Lectures on topological order: Long range entanglement and topological excitations

Xiao-Gang Wen, Boulder summer school

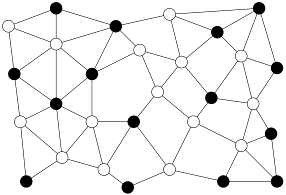
2016/8

Landau symmetry breaking theory does not describe all quantum phases.

Why? What do the phases beyond Landau symmetry breaking theory look like?

Local quantum systems and gapped quantum systems

- A **local quantum system** is described by $(\mathcal{V}_N, H_N)$
  $\mathcal{V}_N$: a Hilbert space with a tensor structure $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_i$
  H_N: a local Hamiltonian acting on $\mathcal{V}_N$:
  $H_N = \sum \hat{O}_{ij}$



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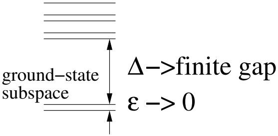

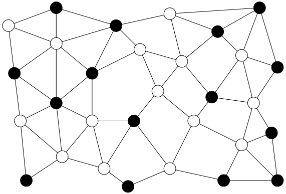

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- A **gapped quantum system** (a concept for $N \to \infty$ limit):
  $\{(\mathcal{V}_{N_1}, H_{N_1}); (\mathcal{V}_{N_2}, H_{N_2}); (\mathcal{V}_{N_3}, H_{N_3}); \cdots \}$ with gapped spectrum.

\- *A gapped quantum system is not a single Hamiltonian, but a sequence of Hamiltonian with larger and larger sizes.*

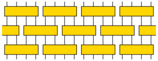
A gapped (ie short-range correlated) quantum phase

- **A gapped state** is a sequence of ground subspaces: $\Psi_{N_1}, \Psi_{N_2}, \cdots$

- **A gapped quantum phase** is an equivalent class of local unitary (LU) transformation of gapped states

$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\, T \int_0^1 dg\ H(g)}\right)|\Psi(0)\rangle$$

$$= \quad |\Psi(0)\rangle$$

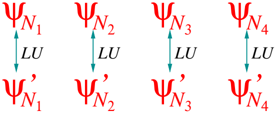


where $H(g) = \sum_i O_i$ is local.

Hastings-Wen cond-mat/0503554; Bravyi-Hastings-Michalakis arXiv:1001.0344

Chen-Gu-Wen arXiv:1004.3835

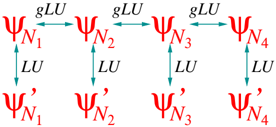
$\Psi_{N_1}, \Psi_{N_2}, \Psi_{N_3}, \Psi_{N_4}, \cdots$
$\Psi'_{N_1}, \Psi'_{N_2}, \Psi'_{N_3}, \Psi'_{N_4}, \cdots$



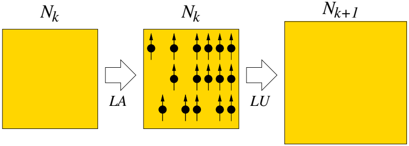
- OK definition with translation symmetry, since there is natural way $N_i \to N_{i+1}$. Not OK without translation symmetry.

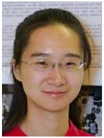
A gapped (short-range correlated) quantum **liquid** phase

- **A gapped quantum liquid phase:**
  $\Psi_{N_1}, \Psi_{N_2}, \Psi_{N_3}, \Psi_{N_4}, \cdots$
  $\Psi'_{N_1}, \Psi'_{N_2}, \Psi'_{N_3}, \Psi'_{N_4}, \cdots$
  $N_{k+1} = sN_k, \ \ s \sim 2$



- $\Psi_{N_{i+1}} \overset{LA}{\sim} \Psi_{N_i} \otimes \Psi^{dp}_{N_{i+1}-N_i}$. Generalized local unitary (gLU) trans.
  where
  $\Psi^{dp}_N = \otimes_{i=1}^{N} |\uparrow\rangle$





Zeng-Wen arXiv:1406.5090

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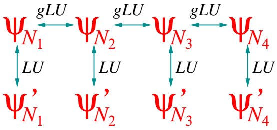

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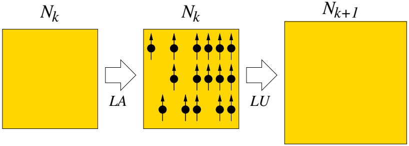

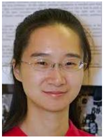

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- gLU transformations allow us to take the thermal dynamical limit ($N_k \to \infty$ limit) without translation symmetry.

Long range entanglement $\rightarrow$ topological order

For gapped systems with no symmetry:

- According to Landau theory, no symm. to break
  $\rightarrow$ all systems belong to one trivial phase

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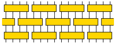
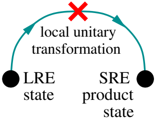
Chen-Gu-Wen arXiv:1004.3835

- **long range entangled (LRE) states**
- **short range entangled (SRE) states**

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$



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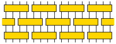
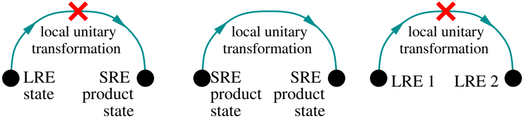
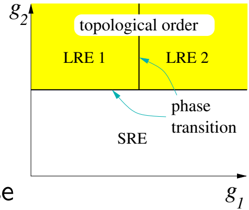
- **long range entangled (LRE) states** → many phases
- **short range entangled (SRE) states** → one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$

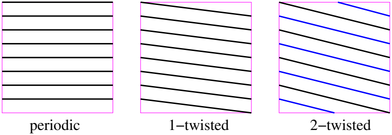




- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases: different patterns of long-range entanglements defined by LU trans.
  = different **topological orders** Wen PRB **40** 7387 (89)

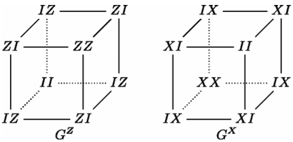
Examples of gapped quantum non-liquid states

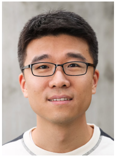
- Stacking 2+1D FQH states → gapped quantum state, but not liquids.
- Layered $\nu = 1/m$ FQH state: Ground state degeneracy can be $GSD = m^{L_z}, m, m^2$



- Haah's cubic code on 3D cubic lattice:

$$H = -\sum_{cubes} (G^Z + G^X),$$

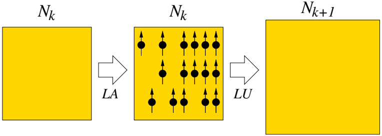


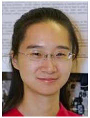


Jeongwan Haah, Phys. Rev. A 83, 042330 (2011) arXiv:1101.1962

More exotic long-range entanglement

- Topo. order = gapped quantum **liquid** Zeng-Wen14; Swingle-McGreevy14
  - → gauge theory
  - → Fermi statistics
  - → quantum field theory
  - → MERA rep. Vidal 06





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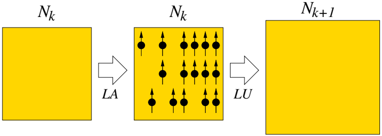
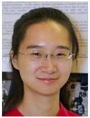
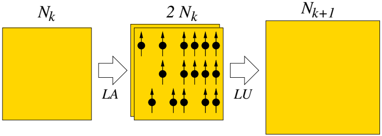




- **s-source** entanglement structure Swingle-McGreevy 14

- Quantum liquid has $s = 1$
- 3D layered FQH: $s = 2$
- d+1D Fermi liquid: $s = \frac{2^d}{2}$
- no MERA rep.



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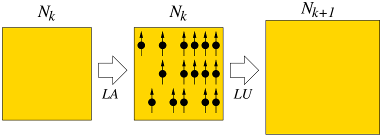

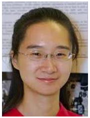

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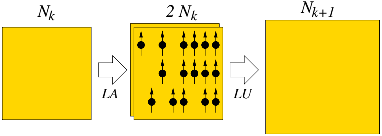
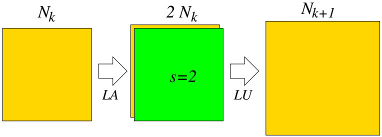
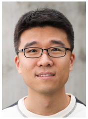


- Haah's cubic code Haah 11

- no MERA rep.
- No quantum field theory description

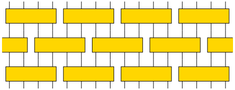




Many-body entanglement goes beyond quantum field theory.

Bosonic/fermionic gapped quantum liquid phases

Both local bosonic and fermionic systems have the following local property: $\mathcal{V}_{\text{tot}} = \otimes_i \mathcal{V}_i$ Gu-Wang-Wen arXiv:1010.1517

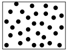
$H' \sim UHU^\dagger, \quad U =$ 

- Bosonic liquid phases are defined by gLU trans. $U = \prod U_{ijk}$:
  (1) $[U_{ijk}, U_{i'j'k'}] = 0$
  (2) U_{ijk} acts within $V_i \otimes V_j \otimes V_k$. e.g. $U_{ijk} = e^{i(b_i b_j b_k^\dagger + h.c.)}$
- Fermionic liquid phases are defined by gLU trans. $U^f = \prod U^f_{ijk}$:
  (1) $[U^f_{ijk}, U^f_{i'j'k'}] = 0$, *but U^f_{ijk} may not act within $V_i \otimes V_j \otimes V_k$*.
  e.g. $U^f_{ijk} = e^{i(t_{ij} c_i c_j + h.c.)}$, where $c_i = \sigma^x_i \prod_{j<i} \sigma^z_j$

Gapped quantum liquids for bosons and fermions have very different mathematical structures

Examples of topological orders (before 2000)

- $\Psi(z_1, z_2, \cdots) = 1$ → equal amplitude superposition of all particle configurations → A product state = superfluid state

$$|\Psi\rangle = \sum_{\text{all conf.}} \Big|$$

$$\Big\rangle = \otimes_z (|0\rangle_z + |1\rangle_z + \cdots)$$

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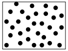
$$|\Psi\rangle = \sum_{\text{all conf.}} \Big|$$  $$\Big\rangle = \otimes_z (|0\rangle_z + |1\rangle_z + \cdots)$$

Laughlin 83

- **Examples**: **I)** scamble the phases

$$\Psi^{\nu=1/2}_{FQH}(z_1, z_2 \cdots) = \Big[\prod (z_i - z_j) \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2} \Big]^2 = [\chi_1(z_i)]^2,$$

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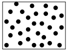
$|\Psi\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle = \otimes_z (|0\rangle_z + |1\rangle_z + \cdots)$

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- **II)** Put $\nu = 1$ state of spin-up(down) electrons $\chi_1(z_i^\uparrow)\chi_1(z_i^\downarrow)$ on lattice, with one electron per site → Chiral spin liquid

Kalmeyer-Laughlin PRL **59** 2095 (87), Wen-Wilczek-Zee PRB **39** 11413 (89)

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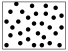
$$|\Psi\rangle = \sum_{\text{all conf.}} \Big|$$  $$\Big\rangle = \otimes_z (|0\rangle_z + |1\rangle_z + \cdots)$$

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- **III)** The square of $\nu = 2$ IHQ wavefunction $[\chi_2(z_i)]^2 \to$ bosonic $\nu = 1$ $SU(2)^f_2$ non-abelian state. $\chi_1[\chi_2]^2$ fermionnic $\nu = \frac{1}{2}$ state

Wen PRL **66** 802 (91). CFT construction: Moore-Read NPB **360** 362 (91)

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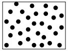
$$|\Psi\rangle = \sum_{\text{all conf.}} \Big|$$  $$\Big\rangle = \otimes_z (|0\rangle_z + |1\rangle_z + \cdots)$$

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Wen PRL **66** 802 (91). CFT construction: Moore-Read NPB **360** 362 (91)

- **IV)** Put an electrons superconducting state on lattice, with one electron per site → Z_2 topological order → Z_2 spin liquid

Read-Sachdev PRL **66** 1773 (91), Wen PRB **44** 2664 (91)

Why Laughlin states have topological order?

K-**matrix states** (generalize Laughlin states):

$$\Psi_K = \prod_{i<j;I} (z_i^I - z_j^I)^{K_{II}} \prod_{i,j;I<J} (z_i^I - z_j^J)^{K_{IJ}} \mathrm{e}^{-\frac{1}{4}\sum |z_i^I|^2}$$

- Quasiparticle excitations are labeled by integer vectors $\boldsymbol{m}$

$$\Psi_\xi = \prod_{i;I} (\xi - z_i^I)^{\boldsymbol{m}_I} \Psi_K,$$

- If $\boldsymbol{m}$ is the I_0^{th} column of K $\to$ Ψ_ξ discribe a missing hole in the I_0^{th} layer, which is a local excitation (not fractionalized).
- Topological excitation is labeled by $\boldsymbol{m}$ mod columns of K. Number of topo. exc. = $\det(K)$.

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$$\Psi_\xi = \prod_{i;I} (\xi - z_i^I)^{\boldsymbol{m}_I} \Psi_K, \qquad \mathcal{L} = \frac{K_{IJ}}{4\pi} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \boldsymbol{m}_I \delta(\xi - x) a_{I0}$$

- If $\boldsymbol{m}$ is the I_0^{th} column of $K \to \Psi_\xi$ discribe a missing hole in the I_0^{th} layer, which is a local excitation (not fractionalized).
- Topological excitation is labeled by $\boldsymbol{m}$ mod columns of K. Number of topo. exc. $= \det(K)$. Statistics: $\theta_{\boldsymbol{m}} = \pi \boldsymbol{m}^T K^{-1} \boldsymbol{m}$.

