But $\boxed{e^{itH} \in \mathcal{U}(N)}$

and $\boxed{e^{iH_a} \in O(N)}$

so that

$$\boxed{e^{itH_s} \in \mathcal{U}(N)/O(N) \quad \text{(coset space)}}$$

This is the meaning of the entry in 4th column of 2nd row of TABLE.

* all other entries work similarly.

done ✓

→: Comment:

The set of time-evolution operators
has a geometrical meaning:

Élie Cartan (1926):
asked an apparently unrelated question:

* generalizations of spheres?

|| constant curvature everywhere,
~~sam~~ one radius of curvature ||

Riemannian spaces (i.e. with a Riemannian metric),
Riemann curvature tensor covariantly constant

* List of constant curvature spaces (= generalizations of spheres)
turns out to be
precisely the set of 10 ~~tetas~~ classes
of time-evolution operators!

Classification – Topology

→: So far, no discussion of Topology.

Three ways to bring in Topology:

3 Approaches to Classification of Topol.

Insulators + Superconductors.

→: Anderson localization [boundary-based: anomaly]

[Schnyder, Ryu, Furusaki, Ludwig: 2008, 2009, 2010]

→: Topology (K-Theory) [bulk-based: band structure]

(Kitaev: 2009)

→: Quantum Anomalies

[Ryu, Moore, Ludwig: 2012]

Basic Ideas underlying classification, simplest example for classification in the bulk

→: Impose translational symmetry

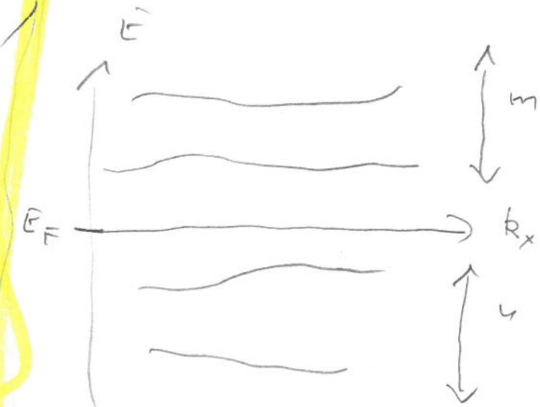
⇒ can label states by momentum eigenvalue $\vec{k}$

$$H(\vec{k})\,|u_a(\vec{k})\rangle = E_a(\vec{k})\,|u_a(\vec{k})\rangle$$

$\vec{k} \in BZ$ (torus)

a = band index

"n" filled, "m" ~~empty~~ empty bands



→: Spectral Flattening

Choose $E_F = 0$

All filled bands (deform): $E = -1$

All empty bands (deform): $E = +1$

$$H(\vec{k}) \overset{\text{deform}}{\rightsquigarrow} \underbrace{Q(\vec{k})}_{\Uparrow} : \quad E_a(\vec{k}) = \begin{cases} +1, & \text{empty bands} \\ -1, & \text{filled bands} \end{cases}$$

"simplified (spectrally flattened) Hamiltonian"

→: Simplest Example: Symmetry Class A

[no symmetry conditions: T = C = S = 0]

–: Diagonalize $Q(\vec{k})$:

$$Q(\vec{k}) = \mathcal{U}(\vec{k})\,\Lambda\,\mathcal{U}^{+}(\vec{k})$$

$$\Lambda = \left[\begin{array}{c|c} \mathbb{1}_m & 0 \\ \hline 0 & -\mathbb{1}_n \end{array}\right], \quad \mathcal{U}(\vec{k}) \in U(m+n)$$

–: Clearly, when

$$\mathcal{U}(\vec{k}) = u_0(\vec{k}) = \left[\begin{array}{c|c} u_1(\vec{k}) & 0 \\ \hline 0 & u_2(\vec{k}) \end{array}\right] \quad \begin{array}{l} u_1(\vec{k}) \in U(m) \\ u_2(\vec{k}) \in U(n) \end{array}$$

$$u_0\,\Lambda\,u_0^{+} = \Lambda \quad (\text{invariant})$$

—: Therefore: $Q(\vec{k}) \in U(m+n)/[U(m) \times U(n)] := G_{m,\,m+n}(\mathbb{C}) =$

= complex Grassmannian

and:

$$Q: \; BZ \to G_{m,\,m+n}(\mathbb{C})$$

$$\vec{k} \to Q(\vec{k})$$

→: If the Brillouin zone BZ was a d-dimensional sphere S_d ("spherical BZ") which it isn't, then the topological properties of these maps are classified by the

Homotopy Groups: $\boxed{\pi_d\left(\frac{U(m+n)}{[U(m)\times U(n)]}\right)}$

which are known.

→: For example in $d=2$ spatial dimensions

$$\boxed{\pi_{d=2}\left(\frac{U(m+n)}{[U(m)\times U(n)]}\right) = \mathbb{Z}}$$

[independent of "m" and "n" when these are sufficiently large]

Maps characterized by different integers cannot be continuously deformed into each other.

These integers just count, e.g., the number of chiral edge states of the Integer Quantum Hall states which are examples of topological insulators in symmetry class A (no symmetries)

→: In $d=3$ spatial dimensions,

$$\boxed{\pi_{d=3}\left(\frac{U(m+n)}{U(m)\times U(n)}\right) = \{1\} = \text{"trivial group"}}$$