Charge-conjugation (particle-hole) symmetry

→: $\hat{\mathcal{C}}$ = implements charge-conjugation on the Fermion Fock Space of 2^{nd} quantization

DEF:

$$\hat{\mathcal{C}}\,\hat{\psi}_A\,\hat{\mathcal{C}}^{-1} = \sum_B (u_c^*)^\dagger_{AB}\,\hat{\psi}^\dagger_B \;; \qquad \hat{\mathcal{C}}\,\hat{\psi}^\dagger_A\,\hat{\mathcal{C}}^{-1} = \sum_B \hat{\psi}_B\,(u_c^*)_{BA} \qquad \text{(Ten-15.1)}$$

$$\hat{\mathcal{C}}\,i\,\hat{\mathcal{C}}^{-1} = +i \quad \text{(unitary)}$$

when u_c = unitary $N \times N$ matrix [in order for this to preserve the canonical commutation relations]

→: 2^{nd} quantized Hamiltonian $\hat{H}$ $\hat{\mathcal{C}}$-invariant, if and only if

$$\boxed{\hat{\mathcal{C}}\,\hat{H}\,\hat{\mathcal{C}}^{-1} = \hat{H}} \qquad (+)$$

One can check that this is equivalent to the condition

(the minus sign arises from Fermi statistics)

$$\boxed{u_c\left(H - \tfrac{1}{2}\mathrm{tr}(H)\right)^t u_c^\dagger = -\left(H - \tfrac{1}{2}\mathrm{tr}(H)\right)} \qquad (++)'$$

→: Eq. $(++)'$ actually implies that $\mathrm{tr}(H) = 0$ [see next page: Ten-15b], so that we have

$$\boxed{u_c\,H^*\,u_c^\dagger = -H} \qquad (++)$$

Note: $H^t = H^*$ since H = Hermitean

use:

$$\boxed{\begin{array}{ll} \hat{\mathcal{C}}\,\hat{\psi}_A\,\hat{\mathcal{C}}^{-1} = \sum_B (u_c^*)^{+}_{AB}\,\hat{\psi}^{+}_B \;; & \hat{\mathcal{C}}\,\hat{\psi}^{+}_A\,\hat{\mathcal{C}}^{-1} = \sum_B \hat{\psi}_B\,(u_c^*)_{BA} \\ \hline \hat{\mathcal{C}}\,\hat{\psi}\,\hat{\mathcal{C}}^{-1} = (u_c^*)^{+}(\hat{\psi}^{+})^{t} \;; & \hat{\mathcal{C}}\,\hat{\psi}^{+}\,\hat{\mathcal{C}}^{-1} = (\hat{\psi})^{t}(u_c^*) \end{array}}$$

$$\hat{\mathcal{C}}\,\hat{H}\,\hat{\mathcal{C}}^{-1} = \underset{\uparrow \text{ p.(Ten-5)}}{\hat{\mathcal{C}}}\left(\sum_{A,B} \hat{\psi}^{+}_A\,\underline{H_{AB}}\,\hat{\psi}_B\right)\hat{\mathcal{C}}^{-1} =$$

using: $\hat{\mathcal{C}}\,i\,\hat{\mathcal{C}}^{-1} = +i$ (unitary)

$$= \sum_{A,B} \left(\hat{\mathcal{C}}\,\hat{\psi}^{+}_A\,\hat{\mathcal{C}}^{-1}\right)\left(\hat{\mathcal{C}}\,H_{AB}\,\hat{\mathcal{C}}^{-1}\right)\left(\hat{\mathcal{C}}\,\hat{\psi}_B\,\hat{\mathcal{C}}^{-1}\right) =$$

$$= \sum_{A,B}\left(\sum_{A'} \hat{\psi}_{A'}\,(u_c^*)_{A'A}\right) H_{AB} \left(\sum_{B'} (u_c^*)^{+}_{BB'}\,\hat{\psi}^{+}_{B'}\right) =$$

$$\left(\hat{\psi}_{A'}\hat{\psi}^{+}_{B'} = -\hat{\psi}^{+}_{B'}\hat{\psi}_{A'} + \delta_{A'B'}\right)$$

$$= (-1)\sum_{A',B'} \hat{\psi}^{+}_{B'}\left(u_c^*\,H\,(u_c^*)^{+}\right)_{A'B'}\hat{\psi}_{A'} + \mathrm{tr}(H) =$$

$$= (-1)\sum_{A',B'} \hat{\psi}^{+}_{B'}\left(\left(u_c^*\,H\,(u_c^*)^{+}\right)^{t}\right)_{B'A'}\hat{\psi}_{A'} + \mathrm{tr}(H) =$$

$$\hookrightarrow = \left(u_c\,H^{t}\,u_c^{+}\right) = \left(u_c\,H^{*}\,u_c^{+}\right)$$

$$\boxed{= \sum_{A',B'} \hat{\psi}^{+}_{B'}\left((-1)\,u_c\,H^{*}\,u_c^{+}\right)_{B'A'}\hat{\psi}_{A'} + \mathrm{tr}(H)}$$