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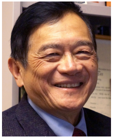

K-matrix states (generalize Laughlin states):

$$\Psi_K = \prod_{i<j;I} (z_i^I - z_j^I)^{K_{II}} \prod_{i,j;I<J} (z_i^I - z_j^J)^{K_{IJ}} \mathrm{e}^{-\frac{1}{4}\sum |z_i^I|^2}$$

- Quasiparticle excitations are labeled by integer vectors $\boldsymbol{m}$

$$\Psi_\xi = \prod_{i;I} (\xi - z_i^I)^{\boldsymbol{m}_I} \Psi_K, \qquad \mathcal{L} = \frac{K_{IJ}}{4\pi} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \boldsymbol{m}_I \delta(\xi - x) a_{I0}$$

- If $\boldsymbol{m}$ is the I_0^{th} column of $K \to \Psi_\xi$ discribe a missing hole in the I_0^{th} layer, which is a local excitation (not fractionalized).
- Topological excitation is labeled by $\boldsymbol{m}$ mod columns of K. Number of topo. exc. $= \det(K)$. Statistics: $\theta_{\boldsymbol{m}} = \pi \boldsymbol{m}^T K^{-1} \boldsymbol{m}$.

K-matrix classification of abelian topological order

- Even K-matrix (all K_{II} are even) classify all 2+1D Abelian topological orders (in a many-to-one way) in local bosonic systems.
- Odd K-matrix (one of the K_{II} is odd) classify all 2+1D Abelian topological orders (in a many-to-one way) in local fermionic systems.

Wen-Zee PRB 46 2290 (92)

Why is the state $[\chi_k(z_i)]^2$ a non-Abelian QH state?

where $\chi_k(z_1, ..., z_N)$ is the IQH wave function of k filled Landau levels.

- What kind of non-Abelian state?
- What is its effective theory
  and edge excitations?

Why is the state $[\chi_k(z_i)]^2 = \chi_k(z_i^{(1)})\chi_k(z_i^{(2)})|_{z_i^{(1)}=z_i^{(2)}}$ a non-Abelian QH state?

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- What kind of non-Abelian state?
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  and edge excitations?

Projective construction:
Split an eletron into partons
and glue them back together

Baskaran-Zou-Anderson

Solid State Comm. **63** 973 (87)

Projective construction: Wen PRL **66** 802 (91); cond-mat/9811111

$$\Phi(z_1,...,z_N) = [\chi_k(z_1,...,z_N)]^n = P[\chi_k(z_1^{(1)},...)\chi_k(z_1^{(2)},...)\cdots]$$

electron $\to$ n-partons, a^{th}-kind partons $z_i^{(a)}$ form $\nu = k$ IQH χ_k

- Effective theory of independent partons

$$H = \frac{1}{2m}\psi_I^\dagger(\partial - iA)^2\psi_I, \qquad I = 1,\cdots,n$$

- Many-body wave function $\Phi(z_i) = \langle 0|\prod \psi_e(z_i)|\chi_k\cdots\chi_k\rangle$
  The electron operator $\psi_e = \psi_1\cdots\psi_n$ is $SU(n)$ singlet,
  if ψ_I form an fundamental representation of $SU(n)$.
- Introduce $SU(n)$ gauge field to glue partons back to electrons:

$$H = \frac{1}{2m}\psi_I^\dagger(\partial - iA\delta_{IJ} - ia_{IJ})^2\psi_J$$

- Effective theory is obtained by integrating out the gapped parton fields:

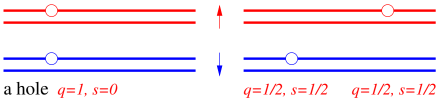
$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}(a_\mu\partial_\nu a_\lambda + \frac{2}{3}a_\mu a_\nu a_\lambda)\epsilon^{\mu\nu\lambda}$$

$SU(n)_k^f$ CS theory. (Level $k=1$ $SU(n)_k^f$ CS theory is abelian.)

Quasiparticle excitations in $[\chi_k(z_i)]^2 = \chi_k(z_i^\uparrow)\chi_k(z_i^\downarrow)|_{z_i^\uparrow = z_i^\downarrow}$

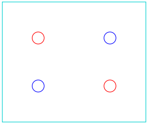
Consider the $[\chi_k(z_i)]^2$ state: $SU(2)^f_k$ Chern-Simons theory

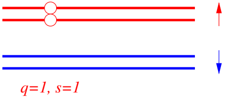
- A charge $q = 1$ hole can be splited into two → two charge $q = 1/2$ quasiparticles.



- The number of four-quasiparticle states: project to $SU(2)$ singlet.
$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes (0 \oplus 1) = 0 \oplus 1 \oplus 1 \oplus (0 \oplus 1 \oplus 2)$

But $SU(2)^f_k$ state has
no quasiparticle with spin $s > \frac{k}{2}$





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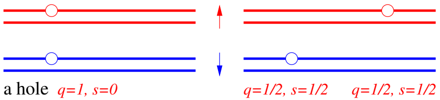

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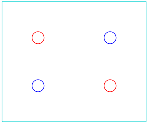
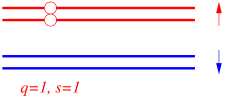




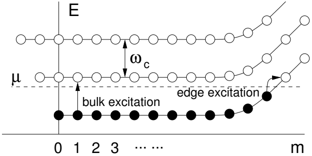
Level-k fusion: $s_1 \otimes s_2 = |s_1 - s_2| \oplus \cdots \oplus \min(s_1 + s_2, k - s_1 - s_2)$

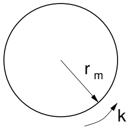
- Level-$k = 1$: $(\frac{1}{2} \otimes \frac{1}{2}) \otimes (\frac{1}{2} \otimes \frac{1}{2}) = (0) \otimes (0) = 0$
- Level-$k = 2$: $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes (0 \oplus 1) = 0 \oplus 1 \oplus 1 \oplus (0)$

Edge excitations in $[\chi_k(z_i)]^n$ state: $U(1) \times SU(k)_n$ CFT

- Edge state: Independent partons → filled Landau levels

$$\mathcal{L} = \psi^\dagger_{\alpha a}(\partial_t - v\partial_x)\psi_{\alpha a},$$
$$\alpha = 1, \cdots, n,$$
$$a = 1, \cdots, k$$





- Excitations are generated by ($a, a^\dagger$ generate exc. in an oscillator)

$U(1): J = \psi^\dagger_{\alpha a}\psi_{\alpha a}$, → $U(1)$ Kac-Moody algebra CFT

$SU(k): J^m = \psi^\dagger_{\alpha a} T^m_{ab}\psi_{\alpha b}$, → $SU(k)_n$ Kac-Moody algebra CFT

$SU(n): j^l = \psi^\dagger_{\alpha a} S^l_{\alpha\beta}\psi_{\beta a}$, → $SU(n)_k$ Kac-Moody algebra CFT

- Glue partons back to electrons = remove the $SU(n)$ excitations.
- Edge excitations are generated by

$U(1): J = \psi^\dagger_{\alpha a}\psi_{\alpha a}$,

$SU(k): J^m = \psi^\dagger_{\alpha a} T^m_{ab}\psi_{\alpha b}$

Edge CFT: $U(1) \times SU(k)_n$ Kac-Moody algebra $c = 1 + \frac{n(k^2-1)}{n+k}$.

- Bulk effective theory $SU(n)^f_k$ CS theory

Another example $\mathcal{S}[\prod(z_i - z_j)^2 \prod(w_i - w_j)^2]$

- Consider with two partons ψ_1, ψ_2, each fills the first Landau level.
  $\to \nu = 1/2$ Laughlin state $\prod(z_i - z_j)^2 = \langle 0| \prod \psi_1(z_i)\psi_2(z_i)|\chi_1\chi_1\rangle$
- Now start with four partons ψ_1, ψ_2, ψ_3, ψ_4, each fills the first Landau level:
  $\prod(z_i - z_j)^2 \prod(w_i - w_j)^2 = \langle 0| \prod \psi_1(z_i)\psi_2(z_i) \prod \psi_3(w_i)\psi_4(w_i)|\chi_1\chi_1\chi_1\chi_1\rangle$
- $\mathcal{S}[\prod(z_i - z_j)^2 \prod(w_i - w_j)^2] = \langle 0| \prod \psi_e(Z_i)|\chi_1\chi_1\chi_1\chi_1\rangle$
  where $\psi_e(Z_i) = \psi_1(Z_i)\psi_2(Z_i) + \psi_3(Z_i)\psi_4(Z_i)$.
- Under $SO(8)$ trans. between $(\text{Re}\psi_i, \text{Im}\psi_i)$, ψ_e is an $SO(5)$ singlet
- **Effective theory** $H = \psi_i^\dagger(\partial - A\delta_{ij} - a_{ij})^2\psi_j \to SO(5)$ CS theory
- **Edge states**: Wen cond-mat/9811111
  Independent partons $\to$ 4 Dirac fermions = 8 Majorana fermions
  After projection $\to$ 8-5 chiral Majorana fermions.
- $\mathcal{S}[\prod(z_i - z_j)^2 \prod(w_i - w_j)^2]$ is the bosonic Pfaffian state.
  $\Psi_{S(220)} = \mathcal{S}[\prod(z_i - z_j)^2 \prod(w_i - w_j)^2] = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)$

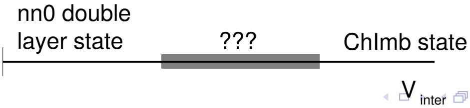
Moore-Read NPB **360** 362 (91); Rezayi-Wen-Read arXiv:1004.2563

How to realize non-Abelian QH states in experiments?

Wen cond-mat/9908394; Rezayi-Wen-Read arXiv:1004.2563

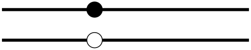
nnm bi-layer state with no interlayer tunneling

- (nnm) state
$$\Phi_{nnm} = \prod (z_i - z_j)^n (w_i - w_j)^n (z_i - w_i)^m e^{-\frac{1}{4}\sum |z_i|^2 + |w_i|^2}$$
where $n = odd$ for fermionic electron and $n = even$ for bosonic "electron".
- (nnm) state $\sim$ $(n-m, n-m, 0)$ state: $\Phi_{nnm} = \chi_1^m \Phi_{n-m,n-m,0}$
Will consider only $(n-m, n-m, 0)$ state, but results apply to (n, n, m) state as well:
$(220) \sim (331)$ state with $\nu = 1/2$ and (330) with $\nu = 2/3$
- Intralayer repulsion $V_{\text{intra}} = 1$, increase interlayer repulsion

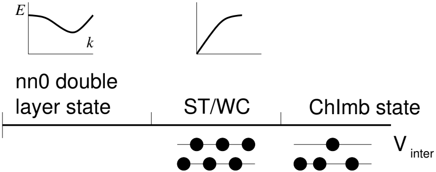


Two possibilities

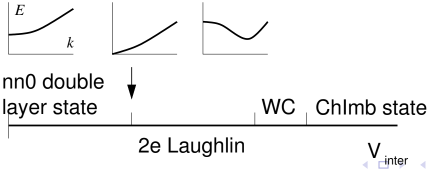
Interlayer-exciton = charge $-\frac{1}{n}$ quasiparticle in one layer + charge $\frac{1}{n}$ quasihole in the other layer



- Interlayer-exciton condensation at $\boldsymbol{k} \neq 0$



- Interlayer-exciton condensation at $\boldsymbol{k} = 0$



Why 2e-Laughlin state? – Hierarchical construction

- $(nn0)$ is described by $U(1) \times U(1)$ CS theory

$$\mathcal{L} = \frac{1}{4\pi} a_{I\mu} \partial_\nu a_{J\lambda} K^{IJ} \epsilon^{\mu\nu\lambda}, \qquad I, J = 1, 2, \qquad K = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$$

- The interlayer exciton (with statistics $\theta = 2\pi/n$) is described by

$$\mathcal{L} = \frac{1}{4\pi} a_I \partial a_J K^{IJ} + \boldsymbol{m}^I a_{I\mu} j^\mu(x), \quad \boldsymbol{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

- Exciton condensation $\mathcal{L} = (j^0)^2 - \boldsymbol{j}^2$ with $\partial_\mu j^\mu = 0$: $j^\mu = \frac{\partial_\nu \tilde{a}_\lambda}{2\pi} \epsilon^{\mu\nu\lambda}$

$$\mathcal{L} = \frac{1}{4\pi} a_I \partial a_J K^{IJ} + \frac{1}{2\pi} \boldsymbol{m}^I a_I \partial \tilde{a} + \frac{1}{8\pi^2 \chi} (\tilde{B}^2 - \frac{1}{v_s^2} \tilde{\boldsymbol{E}}^2)$$

- $\to$ new FQH state:

$$K_{\text{new}} = \begin{pmatrix} K & \boldsymbol{m} \\ \boldsymbol{m}^T & 0 \end{pmatrix} = \begin{pmatrix} n & 0 & 1 \\ 0 & n & -1 \\ 1 & -1 & 0 \end{pmatrix} = W \begin{pmatrix} 2n & 0 & 0 \\ 0 & n\%2 & 1 \\ 0 & 1 & 0 \end{pmatrix} W^T \sim (2n)$$

K and $K' = WKW^T$, $W \in SL(\kappa, Z)$, describe the same FQH state.

- New state is $\nu^* = 1/2n$ Laughlin state of charge-2e electron pairs.

Critical theory for quantum phase transition

- Start with GL theory for excitons and anti-excitons:

$$\mathcal{L} = |\partial_\mu \phi|^2 + \alpha|\phi|^2 + \beta|\phi|^4$$

$\alpha = 0$ at the transition.

- GL-CS theory to reproduce statistics $\theta = 2\pi/n$

$$\mathcal{L} = |(\partial - ia_1 + ia_2)\phi|^2 + \alpha|\phi|^2 + \beta|\phi|^4 + \frac{1}{4\pi} a_I \partial a_J K^{IJ}.$$

- CS term does not destroy the critical point of GL theory, but changes the critical exponents
$(nn0) \to$ *2e-Laughlin is a continuous transition between two states with the SAME symmetry*
- When $n = 2$, critical theory is massless Dirac fermion

$$\mathcal{L} = \bar\psi \gamma^\mu \partial_\mu \psi + m\bar\psi\psi$$

$m = 0$ at the transition.

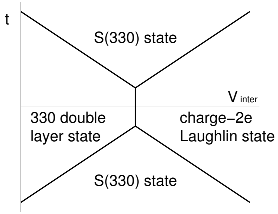
Turn on interlayer tunneling

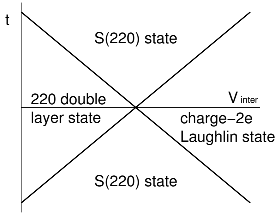
Effective theory near transition

$$\mathcal{L} = |(\partial - ia_1 + ia_2)\phi|^2 + \alpha|\phi|^2 + \beta|\phi|^4 + (t\phi^n \hat{M} + h.c) + \frac{1}{4\pi} a_I \partial a_J K^{IJ}.$$

$$\mathcal{L} = \bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi + (t\psi^T \psi + h.c.), \qquad \text{for } n = 2$$

- When $n = 2$, the $t\psi^T\psi$ term split the massless **Dirac critical point** into two massless **Majorana critical points**.

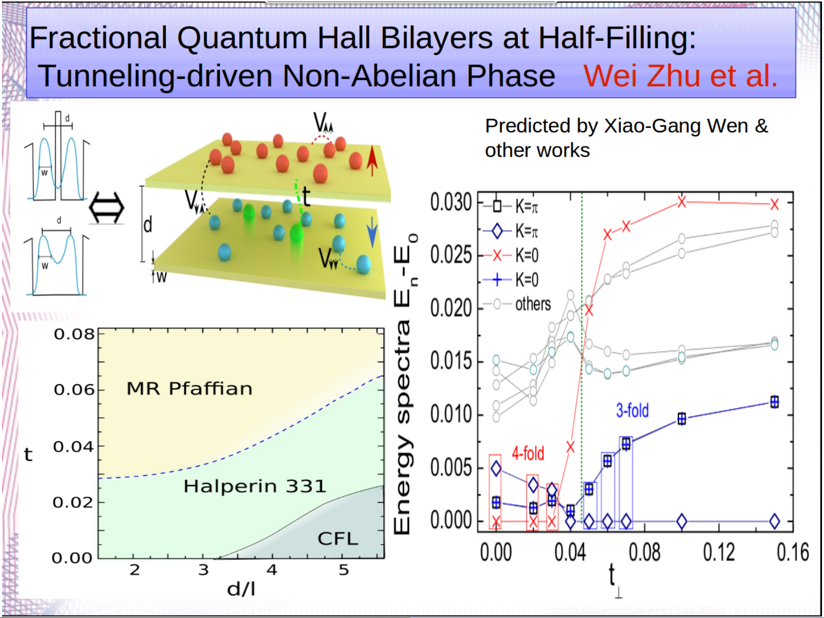




- Weak $p + ip$ superconductor to strong $p + ip$ superconductor is connected by massless Majorana fermion Read-Green cond-mat/9906453

$$\Psi_{S(220)} = \mathcal{S}[\prod(z_i - z_j)^2 \prod(w_i - w_j)^2] = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4} \cdots] \prod(z_i - z_j)$$

Recent numerical result



Projective construction of topo. ordered states on lattice

Consider a spin-$\frac{1}{2}$ system on lattice.

- View spin-$\downarrow$ as zero-boson state and spin-$\uparrow$ as one-boson state
- Split the boson ϕ_i into to fermionic partons $\phi_i = \psi_{i1}\psi_{i2}$, where $\psi_{i\alpha}$ form a 2-dim. rep. of $SU(2)$ and ϕ_i is the $SU(2)$ singlet.

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- Consider the mean-field ground state of a free parton Hamiltonian
  $H_{\text{mean}} = \sum_{\langle ij \rangle} \psi_i^\dagger u_{ij} \psi_j,\ u_{ij} = 2 \times 2 \text{ matrix}; \quad \rightarrow \quad |\Psi_{\text{mean}}^{u_{ij}}\rangle$

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- Project to physical subspace on each site
  $|\downarrow\rangle = |0\rangle, \ |\uparrow\rangle = \psi_{i1}^\dagger\psi_{i2}^\dagger|0\rangle$, both $SU(2)$ singlet.
  Unphysical states $\psi_{i1}^\dagger|0\rangle, \ \psi_{i2}^\dagger|0\rangle$ form a $SU(2)$ doublet.

\- Project into $SU(2)$-singlet subspace on each site:

$$|\Psi_{\text{phy}}^{u_{ij}}\rangle = P_{SU(2)}|\Psi_{\text{mean}}^{u_{ij}}\rangle$$

$|\Psi_{\text{phy}}^{u_{ij}}\rangle$ is a trial wave function with variational parameter u_{ij}.

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- *What is the low energy effective theory that describes the low energy excitations above the many-body state $|\Psi_{phy}\rangle$?*
  Lattice partons ψ_i couple to lattice $SU(2)$ gauge field $a_\mu(x)$:
  $$H_{\text{eff}} = \sum_{\langle ij \rangle} \psi_i^\dagger u_{ij} e^{i a_{ij}} \psi_j + \sum_i \psi_i^\dagger a_0(i) \psi_i$$

Z_2 topological order with time reversal symmetry

- Choose Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91)

$$u_{\mathbf{i},\mathbf{i}+\mathbf{x}} = u_{\mathbf{i},\mathbf{i}+\mathbf{y}} = -\chi\sigma^3, \quad a_0 = c\sigma^1,$$
$$u_{\mathbf{i},\mathbf{i}+\mathbf{x}+\mathbf{y}} = \eta\sigma^1 + \lambda\sigma^2, \qquad u_{\mathbf{i},\mathbf{i}+\mathbf{x}+\mathbf{y}} = \eta\sigma^1 - \lambda\sigma^2$$

$P_{SU(2)}|\Psi^{u_{ij}}_{\text{mean}}\rangle$ has all the symmetry: spin rotation+time reversal.

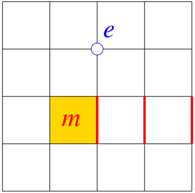
- $H_{\text{eff}} = \sum_{\langle ij\rangle} \psi_i^\dagger u_{ij}\psi_j + \sum_i \psi_i^\dagger a_0\psi_i$ will be fully gapped.
  $\to$ The fermions are all gapped. *The potential gapless excitations may come from the $SU(2)$ gauge fluctuations*.
- a_0 and $SU(2)$ flux $\Phi_i = u_{ij}u_{jk}u_{ki}$ behave like Higgs fields.

$$a_0 \to U a_0 U^\dagger, \quad \Phi_i \to U\Phi_i U^\dagger, \quad U \in SU(2).$$

- If they are invariant under the $SU(2)$ transformation $\to$ The $SU(2)$ is unbroken $\to$ gapless gluon.
- If they are not invariant under the $SU(2)$ transformation $\to$ Break $SU(2)$ to smaller gauge group.

- In our case, a_0 and Φ_i break the $SU(2)$ down to Z_2
  $\to$ Z_2 gauge theory is gapped $\to$ Z_2 **topological order**.

Quasiparticle excitations in the Z_2 topological order

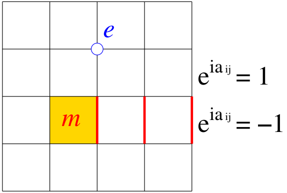


- The pure Z_2 gauge theory:
- Z_2 charge e: boson.
- Z_2 vortex m: boson.
  e and m have mutual π statistics.
- e-m bound state f: fermion.

- Our Z_2 topological order = dressed Z_2 gauge theory, *which also has spin rotation, time reversal and all the lattice symmetry*:
- Z_2 charge e: spin-$\frac{1}{2}$ fermion.
- Z_2 vortex m: spin-0 boson (fermion?).
- e-m bound state f: spin-$\frac{1}{2}$ boson (fermion?).

- We have two possibilities: (2 bosons 1 fermion) or (3 fermions).

Quasiparticle excitations in the Z_2 topological order



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- Z_2 charge e: spin-$\frac{1}{2}$ fermion.
- Z_2 vortex m: spin-0 boson (fermion?).
- e-m bound state f: spin-$\frac{1}{2}$ boson (fermion?).

- We have two possibilities: (2 bosons 1 fermion) or (3 fermions).

The above is the history before 2000

(3 fermions) has a time reversal anomaly, and is not possible.

Examples of topological orders (after 2000)

To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

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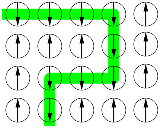
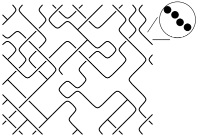
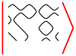
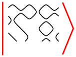
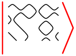


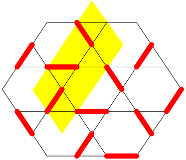


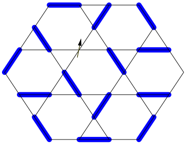
- *sum* over a subset of spin config.:

$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum$ 

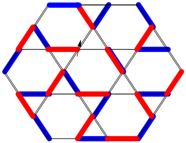
$|\Phi^{DS}_{\text{loops}}\rangle = \sum(-)^{\#\text{ of loops}}$ 

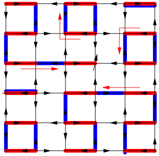
$|\Phi^{\theta}_{\text{loops}}\rangle = \sum(e^{i\theta})^{\#\text{ of loops}}$ 



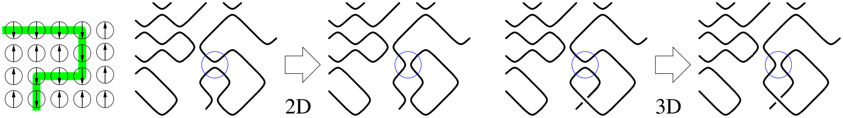


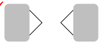
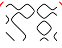
- Can the above wavefunction be the ground states of local Hamiltonians?

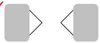
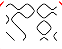




Sum over a subset: local rule → global wave function



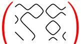
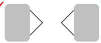
- Local rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $=$ Φ_{str} ()
  → Global wave function Φ_{str} () $= 1$

- Local rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $= -\Phi_{str}$ ()
  → Global wave function Φ_{str} () $= (-)^{\#\text{ of loops}}$

- Two topo. orders: Z_2 topo. order Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91), Moessner-Sondhi PRL 86 1881 (01) and double-semion topo. order. Freedman etal cond-mat/0307511, Levin-Wen cond-mat/0404617

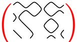
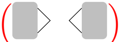
Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
- Ends of strings are point-like excitations,
  which can carry spin-1/2 and Fermi statistics?
  Fidkowski-Freedman-Nayak-Walker-Wang cond-mat/0610583

- $\Phi_{\text{str}}\Big($  $\Big) = 1$ string liquid $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$

  360° rotation: ⫯ → ⫯ and ⫯ = ⫯ → ⫯: $R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

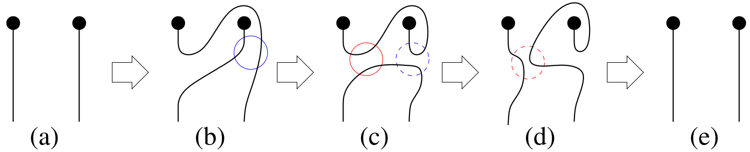
  ⫯ + ⫯ ≡ e spin 0 mod 1. ⫯ − ⫯ ≡ em spin $1/2$ mod 1.

- $\Phi_{\text{str}}\Big($  $\Big) = (-)^{\#\text{ of loops}}$ string liquid $\Phi_{\text{str}}\Big($  $\Big) = -\Phi_{\text{str}}\Big($  $\Big)$

  360° rotation: ⫯ → ⫯ and ⫯ = −⫯ → −⫯: $R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  ⫯ + i⫯ ≡ s_- spin $-\frac{1}{4}$ mod 1. ⫯ − i⫯ ≡ s_+ spin $\frac{1}{4}$ mod 1.

Spin-statistics theorem

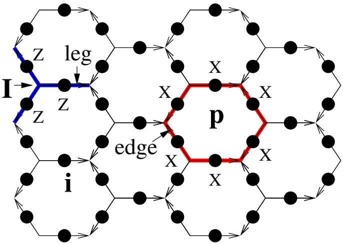


- (a) → (b) = exchange two string-ends.
- (d) → (e) = 360° rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

→ **Spin-statistics theorem**

String operators in Z_2 topological order (Z_2 gauge theory)

- Toric code model: Kitaev quant-ph/9707021
  $H = -U \sum_{\mathbf{I}} Q_{\mathbf{I}} - g \sum_{\mathbf{p}} F_{\mathbf{p}}$
  $Q_{\mathbf{I}} = \prod_{\text{legs of } \mathbf{I}} \sigma^z_{\mathbf{i}}$,
  $F_{\mathbf{p}} = \prod_{\text{edges of } \mathbf{p}} \sigma^x_{\mathbf{i}}$



- Topological excitations:
  e-type: $Q_{\mathbf{I}} = 1 \to Q_{\mathbf{I}} = -1$
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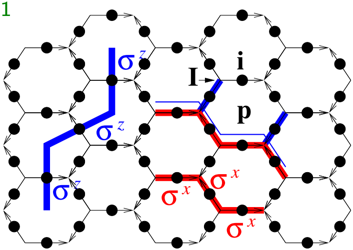

- Topological excitations:
  e-type: $Q_I = 1 \to Q_I = -1$
  m-type: $F_p = 1 \to F_p = -1$

- Type-*e* string operator $W_e = \prod_{\text{str}} \sigma_i^x$
- Type-*m* string operator $W_m = \prod_{\text{str}^*} \sigma_i^z$
- Type-*f* string op. $W_f = \prod_{\text{str}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$

- $[H, W_e^{\text{clsd}}] = [H, W_m^{\text{clsd}}] = 0$. $\to$ Closed strings cost no energy
- $[Q_I, W_e^{\text{open}}] \neq 0$ flip $Q_I \to -Q_I$, $[F_p, W_m^{\text{open}}] \neq 0$ flip $F_p \to -F_p$
  $\to$ open-string create a pair of topo. excitations at their ends.