So: $\boxed{\hat{\mathcal{C}}\,\hat{H}\,\hat{\mathcal{C}}^{-1} = \hat{H} \iff u_c\,H^{*}\,u_c^{+} = -H}$

$\Rightarrow$: Taking the Trace of Eq. (++)' on p. Ten-15,

we see:

$$\boxed{\operatorname{tr}\left[u_c \left(H - \tfrac{1}{2}\operatorname{tr}(H)\right)^t u_c^+\right] = -\operatorname{tr}\left(H - \tfrac{1}{2}\operatorname{tr}(H)\right)}$$

$\Rightarrow$: [H is the single-particle Hamiltonian, a $N \times N$-matrix]

$$\boxed{\left[\operatorname{tr}(H) - \tfrac{N}{2}\operatorname{tr}(H)\right] = -\left[\operatorname{tr}(H) - \tfrac{N}{2}\operatorname{tr}(H)\right]}$$

$$\boxed{2\operatorname{tr}(H) = N \cdot \operatorname{tr}(H)}$$

Since we are interested in the case $N \gg 1$, in particular $N > 2$, we conclude that

$$\boxed{\operatorname{tr}(H) = 0}$$

→: Convenient to introduce

$$\boxed{C := \left(\hat{\mathcal{C}}\right)_{1^{st}\,\text{quantized}}} \qquad \text{(Ten-16.1)}$$

Then (++) (p. Ten-15) can be written as

$$\boxed{C H C^{-1} = -H, \quad \text{where } C = u_c K} \qquad \text{(Ten-16.2)}$$

→: One immediately checks that $\hat{\mathcal{C}}^2$, which is a unitary operator $\hat{u}$ [as on p. (Ten-6)], has an associated unitary matrix

$$\boxed{u = u_c u_c^*} \qquad \text{(Ten-16.3)}$$

[This follows also from Eq. (Ten-16.2) above:

$C^2 = u_c K u_c K = u_c (K u_c K^{-1}) = u_c u_c^*$.]

→ It follows from Eq. (+) / p. (Ten-15):

$$\boxed{\hat{\mathcal{C}}^{2}\hat{H}\hat{\mathcal{C}}^{-2} = \hat{\mathcal{C}}\left(\hat{\mathcal{C}}\hat{H}\hat{\mathcal{C}}^{-1}\right)\hat{\mathcal{C}}^{-1} = \hat{H}}$$

This implies the corresponding statement for the 1st-quantized operators:

$$\boxed{(\mathcal{U}_c\mathcal{U}_c^{*})\,H = H\,(\mathcal{U}_c\mathcal{U}_c^{*})}$$

As for time-reversal, we conclude that

$$\boxed{\mathcal{U}_c\mathcal{U}_c^{*} = \pm 1}$$

so that $\boxed{C^2 = \pm 1}$

Conclusion:

There are three ways a Hamiltonian can respond to charge-conjugation symmetry:

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$$\boxed{\underbrace{C^2(=C)}_{\substack{\text{sometimes}\\ \text{we write}\\ \text{for short "}C\text{"}}} = \begin{cases} 0 & H \text{ is not } \cancel{\text{time-reversal}}\ \text{charge-conjugation symmetric} \\ +1 & H \text{ is charge-conjugation symmetric and } C^2 = +1 \\ -1 & \text{" " " " " " } C^2 = -1 \end{cases}}$$

∴ When $C^2 = -1$,

$$\boxed{\hat{\mathcal{C}}^2 \hat{\psi}_A \hat{\mathcal{C}}^{-2} = (-1)\hat{\psi}_A, \quad \hat{\mathcal{C}}^2 \hat{\psi}_A^{+} \hat{\mathcal{C}}^{-2} = (-1)\hat{\psi}_A^{+}}$$

(since $\eta = \eta_c \eta_c^* = -1$)

As for time-reversal, since a state in the Fock space with q-fermions is created from the Fock space vacuum by applying q fermion creation operators, we have

$$\boxed{\begin{aligned} &\hat{\mathcal{C}}^2 = (-1)^{\hat{Q}} = \text{Fermion number parity operator} \\ &\hat{Q} = \sum \hat{\psi}_A^{+} \psi_A \end{aligned}}$$

→: Consider in more detail the action of charge-conjugation $\hat{\mathcal{C}}$ on the Fock-space $\mathcal{F}$.