String operators in Z_2 topological order (Z_2 gauge theory)

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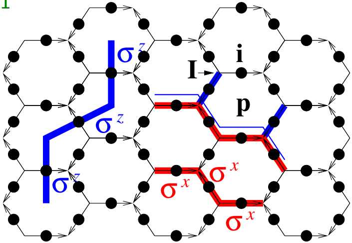

- Topological excitations:
  e-type: $Q_I = 1 \to Q_I = -1$
  m-type: $F_p = 1 \to F_p = -1$

- Type-e string operator $W_e = \prod_{\text{str}} \sigma_i^x$ $\to$ e-type. $e \times e = 1$
- Type-m string operator $W_m = \prod_{\text{str}*} \sigma_i^z$ $\to$ m-type. $m \times m = 1$
- Type-f string op. $W_f = \prod_{\text{str}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$ $\to$ f-type $= e \times m$

- $[H, W_e^{\text{clsd}}] = [H, W_m^{\text{clsd}}] = 0$. $\to$ Closed strings cost no energy
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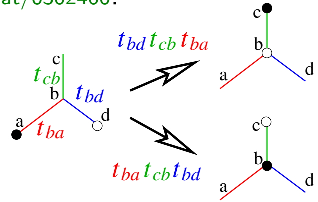
  $\to$ open-string create a pair of topo. excitations at their ends.

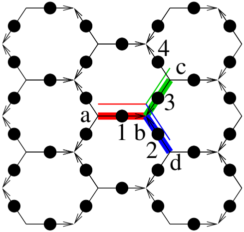
- **Fusion algebra** of string operators $\to$ fusion of topo. excitations:
  $W_e^2 = W_m^2 = W_f^2 = W_e W_m W_f = 1$ when strings are parallel

Statistics of ends of strings

- The statistics is determined by particle hopping operators

Levin-Wen cond-mat/0302460:





- An open string operator is a hopping operator of the 'ends'. The algebra of the open string operator determine the statistics.
- For type-e string: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x$
  We find $t_{bd} t_{cb} t_{ba} = t_{ba} t_{cb} t_{bd}$
  The ends of type-e string are bosons
- For type-f strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \underline{\sigma_3^x} \sigma_4^z$, $t_{bd} = \sigma_2^x \underline{\sigma_3^z}$
  We find $t_{bd} t_{cb} t_{ba} = -t_{ba} t_{cb} t_{bd}$
  The ends of type-f strings are fermions

- *Works for abelian anyons and non-abelian anyons.*

How to label all topological orders systematically?

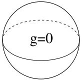
What are the probes (topological invariants) that allow us to distingush all topological orders?

Systematic theory of topo. orders from topo. invariants

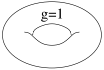
Topological order describes the order in gapped quantum liquds.
We conjectured that 2+1D topological order can be completely defined via only two topological properties:

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

- (1) Ψ_{grnd} = space of degenerate ground states, which is robust against any local perturbations.
  Topological degeneracy:
  $$D_g \equiv \dim \Psi_{\text{grnd}},$$
  depends on topology of space
  Wen PRB 40, 7387 (89), Wen-Niu PRB 41, 9377 (90)



Deg.=1



Deg.=D_1

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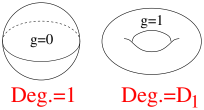
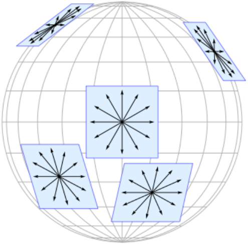


- (2) **Vector bundle on the moduli space**
  i. Consider a torus Σ_1 w/ metrics g_{ij}. **ii**. Different metrics g_{ij} form the moduli space $\mathcal{M} = \{g_{ij}\}$. **iii**. The LI states depend on spacial metrics: $\Psi_\alpha(g_{ij}) \to$ a vector bundle over $\mathcal{M}$ with fiber $\Psi_\alpha(g_{ij})$.

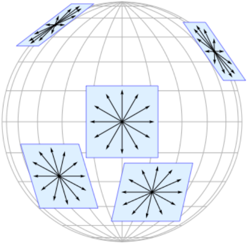
Vector bundle on the moduli space

- Local curvature detects grav. Chern-Simons term $\mathrm{e}^{\mathrm{i}\,\frac{2\pi c}{24}\int_{M^2\times S^1}\omega_3}$



Tangent bundle on a 2-sphere

Vector bundle on the moduli space



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- Loops $\pi_1(\mathcal{M}) = SL(2,\mathbb{Z})$:
  90° rotation $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$
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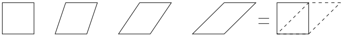

S, T generate a rep. of modular group: $S^2 = (ST)^3 = C,\ C^2 = 1$

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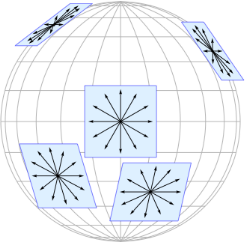
Topological invariants that define LRE and topo. orders

Vector bundle on the moduli space



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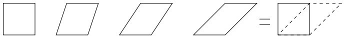

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Conjecture: **The vector bundles from all genus-g Σ_g (*ie* the data (S, T, c), ...) completely characterize the topo. orders**

Conjecture: **The vector bundle for genus-1 Σ_1 (*ie* the data (S, T, c)) completely characterize the topo. orders**

Measure topo. order: Universal wavefunction overlap

Moradi-Wen arXiv:1401.0518, He-Moradi-Wen arXiv:1401.5557

- Ground states $|\Psi_\alpha\rangle$ on torus T^2 under $\hat S$ and $\hat T$.
  The non-Abelian geometric phases S, T via overlap

$$S_{\alpha\beta} e^{-f_S L^2 + o(L^{-1})} = \langle\Psi_\alpha|\hat S|\Psi_\beta\rangle$$
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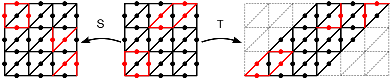

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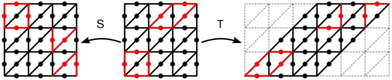

- For Z_2 topo. order:

  $\Psi_1($$) = g^{\text{string-length}}$

  $\Psi_2($$) = (-)^{W_x} g^{\text{str-len}}$

  $\Psi_3($$) = (-)^{W_y} g^{\text{str-len}}$

  $\Psi_4($$) = (-)^{W_x+W_y} g^{\text{str-len}}$

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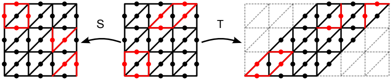

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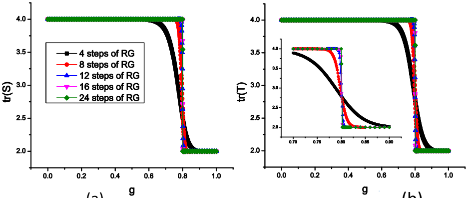
  $\Psi_4($  $) = (-)^{W_x+W_y} g^{\text{str-len}}$

- $g < 0.8$ small-loop phase
  $|\Psi_\alpha\rangle$ are the same state
- $g > 0.8$ large-loop phase
  $|\Psi_\alpha\rangle$ are four diff. states



(a) (b)

g

g=0.802

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c)

Measure topo. order: Universal wavefunction overlap

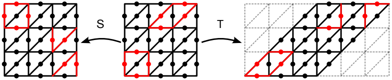
Moradi-Wen arXiv:1401.0518, He-Moradi-Wen arXiv:1401.5557

- Ground states $|\Psi_\alpha\rangle$ on torus T^2 under $\hat{S}$ and $\hat{T}$.

  The non-Abelian geometric phases S, T via overlap

  $S_{\alpha\beta} e^{-f_S L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{S} | \Psi_\beta \rangle$

  $T_{\alpha\beta} e^{-f_T L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{T} | \Psi_\beta \rangle$



- For Z_2 topo. order:

  $\Psi_1($  $) = g^{\text{string-length}}$

  $\Psi_2($  $) = (-)^{W_x} g^{\text{str-len}}$

  $\Psi_3($  $) = (-)^{W_y} g^{\text{str-len}}$

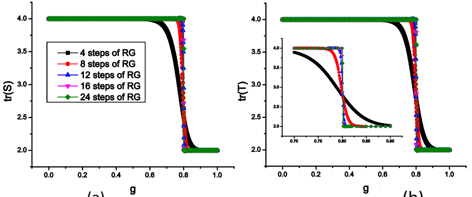
  $\Psi_4($  $) = (-)^{W_x + W_y} g^{\text{str-len}}$

- $g < 0.8$ small-loop phase

  $|\Psi_\alpha\rangle$ are the same state

- $g > 0.8$ large-loop phase

  $|\Psi_\alpha\rangle$ are four diff. states



g

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(c)

- For double-semion topo. order:

  $\Psi($  $) = (-)^{\#\text{ of loop}}$

$$S = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1 \end{pmatrix}, \quad T = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix}.$$

Using group theory, we can classify all 230 crystal structures.

How to classify all 2+1D topological orders?

Classify 2+1D topo. orders (*ie* patterns of entanglement)

via the topological invariants (S, T, c)

- A 2+1D topological order $\to$ a (S, T, c)
- An arbitary (S, T, c) $\not\to$ a 2+1D topological order
- (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ 2+1D topo. orders

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  assuming each (S, T, c) $\to$ *one topological order, otherwise*
  (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ several topo. orders

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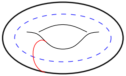
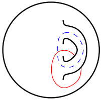
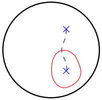




- How to find the conditions, beyond $S^2 = (ST)^3, S^4 = 1$?

  Study topological excitations above the ground states.

  ie consider vector bundle from the degenerate ground states on Σ_g with punctures (quasiparticles).

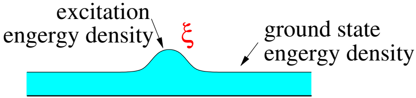
- In particular, the vector bundles from the degenerate ground states on $\Sigma_0 = S^2$ with punctures (quasiparticles)

  $\to$ unitary modular tensor category theory (UMTC)

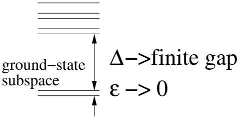
Local and topological quasiparticle excitations

In a system: $H = \sum_{\boldsymbol{x}} H_{\boldsymbol{x}}$

- an excitation $\Psi_{\text{exc}}(\xi, \xi', \cdots)$ $=$ gapped ground space of $H + \delta H_{\xi}^{\text{trap}} + \delta H_{\xi'}^{\text{trap}} + \cdots$



- **Local quasiparticle excitation** at ξ: $\Psi_{\text{exc}}(\xi, \xi', \cdots) = O_\xi \Psi_{\text{exc}}(\xi', \cdots)$ can be created by local operator O_ξ



- **Topological quasiparticle excitation** at ξ: $\Psi_{\text{exc}}(\xi, \xi', \cdots) \neq O_\xi \Psi_{\text{exc}}(\xi', \cdots)$ cannot be created by local O_ξ

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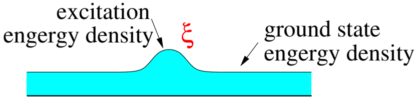

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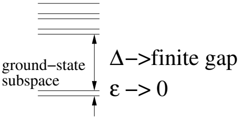


- **Topological quasiparticle excitation** at ξ: $\Psi_{\text{exc}}(\xi,\xi',\cdots)\neq O_\xi\Psi_{\text{exc}}(\xi',\cdots)$ cannot be created by local O_ξ
- **Topological types**: Consider two exictations at ξ from different traps: $\delta H^{\text{trap}}_{\xi}$ and $\delta\tilde H^{\text{trap}}_{\xi}$: $\Psi_{\text{exc}}(\xi,\xi',\cdots)$ and $\tilde\Psi_{\text{exc}}(\xi,\xi',\cdots)$

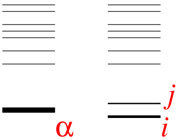
- if $\Psi_{\text{exc}}(\xi,\cdots)=O_\xi\tilde\Psi_{\text{exc}}(\xi,\cdots)\to$ they belong to the same type.

- if $\delta H^{\text{trap}}_{\xi}$ and $\delta\tilde H^{\text{trap}}_{\xi}$ can deform into each other without closing the gap, then the traped excitations at ξ belong to the same type.

- **With symmetry** $\to O_\xi,\delta H^{\text{trap}}_{\xi}$ to be symmetric local operators.

Simple/composite excitations and fusion ring

- **simple excitation** at ξ: The ground space $\Psi^{\text{simple}}_{\text{exc}}(\xi, \cdots)$ is robust against local perturbation near ξ $\to$ type i.
- **composite excitation** at ξ: The ground space $\Psi_{\text{exc}}(\xi, \cdots)$ (the degeneracy) can be splitted by local perturbation near ξ, *ie* contain accidental degeneracy $\to$ type $\alpha = i \oplus j$.



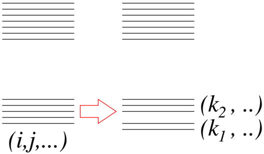
Fusion space $= \Psi_{\text{exc}}(\xi_1, \xi_2, \cdots) = \mathcal{V}_{\text{fus}}(i_1, i_2, \cdots)$

Fusion ring of (non-Abelian) topological excitations

- For simple i, j, if we view (i, j) as one particle, it may correspond to a composite particle:

$$\mathcal{V}_{\text{fus}}(i, j, l_1, l_2, \cdots) = \oplus_n \mathcal{V}_{\text{fus}}(k_n, l_1, l_2, \cdots)$$
$$i \otimes j = k_1 \oplus k_2 \oplus \cdots = \oplus_k N^{ij}_k k$$

$\to$ the *fusion ring (Grothendieck ring)*.



- **Associativity**:

$$(i \otimes j) \otimes k = i \otimes (j \otimes k) = \oplus_l N^{ijk}_l l, \quad N^{ijk}_l = \sum_m N^{ij}_m N^{mk}_l = \sum_n N^{jk}_n N^{in}_l$$

The F-symbol: $F^{ijk;m\alpha\beta}_{l;n\gamma\lambda}$

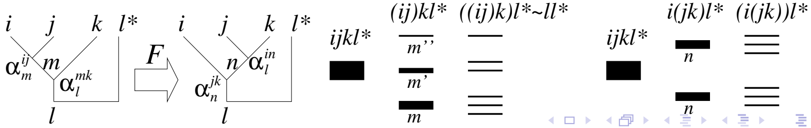
- Consider fusion space: $\mathcal{V}_{\text{fus}}(i,j,\cdots)$ – the ground space of $H+\delta H^{\text{trap}}_{\xi_i}+\delta H^{\text{trap}}_{\xi_j}+\cdots$

  The fusion $i\otimes j=\otimes N^{ij}_l l$ give rise to a choice of basis of $\mathcal{V}_{\text{fus}}(i,j,\cdots)$: $|l,\alpha^{ij}_l;\cdots\rangle$, where $\alpha^{ij}_l=1,2,\cdots,N^{ij}_l$.
- Consider fusion space: $\mathcal{V}_{\text{fus}}(i,j,k;\cdots)$, two ways of fusion give rise to two choices of basis:

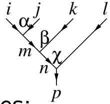
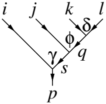
- $|i,j,k;\cdots\rangle \to |m,\alpha^{ij}_m;k;\cdots\rangle \to |m,\alpha^{ij}_m;l,\alpha^{mk}_l;\cdots\rangle$
- $|i,j,k;\cdots\rangle \to |i;n,\alpha^{jk}_n;\cdots\rangle' \to |l,\alpha^{in}_l;n,\alpha^{jk}_n;\cdots\rangle'$

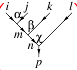
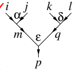
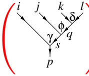
- The F-symbol is unitary matrix that relate the two basis

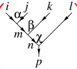
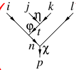
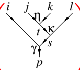
$$|l,\alpha^{in}_l;n,\alpha^{jk}_n;\cdots\rangle' = \sum_{m\alpha^{ij}_m\alpha^{mk}_l} F^{ijk;m\alpha^{ij}_m\alpha^{mk}_l}_{l;n\alpha^{jk}_n\alpha^{in}_l} |m,\alpha^{ij}_m;l,\alpha^{mk}_l;\cdots\rangle$$

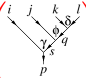


Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ and UFC

Two different ways of fusion  and  are related via two different paths of F-moves:

Φ () $= \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} \Phi$ () $= \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma} \Phi$ (),

Φ () $= \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} \Phi$ () $= \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} \Phi$ ()

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} \Phi$ ().

The two paths should lead to the same unitary trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity. Moore-Seiberg 89

$N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ → **Unitary fusion category (UFC)**

Theory of quasiparticles = fusion category theory

A simples example with symmetry G:

e.g.: Each site has 4 states: spin-0 and spin-1. Hamiltonian $H = \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_i$. Ground state $= \otimes_i |0\rangle$ is a product state.

- Type-i simple excitation defined by G-symmetric trap = i^{th} irreducible represetation of G.
- The fusion $i \otimes j = \oplus_k N^{ij}_k k$ is the fusion of representations. For $G = SO(3)$: $i = 0, 1, 2, \cdots$ is the spin-s:

  $$i \otimes j = |i - j| \oplus |i - j| + 1 \oplus \cdots \oplus i + j$$
- The $F^{ijk;m\alpha\beta}_{l;n\gamma\lambda}$ is nothing but the well known 6j-symbol, that relate two different ways of fusing three representations.

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- Braiding: all the particles are bosons with **trivial mutual statistics**
Theory of quasiparticles = braided fusion category theory
- The above braided fusion category theory is called **symmetric fusion category (SFC)** (described by $N_k^{ij}, F^{ijk;m\alpha\beta}_{l;n\gamma\lambda}$).
- SFC is a way to described symmetry group without using symmetry breaking probe: $SFC \leftrightarrow G$.

Quantum dimension and “fractional” degree of freedom

Vector space fractionalization:

- In general, $\dim[\mathcal{V}_{\text{fus}}(i, i, i, \cdots)] \neq (\text{integer})^n$.
  Quasiparticle i may carry fractional degree freedom.

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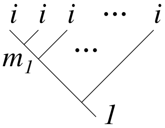
  Quasiparticle i may carry fractional degree freedom.

- $\dim[\mathcal{V}_{\text{fus}}(i,i,\cdots,i)] = \sum_{m_i} N^{ii}_{m_1} N^{m_1 i}_{m_2} \cdots N^{m_{n-2} i}_{1} = (\boldsymbol{N}^i)^{n-1}_{i1} \sim d_i^n$
  where the matrix $(\boldsymbol{N}^i)_{jk} = N^{ji}_{k}$, and d_i the largest eigenvalue of $\boldsymbol{N}^i$:

$$\dim[\mathcal{V}_{\text{fus}}(i,i)] = N^{ii}_{1}, \quad \dim[\mathcal{V}_{\text{fus}}(i,i,i)] = N^{ii}_{m_1} N^{m_1 i}_{1},$$

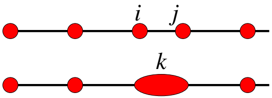
$$\dim[\mathcal{V}_{\text{fus}}(i,i,i,i)] = N^{ii}_{m_1} N^{m_1 i}_{m_2} N^{m_2 i}_{1}.$$



- d_i is called the *quantum dimension* of the quasiparticle i.
  Abelian particle $\to$ $d_i = 1$. Non-Abelian particle $\to$ $d_i \neq 1$.

Theory of topological excitations = braided fusion category

- In 1D, the set of particles → UFC
  All 1D topo. orders are described by UFC.
  All anomalous → boundary of 2D system.



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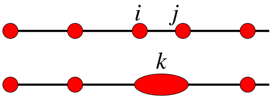
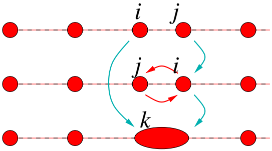


- Above 1D, particles can braid → unitary braided fusion category
- Braiding requires that
  $N_k^{ij} = N_k^{ji}$.



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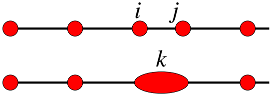

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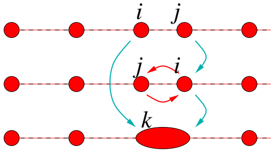
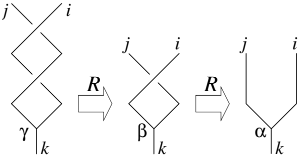


- Braiding → **mutual statistics** $\mathrm{e}^{\mathrm{i}\theta^{(k)}_{ij}}$
  and non-trivial fractional **spin** s_i

  2π rotation of (i, j) = 2π rotation of k
  2π rotation of (i, j) = 2π rotation
  of i and j and exchange i, j twice

$$\mathrm{e}^{\mathrm{i}2\pi s_i}\,\mathrm{e}^{\mathrm{i}2\pi s_j}\,\mathrm{e}^{\mathrm{i}\theta^{(k)}_{ij}} = \mathrm{e}^{\mathrm{i}2\pi s_k}$$



A unitary braided fusion category (UBFC) is a set of topological types with fusion and braiding, which is described by data (N^{ij}_k, s_i)

Relation between (S, T, c) and (N^{ij}_k, s_i, c)

Conjecture: **A bosonic topological order [*ie* a non-degenerate UBFC $\equiv$ an unitary modular tensor category (UMTC)] is fully characterized by data (S, T, c) or by data (N^{ij}_k, s_i, c) .**

- From (S, T, c) to (N^{ij}_k, s_i, c): E. Verlinde NPB **300** 360 (88)

$$N^{ij}_k = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}}, \qquad e^{i2\pi s_i} e^{-i2\pi \frac{c}{24}} = T_{ii}.$$

- From (N^{ij}_k, s_i, c) to (S, T, c):

$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N^{ij}_k e^{2\pi i (s_i + s_j - s_k)} d_k, \qquad T_{ii}. = e^{i2\pi s_i} e^{-i2\pi \frac{c}{24}}$$

Conditions on (N^{ij}_k, s_i, c) $\leftrightarrow$ Conditions on (S, T, c)
$\rightarrow$ A theory of unitary modular tensor category (UMTC)

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- From (S, T, c) to (N^{ij}_k, s_i, c): E. Verlinde NPB **300** 360 (88)

$$N^{ij}_k = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}}, \qquad e^{i2\pi s_i} e^{-i2\pi \frac{c}{24}} = T_{ii}.$$

- From (N^{ij}_k, s_i, c) to (S, T, c):

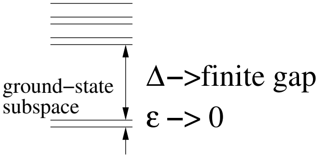
$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N^{ij}_k e^{2\pi i (s_i + s_j - s_k)} d_k, \qquad T_{ii}. = e^{i2\pi s_i} e^{-i2\pi \frac{c}{24}}$$

Conditions on (N^{ij}_k, s_i, c) $\leftrightarrow$ Conditions on (S, T, c)
$\rightarrow$ **A theory of unitary modular tensor category (UMTC)**
simplified theory of UMTC Rowell-Stong-Wang arXiv:0712.1377

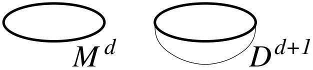
- **The standard point of view**:
UMTC's are fully characterized by $(N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\gamma\lambda}, R^{ij;\alpha}_{k;\beta})$ (but not one-to-one). Conditions on those data + the equivalent relations $\rightarrow$ a theory of UMTC. *hard to work with*

$d+1$D Topological quantum field theory

- d-dim. closed manifold M^d
  $\to$ Hilbert space $\mathcal{V}_{M^d} = \{|\alpha\rangle\}$.
  Subspace of ground states on M^d

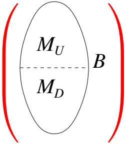


- $d+1$-dim. open manifold D^{d+1}
  $\to$ a vector $|D^{d+1}\rangle$ in $\mathcal{V}_{\partial D^{d+1}}$.



- Partition function on closed space-time N^{d+1}
  $=$ a vector $Z(N^{d+1}) \in \mathcal{V}_{\partial N^{d+1} = \emptyset} = \mathbb{C}$ (*ie* a complex number)
- Surgery formula:

$$\langle M_U|M_D\rangle = Z$$

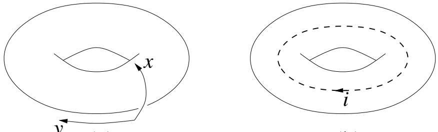


The relations between (N_k^{ij}, s_i, c) and (S, T, c)

- Number of particle types (dimensions of N_k^{ij}, s_i) $=$ ground state degeneracy on torus (dimensions of S, T).
  Type-i particle is created as the end of type-i string operator, which also describe particle-anti-particle tunneling process.

- A particular ground state $|W_1\rangle$ on torus is obtained via the time evolution on space-time of a solid torus. Other ground state $|W_i\rangle$ is obtained by inserting a loop of type-i string operator W_i.

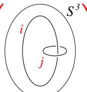
$|W_1\rangle$ $\qquad$ $|W_i\rangle = \hat{W}_i^{\text{str}} |W_1\rangle$



(a) $\qquad$ (b)

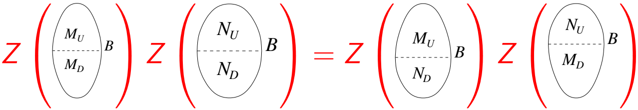
- S-matrix and link loops:

$$S_{ij} = \langle W_i | \hat{S} | W_j \rangle = Z \left(\quad \right) = S_{ji}$$



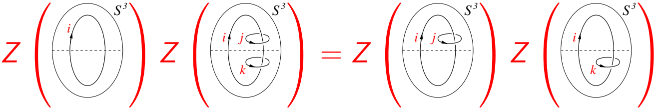
Witten CMP **121** 351 (89); Wang-Wen-Yau arXiv:1602.05951

- A surjery formula $\langle M_U|M_D\rangle\langle N_U|N_D\rangle = \langle M_U|N_D\rangle\langle N_U|M_D\rangle$



provided that the ground state degeneracy on the space-B is one.

- $\rightarrow \quad \langle W_i|\hat S|1\rangle\langle W_i|\hat S|W_j\otimes k\rangle = \langle W_i|\hat S|W_j\rangle\langle W_i|\hat S|W_k\rangle$

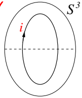


where we have used the string operator algebra

$$\hat W_j^{\text{str}}\hat W_k^{\text{str}} = \sum_i N_i^{jk}\hat W_i^{\text{str}} \;\rightarrow\; |W_jW_k\rangle = \sum_l N_l^{jk}|W_l\rangle.$$

- Verlinde formula: $\sum_l S_{i1}S_{il}N_l^{jk} = S_{ij}S_{ik}$

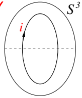
The relation between quantum dimension d_i and S

- Z () $= S_{1i} = \langle W_{i\to\bar{i}} | W_{i\to\bar{i}} \rangle > 0$
- Let vector $\boldsymbol{v}_i = (S_{i1}, S_{i2}, \cdots)$. Verlinde formula can be rewritten as

$$\boldsymbol{N}^k \boldsymbol{v}_i = \lambda_i^k \boldsymbol{v}_i, \quad \lambda_i^k = \frac{S_{ik}}{S_{i1}}$$

Since $\boldsymbol{v}_1$ has positive components, λ_1^k is the largest eigenvalue of $\boldsymbol{N}^k \to \frac{S_{1k}}{S_{11}} = d_i$. Using $\sum_i S_{1i}^2 = 1$, we find

$$S_{1i} = S_{i1} = d_i/D, \qquad D^2 = \sum_i d_i^2.$$

- $Z\Big($ 

$\Big) = S_{1i} = \langle W_{i\to\bar{i}} | W_{i\to\bar{i}} \rangle > 0$

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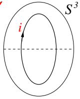
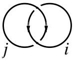
- We also find

$Z\Big($ 

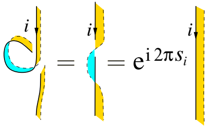
$\Big) = S_{1i} = \frac{S_{1i}}{S_{11}} Z(S^3) = d_i Z(S^3);$

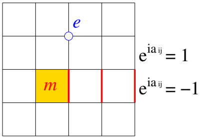


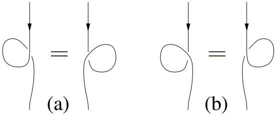
$= \frac{S_{ij}}{S_{11}} = S_{ij} D$

The relation between fractional spin s_i and T

- A particle is not an ideal point. It has internal structure. We can use the framing to represent the internal structure.
- (a) $e^{i2\pi s_i}$
  (b) $e^{-i2\pi s_i}$

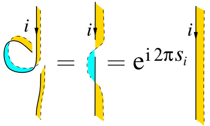


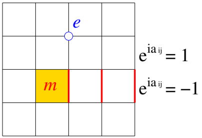




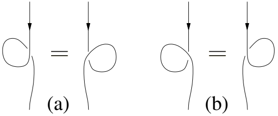
The relation between fractional spin s_i and T

- A particle is not an ideal point. It has internal structure. We can use the framing to represent the internal structure.