$\mathcal{F}$ is a direct sum over q-particle Fock spaces

= Fock spaces

$$\boxed{\mathcal{F} = \bigoplus_{q=0}^{N} \mathcal{F}_q}$$

where

"q" is the eigenvalue of the particle-number operator

$$\boxed{\hat{Q} = \sum_{A=1}^{N} \hat{\psi}_A^{+} \hat{\psi}_A}$$

-: $\boxed{\mathcal{F}_{q=0}}$ is spanned by the Fock vacuum $|0\rangle$

$$\hat{\psi}_A |0\rangle = 0 \; ; \quad A = 1, \dots, N$$

$$\boxed{\hat{Q}\,|0\rangle = \overset{q=0}{0} \cdot |0\rangle},$$

and

-: $\boxed{\mathcal{F}_{q=N}}$ is spanned by the 'completely filled' state

$$\boxed{|q=N\rangle = \hat{\psi}_1^{+} \hat{\psi}_2^{+} \hat{\psi}_3^{+} \dots \hat{\psi}_N^{+} |0\rangle} : \boxed{\hat{Q}\,|q=N\rangle = N\,|q=N\rangle}$$

→: Then:

$$\boxed{\hat{\mathcal{C}} : \mathcal{F}_q \to \mathcal{F}_{N-q} \quad (q = 0, 1, 2, \dots, N)}$$

→: Note: when N = even, then

$$\boxed{\hat{\mathcal{C}} : \mathcal{F}_{q=\frac{N}{2}} \to \mathcal{F}_{q=\frac{N}{2}}}$$

→: **In particular:**

$$\boxed{\hat{\mathcal{C}}\,\hat{Q}\,\hat{\mathcal{C}}^{-1}} = \hat{\mathcal{C}}\left(\sum_{A=1}^{N}\hat{\psi}_A^{+}\hat{\psi}_A\right)\hat{\mathcal{C}}^{-1} =$$

$$= \sum_{A=1}^{N}\left(\hat{\mathcal{C}}\,\hat{\psi}_A^{+}\,\hat{\mathcal{C}}^{-1}\right)\left(\hat{\mathcal{C}}\,\hat{\psi}_A\,\hat{\mathcal{C}}^{-1}\right) =$$

$$= \boxed{\sum_{A=1}^{N}}\ \sum_{B}\sum_{B'}\ \hat{\psi}_B\ \boxed{(\mathcal{U}_c^{*})_{BA}\ (\mathcal{U}_c^{*})^{+}_{AB'}}\ \hat{\psi}_{B'}^{+} =$$

$$= \delta_{BB'}$$

$$= \sum_{B}\underbrace{\hat{\psi}_B\,\hat{\psi}_B^{+}}_{=-\hat{\psi}_B^{+}\hat{\psi}_B+1} = -\hat{Q} + N = \boxed{N - \hat{Q}}$$

→: Reconsider the action of $\hat{\mathcal{C}}$ on creation / annihilation operators [p. (Ten - 15)]:

e.g.:

$$\boxed{\hat{\mathcal{C}}\,\hat{\psi}_A^{+}} = \sum_B \hat{\psi}_B \, (\mathcal{U}_C^*)_{BA} \, \hat{\mathcal{C}} = \boxed{\sum_B (\mathcal{U}_C^*)_{BA} \, \hat{\psi}_B \, \hat{\mathcal{C}}}$$

$$\hat{\mathcal{C}}\,\hat{\psi}_A^{+} : \mathcal{F}_q \xrightarrow{\hat{\psi}_A^{+}} \mathcal{F}_{q+1} \xrightarrow{\hat{\mathcal{C}}} \mathcal{F}_{N-(q+1)}$$

$$\hat{\psi}_B\,\hat{\mathcal{C}} : \mathcal{F}_q \xrightarrow{\hat{\mathcal{C}}} \mathcal{F}_{N-q} \xrightarrow{\hat{\psi}_B} \mathcal{F}_{N-q-1}$$