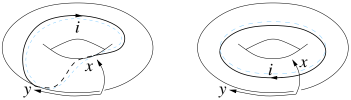




- (a) $\mathrm{e}^{\mathrm{i}2\pi s_i}$
  (b) $\mathrm{e}^{-\mathrm{i}2\pi s_i}$



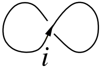
- $\hat{T}$ is a 2π twist of the particle world line:
  $$\hat{T}|W_i\rangle = \mathrm{e}^{\mathrm{i}2\pi s_i}|W_i\rangle$$



- But $\hat{T}$ also change the metrics of the solid tours $\to$ i independent phase from the gravitational CS term $\mathrm{e}^{\mathrm{i}\,\frac{2\pi c}{24}\int_{M^2\times S^1}\omega_3}$

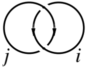
$$\hat{T}|W_i\rangle = \mathrm{e}^{\mathrm{i}2\pi s_i}\,\mathrm{e}^{-\mathrm{i}2\pi c/24}|W_i\rangle$$

From (N^{ij}_k, s_i, c) to (S, T, c) – Graphic calculus

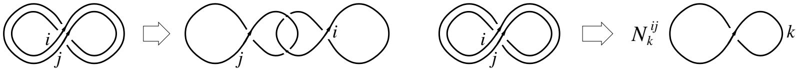
- 

$= e^{i\,2\pi s_i}$ 

$= e^{i\,2\pi s_i} d_i$

- 

$= \frac{S_{ij}}{S_{11}} = S_{ij} D$

- 

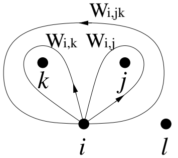
$e^{i\,2\pi(s_i+s_j)} S_{ij} D \quad = \quad \sum_k N^{ij}_k e^{i\,2\pi s_k} d_k$

The above can be rewritten as

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k e^{2\pi i (s_i+s_j-s_k)} d_k,$$

A relation between N^{ij}_k and s_i

Anderson-Moore CMP **117** 441 (88); Vafa PLB **206**, 421 (88)



$$\det(W_{i,jk}) = \det(W_{i,j})\det(W_{i,k})$$

$$\det(W_{i,j}) = \prod_r \left(\frac{e^{i2\pi s_r}}{e^{i2\pi s_i}e^{i2\pi s_j}}\right)^{N^{ij}_r N^{rk}_{\bar l}},$$

$$\det(W_{i,k}) = \prod_r \left(\frac{e^{i2\pi s_r}}{e^{i2\pi s_i}e^{i2\pi s_k}}\right)^{N^{ik}_r N^{rj}_{\bar l}},$$

$$\det(W_{i,jk}) = \prod_r \left(\frac{e^{i2\pi s_{\bar l}}}{e^{i2\pi s_i}e^{i2\pi s_r}}\right)^{N^{jk}_r N^{ri}_{\bar l}}.$$

$W_{i,j}$, $W_{i,k}$, $W_{i,jk}$ are diagonal with the dimension of the fusion space $\mathcal{V}_{\text{fus}}(i,j,k,l)$: $\sum_r N^{ij}_r N^{rk}_{\bar l} = \sum_r N^{ik}_r N^{rj}_{\bar l} = \sum_r N^{jk}_r N^{ri}_{\bar l}$

$$\to \quad \sum_r V^r_{ijkl} s_r = 0 \text{ mod } 1$$

$$V^r_{ijkl} = N^{ij}_r N^{kl}_{\bar r} + N^{il}_r N^{jk}_{\bar r} + N^{ik}_r N^{jl}_{\bar r} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{\bar m}$$

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

Rowell-Stong-Wang arXiv:0712.1377, Wen arXiv:1506.05768

- **Fusion ring**: N^{ij}_k are non-negative integers that satisfy

$$N^{ij}_k = N^{ji}_k, \quad N^{1i}_j = \delta_{ij}, \quad \sum_{k=1}^{N} N^{ik}_1 N^{kj}_1 = \delta_{ij},$$

$$\sum_{m=1}^{N} N^{ij}_m N^{mk}_l = \sum_{m=1}^{N} N^{im}_l N^{jk}_m \text{ or } \boldsymbol{N}^i \boldsymbol{N}^k = \boldsymbol{N}^k \boldsymbol{N}^i$$

where $i, j, \cdots = 1, 2, \cdots, N$, and the matrix $\boldsymbol{N}^j$ is given by $(\boldsymbol{N}^j)_{ik} = N^{ij}_k$. N^{ij}_1 defines a charge conjugation $i \to \bar{i}$: $N^{ij}_1 = \delta_{\bar{i}j}$. We refer N as the rank.
There are only finite numbers of solutions for each fixed N, D.

- N^{ij}_k and s_i satisfy $\sum_r V^r_{ijkl} s_r = 0 \mod 1$

$$V^r_{ijkl} = N^{ij}_r N^{kl}_{\bar{r}} + N^{il}_r N^{jk}_{\bar{r}} + N^{ik}_r N^{jl}_{\bar{r}} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{\bar{m}}$$

This determines s_i to be a rational number. There are only finite sets of solutions.

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

From $(N^{ij}_k, s_i, c) \to (S, T)$

- Let d_i be the largest eigenvalue of the matrix $\boldsymbol{N}^i$. Let

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k \mathrm{e}^{2\pi \mathrm{i}(s_i + s_j - s_k)} d_k, \quad D^2 = \sum_i d_i^2.$$

Then, S satisfies

$$S_{11} > 0, \quad \sum_k S_{kl} N^{ij}_k = \frac{S_{li} S_{lj}}{S_{1l}}, \quad S = S^\dagger C, \quad C_{ij} \equiv N^{ij}_1.$$

- Let $T_{ij}. = \mathrm{e}^{\mathrm{i}2\pi s_i} \mathrm{e}^{-\mathrm{i}2\pi \frac{c}{24}} \delta_{ij}$ then ($SL(2,\mathbb{Z})$ modular representation)

$$S^2 = (ST)^3 = C.$$

- Let $\nu_i = \frac{1}{D^2} \sum_{jk} N^{jk}_i d_j d_k \mathrm{e}^{4\pi \mathrm{i}(s_j - s_k)}$. Then $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$. Rowell-Stong-Wang arXiv:0712.1377

2+1D bosonic topo. orders (up to E_8-states) via (N^{ij}_k, s_i, c)

$\zeta_n^m = \frac{\sin(\pi(m+1)/(n+2))}{\sin(\pi/(n+2))}$ Rowell-Stong-Wang arXiv:0712.1377; Wen arXiv:1506.05768

N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.	N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.
1_1^B	1	0					
2_1^B	$1, 1$	$0, \frac{1}{4}$	$\prod(z_i - z_j)^2$	2_{-1}^B	$1, 1$	$0, -\frac{1}{4}$	$\prod(z_i^* - z_j^*)^2$
$2_{14/5}^B$	$1, \zeta_3^1$	$0, \frac{2}{5}$	Fibonacci TO	$2_{-14/5}^B$	$1, \zeta_3^1$	$0, -\frac{2}{5}$	
3_2^B	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$	(221) double-layer	3_{-2}^B	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$	
$3_{8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$		$3_{-8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$	
$3_{1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$	Ising TO	$3_{-1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$	
$3_{3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	$\mathcal{S}(220)$, Ψ_{Pfaffian}	$3_{-3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$	
$3_{5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	$\Psi_{\nu=2}^2$ $SU(2)_2^f$	$3_{-5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$	
$3_{7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$		$3_{-7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$	
$4_0^{B,a}$	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$	$(1, e, m, f)$ Z_2-gauge	4_4^B	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
4_1^B	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$	$\prod(z_i - z_j)^4$	4_{-1}^B	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$	
4_2^B	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	(220) double-layer	4_{-2}^B	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$	
4_3^B	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$		4_{-3}^B	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$	
$4_0^{B,b}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$	double semion	$4_{9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$	
$4_{-9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$		$4_{19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$	
$4_{-19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$	$\Psi_{\nu=3}^2$ $SU(2)_3^f$	$4_0^{B,c}$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, -\frac{2}{5}, 0$	Fibonacci^2
$4_{12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$		$4_{-12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$	
$4_{10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$		$4_{-10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$	
5_0^B	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	(223) DL	5_4^B	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	
$5_2^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$		$5_2^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	
$5_{-2}^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$		$5_{-2}^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	
$5_{16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$		$5_{-16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	
$5_{18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$		$5_{-18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	

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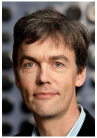

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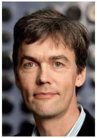

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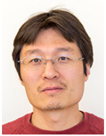

  Levin arXiv:1301.7355, Kong-Wen arXiv:1405.5858
- **Remote detectable = Realizable (anomaly-free)**:
  Every non-trivial topo. excitation i can be remotely detected by at least one other topo. excitation j via the non-zero mutual braiding $\theta^{(k)}_{ij} \neq 0 \to S_{ij} = \frac{1}{D}\sum_k N^{ij}_k \mathrm{e}^{-\mathrm{i}\theta^{(k)}_{ij}} d_k$ is unitary (one of conditions) $\to$ the topological order is realizable in the same dimension.
- The centralizer of BFC $\mathcal{C}$ = the set of particles with trivial mutual statistics respecting to all others: $\mathcal{C}^{\text{cen}}_{\mathcal{C}} \equiv \{i \mid \theta^{(k)}_{ij} = 0,\ \forall j, k\}$.
  Remote detectable $\leftrightarrow \mathcal{C}^{\text{cen}}_{\mathcal{C}} = \{1\} \leftrightarrow$ Realizable (anomaly-free)

Bosonic/fermionic topo. orders with/without symmetry

- "Topological" excitations with symmetry: Two particles are equivalent iff they are connected by **symmetric** local operators.
  Equivalent classes = topological types with symmetry

- **Example**: for $G = SO(3)$:

- Trivial "topogical" types: spin-0. (centralizer=SFC)
- Non-trivial "topogical" types: spin-1, spin-2, $\cdots \sim$ irreducible reps.
  (Cannot be created by local symmetric operators, but can be created by local asymmetric operators.)
- Really non-trivial "topogical" types. (Other types)
  (Cannot created by local symmetric operators, nor by local asymmetric operators.)

- *How to classify topological orders with symmetry?*
  How to classify fermionic topo. orders with/without symmetry?
  Consider braided fusion category whose centralizer is non-trivial.
  centralizer = symmetric fusion catgeory (SFC) = symmetry

SFC = Exc. in bosonic/fermionic product states with symmetry = a categorical description of symmetry

Symmetric fusion catgeories (SFC):

- For *bosonic product states*, 1) Particle are bosonic with **trivial mutual statistics (not remotely detectable)**;
  2) Particles are labeled by irrep. R_i.
  Topological types = irreducible representation $R_i \in Rep(G)$
  The fusion and the trivial braiding of R_i define a spectial UBFC, called **symmetric fusion category (SFC)** and denoted as Rep(G)

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- For *fermionic product states*, 1) Some particles are bosonic, and others are fermionic, and all have trivial mutual statistics
  2) Particles are labeled by irrep. R_i. The full symm. group G^f contain fermion-number-parity $\hat{f} = (-)^{\hat{N}_f} \in G^f$.

- Topological types = irreducible representation R_i (ex. spin-s)
  The particle R_i has a Fermi statistics if $\hat{f} \neq 1$ in R_i (ex. spin-1)
  The particle R_i has a Bose statistics if $\hat{f} = 1$ in R_i (ex. spin-$\frac{1}{2}$)
- The fusion and bosonic/fermionic braiding of $R_i \to$ SFC = sRep(G^f)

Classifcation of bosonic/fermionic topo. orders with symm.

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UMTC$_{/\mathcal{E}}$ and topological phases with symmetry/fermion

- To describe topological phases with symmetry/fermion, we need
- a unitary BFC $\mathcal{C}$
- that contains a SFC $\mathcal{E}$,
- such that the particles (objects) in $\mathcal{E}$ are transparent
- and there is no other transparent particles (objects).

$\to$ **Unitary non-degenerate braided fusion category over a SFC** (UMTC$_{/\mathcal{E}}$).

Using the notion of centralizer: $\mathcal{C}_{\mathcal{C}}^{\text{cen}} = \mathcal{E}$, $\mathcal{E}_{\mathcal{C}}^{\text{cen}} = \mathcal{C}$.

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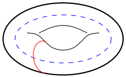
Can $\text{UMTC}_{/\mathcal{E}}$'s classify topological phases with symmetry/fermion?
Answer: **No**.
We also require the symmetry to be gaugable: the $\text{UMTC}_{/\mathcal{E}}$ must have modular extension.

Why do we require **modular extensions**?

- The symmetry G in a physical system is always twistable (on-site) *ie* we can always put the physical system on any 2D manifold with any flat G-connection, still with consistent braiding and fusion.



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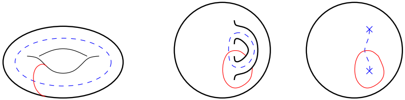

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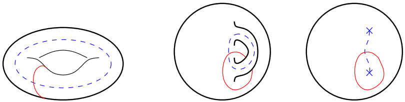


- *We can add extra particles that braid non-trivially with the particles in SFC $\mathcal{E}$, and make the UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ into a unitary non-degenerate braided fusion category (ie an UMTC) $\mathcal{M}$.*
  $\mathcal{M}$ is called the modular extension of $\mathcal{C}$:
  $$\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}, \qquad D^2_{\mathcal{E}} D^2_{\mathcal{C}} = D^2_{\mathcal{M}}$$
  In $\mathcal{M}$, the set of particles that have trivial double-braiding with the particles in $\mathcal{E}$ is given by $\mathcal{C}$. Using centralizer: $\mathcal{C}^{\text{cen}}_{\mathcal{M}} = \mathcal{E}$, $\mathcal{E}^{\text{cen}}_{\mathcal{M}} = \mathcal{C}$.

- **Only UMTC$_{/\mathcal{E}}$'s $\mathcal{C}$ that have modular extensions are realizable by physical 2D bulk systems (maybe with symmetry and/or fermion).**

2+1D fermionic topo. orders (up to $p+\mathrm{i}p$) via (N^{ij}_k, s_i, c)

Classified by **UMTC$_{/\mathcal{E}}$'s with $\mathcal{E}=\{1,f\}$.**

Lan-Kong-Wen arXiv:1507.04673

$N^F_c\binom{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$2^F_0\binom{\zeta_2^1}{0}$	2	$1, 1$	$0, \frac{1}{2}$	$\mathcal{F}_0 = \mathrm{sRep}(Z_2^f)$ fermion product state
$4^F_0\binom{0}{0}$	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2^B_1\binom{0}{0}$ $K=\begin{pmatrix}2&2\\2&1\end{pmatrix}$
$4^F_{1/5}\binom{\zeta_2^1\zeta_3^1}{3/20}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2^B_{-14/5}\binom{\zeta_3^1}{3/20}$
$4^F_{-1/5}\binom{\zeta_2^1\zeta_3^1}{-3/20}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2^B_{14/5}\binom{\zeta_3^1}{-3/20}$
$4^F_{1/4}\binom{\zeta_6^3}{1/2}$	13.656	$1, 1, \zeta_6^2, \zeta_6^2 = 1+\sqrt{2}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_{(A_1,6)}$
$6^F_0\binom{\zeta_2^1}{1/4}$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3^B_{-2}\binom{1}{1/4}$ $K=(3)$, $\Psi_{1/3}(z_i)$
$6^F_0\binom{\zeta_2^1}{-1/4}$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3^B_{2}\binom{1}{-1/4}$ $K=(-3)$, $\Psi^*_{1/3}(z_i)$
$6^F_0\binom{\zeta_6^3}{1/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1 = \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{1/2}\binom{\zeta_6^1}{1/16}$, $\mathcal{F}_{U(1)_2/\mathbb{Z}_2}$
$6^F_0\binom{\zeta_6^3}{-1/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$
$6^F_0\binom{1.0823}{3/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{3/2}\binom{0.7653}{3/16}$
$6^F_0\binom{1.0823}{-3/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{-3/2}\binom{0.7653}{-3/16}$
$6^F_{1/7}\binom{\zeta_2^1\zeta_5^2}{-5/14}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3^B_{8/7}\binom{\zeta_5^2}{-5/14}$
$6^F_{-1/7}\binom{\zeta_2^1\zeta_5^2}{5/14}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3^B_{-8/7}\binom{\zeta_5^2}{5/14}$
$6^F_0\binom{2\zeta_{10}^1}{-1/12}$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1,-10)}$
$6^F_0\binom{2\zeta_{10}^1}{1/12}$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1,10)}$

2+1D bosonic topo. orders with Z_2 symmetry

Classified by **UMTC$_{/\mathcal{E}}$'s with centralizer $\mathcal{E} =$ Rep(Z_2)**.

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$2_0^{\zeta_2^1}$	2	$1, 1$	$0, 0$	$\mathcal{E} = \text{Rep}(Z_2)$
$3_2^{\zeta_2^1}$	6	$1, 1, 2$	$0, 0, \frac{1}{3}$	$K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
$3_{-2}^{\zeta_2^1}$	6	$1, 1, 2$	$0, 0, \frac{2}{3}$	$K = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$
$4_1^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{1}{4}$	$\Psi_{\nu=1/2}^{\text{neutral}} \boxtimes \text{Rep}(Z_2)$
$4_1^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{1}{4}$	$\Psi_{\nu=1/2}^{\text{charged}} \boxtimes \text{Rep}(Z_2)$
$4_{-1}^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{3}{4}, \frac{3}{4}$	$\Psi_{\nu=-1/2}^{\text{neutral}} \boxtimes \text{Rep}(Z_2)$
$4_{-1}^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{3}{4}, \frac{3}{4}$	$\Psi_{\nu=-1/2}^{\text{charged}} \boxtimes \text{Rep}(Z_2)$
$4_{14/5}^{\zeta_2^1}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{2}{5}, \frac{2}{5}$	$2_{14/5}^B \boxtimes \text{Rep}(Z_2)$
$4_{-14/5}^{\zeta_2^1}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{3}{5}, \frac{3}{5}$	$2_{-14/5}^B \boxtimes \text{Rep}(Z_2)$
$4_0^{\zeta_2^1}$	10	$1, 1, 2, 2$	$0, 0, \frac{1}{5}, \frac{4}{5}$	$K = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$
$4_4^{\zeta_2^1}$	10	$1, 1, 2, 2$	$0, 0, \frac{2}{5}, \frac{3}{5}$	$K = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

Lan-Kong-Wen arXiv:1602.05946

2+1D bosonic topo. orders with Z_2 symmetry (conitnue)

$N_c^{\vert\Theta\vert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$2_0^{\zeta_2^1}$	2	$1, 1$	$0, 0$	$\mathcal{E} = \text{Rep}(Z_2)$
$5_0^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0$	SB:4_0^B F:$Z_2 \times Z_2$
$5_1^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}$	SB:4_1^B F:$Z_2 \times Z_2$
$5_2^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}$	SB:4_2^B F:$Z_2 \times Z_2$
$5_3^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}$	SB:4_3^B F:$Z_2 \times Z_2$
$5_4^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	SB:4_4^B $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$
$5_{-3}^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}$	SB:4_{-3}^B F:$Z_2 \times Z_2$
$5_{-2}^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	SB:4_{-2}^B F:$Z_2 \times Z_2$
$5_{-1}^{\zeta_2^1}$	8	$1, 1, 1, 1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{7}{8}$	SB:4_{-1}^B F:$Z_2 \times Z_2$
$5_2^{\zeta_2^1}$	14	$1, 1, 2, 2, 2$	$0, 0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}$	SB:7_2^B
$5_{-2}^{\zeta_2^1}$	14	$1, 1, 2, 2, 2$	$0, 0, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}$	SB:7_{-2}^B
$5_{12/5}^{\zeta_2^1}$	26.180	$1, 1, \zeta_8^2, \zeta_8^2, \zeta_8^4$	$0, 0, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}$	SB:$4_{12/5}^B$
$5_{-12/5}^{\zeta_2^1}$	26.180	$1, 1, \zeta_8^2, \zeta_8^2, \zeta_8^4$	$0, 0, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}$	SB:$4_{-12/5}^B$

SB: 4_0^B $\to$ topo. order after symmetry breaking is Z_2-gauge theory.

2+1D bosonic topo. orders with Z_2 symmetry (conitnue)

The Z_2 symmetry is anomalous, since the following BF categories have no modular extensions:

$N_c^{\vert\Theta\vert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$2_0^{\zeta_2^1}$	2	1, 1	0, 0	$\mathcal{E} = \text{Rep}(Z_2)$
$5_0^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0$	SB:4_0^B F:Z_4 anom.
$5_1^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}$	SB:4_1^B F:Z_4 anom.
$5_2^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}$	SB:4_2^B F:Z_4 anom.
$5_3^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}$	SB:4_3^B F:Z_4 anom.
$5_4^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	SB:4_4^B F:Z_4 anom.
$5_{-3}^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}$	SB:4_{-3}^B F:Z_4 anom.
$5_{-2}^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	SB:4_{-2}^B F:Z_4 anom.
$5_{-1}^{\zeta_2^1}$	8	1, 1, 1, 1, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{7}{8}$	SB:4_{-1}^B F:Z_4 anom.

Z_2-gauge theory with Z_2 symmetry

The first rows of last two tables are identical.
They have identical d_i but different N^{ij}_k

They are Z_2-gauge theory $1, e, m, f$, with Z_2 symmetry: $e \leftrightarrow m$

Fusion rules: $Z_2 \times Z_2$

	1_0	1_1	f_0	f_1	$e \oplus m$
s_i	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
d_i	1	1	1	1	2
$5_0^{\zeta_2^1}$	**1**	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**	**5**
2	**2**	**1**	**4**	**3**	**5**
3	**3**	**4**	**1**	**2**	**5**
4	**4**	**3**	**2**	**1**	**5**
5	**5**	**5**	**5**	**5**	$\mathbf{1 \oplus 2 \oplus 3 \oplus 4}$

Anomaly-free

Z_4

	1_0	1_1	$f_{1/2}$	$f_{3/2}$	$e \oplus m$
s_i	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
d_i	1	1	1	1	2
$5_0^{\zeta_2^1}$	**1**	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**	**5**
2	**2**	**1**	**4**	**3**	**5**
3	**3**	**4**	**2**	**1**	**5**
4	**4**	**3**	**1**	**2**	**5**
5	**5**	**5**	**5**	**5**	$\mathbf{1 \oplus 2 \oplus 3 \oplus 4}$

Anomalous

- F: $Z_2 \times Z_2$ means that the four $d_i = 1$ particles have a fusion described by $Z_2 \times Z_2$.
- F: Z_4 means that the four $d_i = 1$ particles have a fusion described by Z_4:

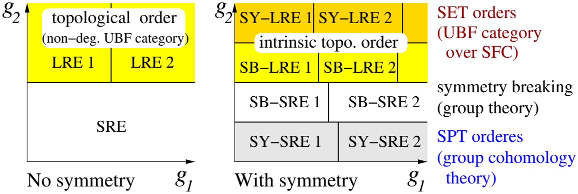
Fermionic topo. orders with mod-4 fermion number conservation: symmetry $G^f = Z_4^f$

Classified by **UMTC$_{/\mathcal{E}}$'s with centralizer $\mathcal{E}$ = sRep(Z_4^f)**:

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
4_0^0	4	$1, 1, 1, 1$	$0, 0, \frac{1}{2}, \frac{1}{2}$	$\mathcal{E}$ = sRep(Z_4^f)
6_0^0	12	$1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{2}{3}$	$K = -\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
6_0^0	12	$1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{5}{6}$	$K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
8_0^0	8	$1, 1, 1, 1, 1, 1, 1, 1$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$2_{-1}^B \boxtimes$ sRep(Z_4^f)
8_0^0	8	$1, 1, 1, 1, 1, 1, 1, 1$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$2_1^B \boxtimes$ sRep(Z_4^f)
$8_{-14/5}^0$	14.472	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{3}{5}, \frac{3}{5}$	$2_{-14/5}^B \boxtimes$ sRep(Z_4^f)
$8_{14/5}^0$	14.472	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{9}{10}, \frac{9}{10}$	$2_{14/5}^B \boxtimes$ sRep(Z_4^f)
8_0^0	20	$1, 1, 1, 1, 2, 2, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{10}, \frac{2}{5}, \frac{3}{5}, \frac{9}{10}$	SB:$10_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{0})$
8_0^0	20	$1, 1, 1, 1, 2, 2, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{3}{10}, \frac{7}{10}, \frac{4}{5}$	SB:$10_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{1/2})$
$10_0^0(\genfrac{}{}{0pt}{}{4}{0})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{\sqrt{8}}{0})$
$10_0^0(\genfrac{}{}{0pt}{}{4}{0})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{\sqrt{8}}{0})$
$10_0^0(\genfrac{}{}{0pt}{}{\sqrt{8}}{1/8})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{5}{8}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{2}{1/8})$
$10_0^0(\genfrac{}{}{0pt}{}{\sqrt{8}}{1/8})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{5}{8}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{2}{1/8})$
$10_0^0(\genfrac{}{}{0pt}{}{0}{0})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{0}{0})$
$10_0^0(\genfrac{}{}{0pt}{}{0}{0})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{0}{0})$
$10_0^0(\genfrac{}{}{0pt}{}{\sqrt{8}}{-1/8})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{7}{8}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{2}{-1/8})$
$10_0^0(\genfrac{}{}{0pt}{}{\sqrt{8}}{-1/8})$	16	$1, 1, 1, 1, 1, 1, 1, 1, 2, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{7}{8}$	SB:$8_0^F(\genfrac{}{}{0pt}{}{2}{-1/8})$

Distinct topo. phases with identical set of bulk excitations

In the presence of symmetry/fermion, there are distinct topological phases, such as SPT phases with the same symmetry, that have identical bulk excitations. But they have different edge structures.



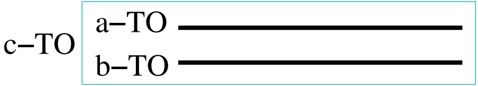
- A UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ only describes the bulk excitations. But it can have *several different* modular extensions. → Distinct topological phases with identical set of bulk excitations, but different edge structures.