Chiral ("sublattice") symmetry

→: As we will see shortly, we will need to consider besides $\hat{\mathcal{T}}$ and $\hat{\mathcal{C}}$, also the combined operation

$$\boxed{\hat{\mathcal{S}} = \hat{\mathcal{T}} \cdot \hat{\mathcal{C}}} \tag{Ten-20.1}$$

(called: "chiral", or "sublattice" symmetry
[these names are largely historical])

→: $\hat{\mathcal{S}}$ is an anti-unitary operator on Fock space (because $\hat{\mathcal{T}}$ is anti-unitary and $\hat{\mathcal{C}}$ is unitary), whose action follows from that of $\hat{\mathcal{T}}$ and $\hat{\mathcal{C}}$:

$$\boxed{\begin{gathered} \hat{\mathcal{S}}\,\hat{\psi}_A\,\hat{\mathcal{S}}^{-1} = \sum_B (\mathcal{U}_S^*)^+_{AB}\,\hat{\psi}^+_B \,; \quad \hat{\mathcal{S}}\,\hat{\psi}^+_A\,\hat{\mathcal{S}}^{+} = \sum_B \hat{\psi}_B\,(\mathcal{U}_S^*)_{BA} \\ \hat{\mathcal{S}}\, i\, \hat{\mathcal{S}}^{-1} = -i \quad (\text{anti-unitary}) \end{gathered}} \tag{Ten-20.2}$$

where the unitary matrix

$$\boxed{\mathcal{U}_S = \mathcal{U}_T\,\mathcal{U}_C^*} \tag{Ten-20.3}$$

→: The 2^{nd} quantized Hamiltonian $\hat{H}$ is invariant under $\hat{S}$, if and only if

$$\boxed{\hat{S}\,\hat{H}\,\hat{S}^{-1} = \hat{H}} \qquad (\text{Ten-21.1})$$

It is immediately checked that this is equivalent to the following condition on the 1^{st} quantized Hamiltonian

$$\boxed{u_S\, H\, u_S^{\dagger} = -H; \quad (u_S = u_T u_C^{*})} \qquad (\#)$$

[As for charge conjugation invariance, the 2^{nd} quantized condition (Ten-21.1) implies $\boxed{\mathrm{Tr}(H) = 0.}$]

→: convenient to introduce (as before)

$$\boxed{S} = \underbrace{\hat{S}\Big|_{1^{st}\text{ quantized}}}_{} = T \cdot C =$$

$$= (u_T K)(u_C K) = \boxed{u_T\, u_C^{*} = u_S}$$

from Eq. (#) above:

$$\boxed{S\, H\, S^{-1} = -H} \qquad (S = u_S), \qquad (\#\#)$$

— : From Eq. (##) / p. Ten-21 $\Rightarrow$

$\Rightarrow$ $\boxed{S^2 = (U_S)^2 \text{ commutes with } H}$

— : As before, we conclude that

$$\boxed{S^2 = (U_S)^2 = e^{i\phi} \cdot \mathbb{1}}$$

(phase, because l.h.s. is unitary)

But recalling that $U_S = U_T U_C^*$,

and observing that the phases of U_T and U_C are completely arbitrary (were not affected on any of the properties mentioned), we can choose the phases of U_T and U_C so that

$$\boxed{e^{i\phi} = \mathbb{1}}.$$

With this choice,

$$\boxed{S^2 = (U_S)^2 = \mathbb{1}}$$

Alternative Definition of $\hat{S}$, and commutation properties between T and C

-: We could have equally well chosen to consider

$$\boxed{\hat{S}' := \hat{C} \cdot \hat{J}}$$

(instead of $\hat{S} = \hat{J} \cdot \hat{C}$).

-: The corresponding 1^{st} quantized version would be

$$\boxed{S'} = \hat{S}'\Big|_{1^{st}\text{ quantized}} = C \cdot T =$$

$$= (U_C K)(U_T K) = \boxed{U_C U_T^*}$$

-): Now observe:

(a): $T = U_T K$; $T^2 = \epsilon_T \cdot \mathbb{1}$; $\epsilon_T = \pm 1$

=): $U_T U_T^* = \epsilon_T$ =): $U_T = \epsilon_T (U_T^*)^\dagger = \epsilon_T (U_T)^t$

or: $\boxed{(U_T)^t = \epsilon_T U_T}$ (U_T symmetric when $T^2 = +1$; U_T antisymmetric when $T^2 = -1$)

(b): $C = U_C K$; $C^2 = \epsilon_C \cdot \mathbb{1}$; $\epsilon_C = \pm 1$

=): $U_C U_C^* = \epsilon_C$ =): $U_C = \epsilon_C (U_C^*)^\dagger = \epsilon_C (U_C)^t$

or: $\boxed{(U_C)^t = \epsilon_C U_C}$ (U_C symmetric when $C^2 = +1$; U_C antisymmetric when $C^2 = -1$)

– : Now consider:

$$S = TC = \mathcal{U}_T K \mathcal{U}_C K = \mathcal{U}_T \mathcal{U}_C^*$$

$$S' = CT = \mathcal{U}_C K \mathcal{U}_T K = \mathcal{U}_C \mathcal{U}_T^*$$

$$\Rightarrow: \quad S^+ = (\mathcal{U}_C^*)^+ (\mathcal{U}_T)^+ = \mathcal{U}_C^{\ t}\ \mathcal{U}_T^{*\,t} = \epsilon_C \epsilon_T \underbrace{\mathcal{U}_C \mathcal{U}_T^*}_{= S'} = \epsilon_C \epsilon_T S'$$

$$\cancel{\Rightarrow: \quad S^{*t} = \mathcal{U}_C^{\ t}\ \mathcal{U}_T^{*\,t} = \epsilon_C \epsilon_T \mathcal{U}_C \mathcal{U}_T^* = \epsilon_C \epsilon_T S'}$$

Hence:

$$\boxed{S^+ = \epsilon_C \epsilon_T S'}$$

but: $S^2 = \mathbb{1}$ $\Rightarrow$ $S = S^{-1} = S^+$

(because $S = \mathcal{U}_T \mathcal{U}_C^* =$ unitary)

$$\Rightarrow: \quad \boxed{S = \epsilon_C \epsilon_T S'} \quad \text{or:} \quad \boxed{\mathcal{U}_S = \epsilon_C \epsilon_T \mathcal{U}_{S'}} \quad (*)$$

this implies therefore that:

$$\boxed{\underbrace{TC}_{= S} = \epsilon_C \epsilon_T \underbrace{CT}_{= S'}} \quad (**)$$

Eq. (*) / p. Ten-24 implies:

$$\boxed{\hat{\varsigma} = \epsilon_C \epsilon_T \hat{\varsigma}'}$$

[because the unitary matrices U_ς and $U_{\varsigma'}$ completely fix the respective 2nd quantized operators $\hat{\varsigma}$ and $\hat{\varsigma}'$; see Eq. (Ten-20.1).]

$\boxed{\text{Comment (i): We need to consider only one time-reversal, and one charge-conjugation operator}}$

→: $\boxed{\text{time-reversal:}}$

* Assume there were two different time-reversal operators,

$$\boxed{T_1 = u_{T_1} \cdot K} \quad \text{and} \quad \boxed{T_2 = u_{T_2} \cdot K}$$

* Then, the composition

$$T_1 \cdot T_2 = u_{T_1} \cdot K \cdot u_{T_2} K = u_{T_1} u_{T_2}^*$$

is a unitary operator.

* This means that

$$T_2 = u_{T_2} \cdot K = \underbrace{\left(u_{T_2} (u_{T_1})^\dagger\right)}_{= u_{12}} \underbrace{\left(u_{T_1} \cdot K\right)}_{= T_1} =$$

$$= u_{12} \, T_1 \quad \text{where} \quad \boxed{u_{12} = u_{T_2} (u_{T_1})^\dagger}$$

* Hence we can enlarge the existing group G_0 of symmetries that are unitarily realized on the single-particle Hilbert space by the group element u_{12} to a new group G_0' [$G_0' = G_0$ if u_{12} is already in G_0].

$\Downarrow$

[this may reorganize the notion of "blocks".]

Upon repeating all the previous analysis using G_0' instead of G_0, there will be no difference in extending the unitarily realized symmetries by the anti-unitary symmetry elements T_1 or T_2.

[Recall that the structure of block Hamiltonians does not depend on G_0.]

So we do not get a new result when we include a second time-reversal operator T_2, in addition to T_1.

→: $\boxed{\text{charge-conjugation}}$:

A completely analogous argument holds in the case of two (anti-unitary) charge-conjugation operators C_1 and C_2

→ **Chiral symmetry:**

However, it is not possible to dispose of the anti-unitary charge-conjugation operator C, while keeping only T (or vice versa):

while

$$\boxed{S = T \cdot C}$$

is a unitary operator, invariance under chiral symmetry implies that S anti-commutes with H.

S will not commute with H.

Therefore, it is not possible to dispose of C in favor of T (or vice versa) by augmenting the group G_0 by the element S.

We must keep S explicitly in our analysis.