The main conjecture: Lan-Kong-Wen arXiv:1602.05946

- **The triple $(\mathbf{Rep}(G) \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M})$ classifies 2+1D bosonic topological phase with symmetry G.**
- **The triple $(\mathbf{sRep}(G^f) \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M})$ classifies 2+1D fermionic topological phase with symmetry G^f.**

From physical picture to mathematical theorem

- Stacking two topological phases a, b with symmetry G give rise to a third topological phase $c = a \boxtimes_{\text{stack}} b$ with symmetry G



- **For a fixed SFC $\mathcal{E}$, there exists a "tensor product" $\boxtimes_{\mathcal{E}}$, under which the triple $(\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M})$ form a commutative monoid**

$$(\mathcal{E} \hookrightarrow \mathcal{C}_1 \hookrightarrow \mathcal{M}_1) \boxtimes_{\mathcal{E}} (\mathcal{E} \hookrightarrow \mathcal{C}_2 \hookrightarrow \mathcal{M}_2) \equiv (\mathcal{E} \hookrightarrow \mathcal{C}_3 \hookrightarrow \mathcal{M}_3)$$

- $\boxtimes_{\mathcal{E}}$ is different from the Deligne tensor product $\boxtimes$:

$$(\mathcal{E} \hookrightarrow \mathcal{C}_1 \hookrightarrow \mathcal{M}_1) \boxtimes (\mathcal{E} \hookrightarrow \mathcal{C}_2 \hookrightarrow \mathcal{M}_2)$$
$$\equiv (\mathcal{E} \boxtimes \mathcal{E} \hookrightarrow \mathcal{C}_1 \boxtimes \mathcal{C}_2 \hookrightarrow \mathcal{M}_1 \boxtimes \mathcal{M}_2)$$

which has a symmetry $G \times G$. Need to be reduced to G (or $\mathcal{E}$).

- Lan-Kong-Wen arXiv:1602.05936 **has constructed $\boxtimes_{\mathcal{E}}$ using condensable algebra $L_{\mathcal{E}} = \oplus_{a \in \mathcal{E}} a \boxtimes \bar{a}$:**

$$\mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})^0_{L_{\mathcal{E}}}, \quad \mathcal{C}_3 = (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^0_{L_{\mathcal{E}}}, \quad \mathcal{M}_3 = (\mathcal{M}_1 \boxtimes \mathcal{M}_2)^0_{L_{\mathcal{E}}}$$

eg, $\mathcal{M}_3$ is the category of local $L_{\mathcal{E}}$-modules in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$

From physical picture to mathematical theorem

- $\{(\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M})\}$ describes topological phases with symmetry $\mathcal{E}$. Its subset $\{(\mathcal{E} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{M})\}$ describes symmetry protected trivial (SPT) phases, which forms an abelian group under the stacking.

- **For a fixed SFC $\mathcal{E}$, the modular extensions of $\mathcal{E}$ form an abelian group.** $\boxtimes_{\mathcal{E}}$ is the group product, the Drinfeld center $Z(\mathcal{E})$ is the identity, and the "complex conjugate" is the inverse.

- A special case: $\{(\text{Rep}(G) \hookrightarrow \mathcal{M})\} = \mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$

From physical picture to mathematical theorem

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- A special case: $\{(\text{Rep}(G) \hookrightarrow \mathcal{M})\} = \mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$
- **The modular extensions of Rep(G), (Rep$(G) \hookrightarrow \mathcal{M}$), classifies 2+1D bosonic SPT phases with symmetry G.**
- **The $c = 0$ modular extensions of sRep(G^f), (sRep$(G^f) \hookrightarrow \mathcal{M}$), classifies 2+1D fermionic SPT phases with symmetry G^f.**

From physical picture to mathematical theorem

- $\{(\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M})\}$ describes topological phases with symmetry $\mathcal{E}$. Its subset $\{(\mathcal{E} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{M})\}$ describes symmetry protected trivial (SPT) phases, which forms an abelian group under the stacking.
  - **For a fixed SFC $\mathcal{E}$, the modular extensions of $\mathcal{E}$ form an abelian group.** $\boxtimes_{\mathcal{E}}$ is the group product, the Drinfeld center $Z(\mathcal{E})$ is the identity, and the "complex conjugate" is the inverse.
  - A special case: $\{(\text{Rep}(G) \hookrightarrow \mathcal{M})\} = \mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$
  - **The modular extensions of $\textbf{Rep}(G)$, $(\textbf{Rep}(G) \hookrightarrow \mathcal{M})$, classifies 2+1D bosonic SPT phases with symmetry G.**
  - **The $c = 0$ modular extensions of $\textbf{sRep}(G^f)$, $(\textbf{sRep}(G^f) \hookrightarrow \mathcal{M})$, classifies 2+1D fermionic SPT phases with symmetry G^f.**

- There can be several topological phases that have identical bulk excitations. They are related by stacking SPT phases.

From physical picture to mathematical theorem

- $\{(\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M})\}$ describes topological phases with symmetry $\mathcal{E}$. Its subset $\{(\mathcal{E} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{M})\}$ describes symmetry protected trivial (SPT) phases, which forms an abelian group under the stacking.
  - **For a fixed SFC $\mathcal{E}$, the modular extensions of $\mathcal{E}$ form an abelian group.** $\boxtimes_{\mathcal{E}}$ is the group product, the Drinfeld center $Z(\mathcal{E})$ is the identity, and the "complex conjugate" is the inverse.
  - A special case: $\{(\text{Rep}(G) \hookrightarrow \mathcal{M})\} = \mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$
  - **The modular extensions of $\textbf{Rep}(G)$, $(\textbf{Rep}(G) \hookrightarrow \mathcal{M})$, classifies 2+1D bosonic SPT phases with symmetry G.**
  - **The $c = 0$ modular extensions of $\textbf{sRep}(G^f)$, $(\textbf{sRep}(G^f) \hookrightarrow \mathcal{M})$, classifies 2+1D fermionic SPT phases with symmetry G^f.**

- There can be several topological phases that have identical bulk excitations. They are related by stacking SPT phases.
  - **All the modular extensions of a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ are generated by $\boxtimes_{\mathcal{E}}$ing with the modular extensions of $\mathcal{E}$:**

$$(\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}) = (\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}_0) \boxtimes_{\mathcal{E}} (\mathcal{E} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{M}')$$

Bosonic 2+1D SPT phases from modular extensions

- Z_2-SPT phases:

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$2_0^{\zeta_2^1}$	2	$1, 1$	$0, 0$	Rep(Z_2)
4_0^B	4	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$	Z_2 gauge
4_0^B	4	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{3}{4}$	double semion

- S_3-SPT phases:

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$3_0^{\sqrt{6}}$	6	$1, 1, 2$	$0, 0, 0$	Rep(S_3)
8_0^B	36	$1, 1, 2, 2, 2, 2, 3, 3$	$0, 0, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{2}$	S_3 gauge
8_0^B	36	$1, 1, 2, 2, 2, 2, 3, 3$	$0, 0, 0, 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	
8_0^B	36	$1, 1, 2, 2, 2, 2, 3, 3$	$0, 0, 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, 0, \frac{1}{2}$	$(B_4, 2)$
8_0^B	36	$1, 1, 2, 2, 2, 2, 3, 3$	$0, 0, 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{1}{4}, \frac{3}{4}$	
8_0^B	36	$1, 1, 2, 2, 2, 2, 3, 3$	$0, 0, 0, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}, 0, \frac{1}{2}$	$(B_4, -2)$
8_0^B	36	$1, 1, 2, 2, 2, 2, 3, 3$	$0, 0, 0, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{1}{4}, \frac{3}{4}$	

Fermionic 2+1D SPT phases from modular extensions

- Z_2^f-SPT phases (16 modular extensions, 1 with $c = 0$):

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
2_0^0	2	$1, 1$	$0, \frac{1}{2}$	sRep(Z_2^f)
4_0^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, 0, 0$	Z_2 gauge
4_1^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}$	F:Z_4
4_2^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$	F:$Z_2 \times Z_2$
4_3^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}$	F:Z_4
4_4^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	F:$Z_2 \times Z_2$
4_{-3}^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{5}{8}, \frac{5}{8}$	F:Z_4
4_{-2}^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}$	F:$Z_2 \times Z_2$
4_{-1}^B	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{7}{8}, \frac{7}{8}$	F:Z_4
$3_{1/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$	$p + \mathrm{i}p$ SC
$3_{3/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	
$3_{5/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	
$3_{7/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$	
$3_{-7/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{9}{16}$	
$3_{-5/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{11}{16}$	
$3_{-3/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{13}{16}$	
$3_{-1/2}^B$	4	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{15}{16}$	

Fermionic 2+1D SPT phases from modular extensions

- Z_4^f-SPT phases (only 8 modular extensions, 1 with $c = 0$):

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
4_0^0	4	$1, 1, 1, 1$	$0, 0, \frac{1}{2}, \frac{1}{2}$	$\text{sRep}(Z_4^f)$
16_0^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}$	
16_1^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{32}, \frac{1}{32}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{9}{32}, \frac{9}{32}, \frac{17}{32}, \frac{17}{32}, \frac{25}{32}, \frac{25}{32}$	
16_2^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{16}, \frac{1}{16}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{5}{16}, \frac{5}{16}, \frac{9}{16}, \frac{9}{16}, \frac{13}{16}, \frac{13}{16}$	$8_1^B \boxtimes 2_1^B$
16_3^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{32}, \frac{3}{32}, \frac{11}{32}, \frac{11}{32}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{19}{32}, \frac{19}{32}, \frac{27}{32}, \frac{27}{32}$	
16_4^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}$	$4_3^B \boxtimes 4_1^B$
16_{-3}^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{5}{32}, \frac{5}{32}, \frac{13}{32}, \frac{13}{32}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{21}{32}, \frac{21}{32}, \frac{29}{32}, \frac{29}{32}$	
16_{-2}^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{16}, \frac{3}{16}, \frac{7}{16}, \frac{7}{16}, \frac{11}{16}, \frac{11}{16}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{15}{16}, \frac{15}{16}$	$8_{-1}^B \boxtimes 2_{-1}^B$
16_{-1}^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{7}{32}, \frac{7}{32}, \frac{15}{32}, \frac{15}{32}, \frac{23}{32}, \frac{23}{32}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{31}{32}, \frac{31}{32}$	

- Z_8^f-SPT phases: Z_2 class

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$
8_0^0	8	1×8	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}$
64_0^B	64	1×64	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}$
64_0^B	64	1×64	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{7}{16}, \frac{7}{16}, \frac{7}{16}, \frac{7}{16}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{9}{16}, \frac{9}{16}, \frac{9}{16}, \frac{9}{16}, \frac{11}{16}, \frac{11}{16}, \frac{11}{16}, \frac{11}{16}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{13}{16}, \frac{13}{16}, \frac{13}{16}, \frac{13}{16}, \frac{15}{16}, \frac{15}{16}, \frac{15}{16}, \frac{15}{16}$

Fermionic 2+1D SPT phases from modular extensions

- $Z_2^f \times Z_2$-SPT phases (128 modular extensions, 8 with $c = 0$):

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
4_0^0	4	$1, 1, 1, 1$	$0, 0, \frac{1}{2}, \frac{1}{2}$	$\mathrm{sRep}(Z_2 \times Z_2^f)$
9_0^B	16	$1 \times 4, \zeta_2^1, \zeta_2^1, \zeta_2^1, \zeta_2^1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}, \frac{15}{16}, 0$	$3_{-1/2}^B \boxtimes 3_{1/2}^B$
9_0^B	16	$1 \times 4, \zeta_2^1, \zeta_2^1, \zeta_2^1, \zeta_2^1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}, \frac{15}{16}, 0$	$3_{-1/2}^B \boxtimes 3_{1/2}^B$
9_0^B	16	$1 \times 4, \zeta_2^1, \zeta_2^1, \zeta_2^1, \zeta_2^1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{16}, \frac{5}{16}, \frac{11}{16}, \frac{13}{16}, 0$	$3_{-3/2}^B \boxtimes 3_{3/2}^B$
9_0^B	16	$1 \times 4, \zeta_2^1, \zeta_2^1, \zeta_2^1, \zeta_2^1, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{16}, \frac{5}{16}, \frac{11}{16}, \frac{13}{16}, 0$	$3_{-3/2}^B \boxtimes 3_{3/2}^B$
16_0^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$4_0^B \boxtimes 4_0^B$
16_0^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}$	$4_{-1}^B \boxtimes 4_1^B$
16_0^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}$	$4_{-1}^B \boxtimes 4_1^B$
16_0^B	16	1×16	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}$	$8_{-1}^B \boxtimes 2_1^B$

Bosonic 2+1D Z_2-SET phases from modular extensions

- Z_2-SET phases (Z_2-gauge with Z_2 symmetry $e \leftrightarrow m$)

  4 modular extensions, 2 distinct phases:

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$5_0^{\zeta_2^1}$	8	$1 \times 4, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0$	
9_0^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{15}{16}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}$	$3_{-1/2}^B \boxtimes 3_{1/2}^B$
9_0^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{3}{16}, \frac{13}{16}, \frac{11}{16}, \frac{5}{16}$	$3_{3/2}^B \boxtimes 3_{-3/2}^B$
9_0^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{16}, \frac{15}{16}, \frac{9}{16}, \frac{7}{16}$	$3_{1/2}^B \boxtimes 3_{-1/2}^B$
9_0^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{13}{16}, \frac{3}{16}, \frac{5}{16}, \frac{11}{16}$	$3_{-3/2}^B \boxtimes 3_{3/2}^B$

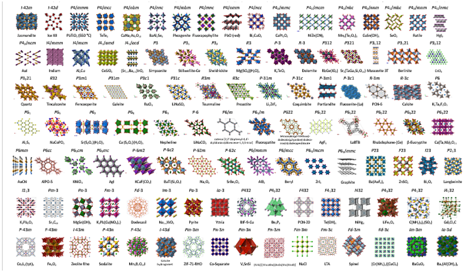
- Z_2-SET phases (Z_2-gauge with Z_2 symmetry $e \leftrightarrow m$, plus fermion condensation to $\nu = 1$ IQH state)

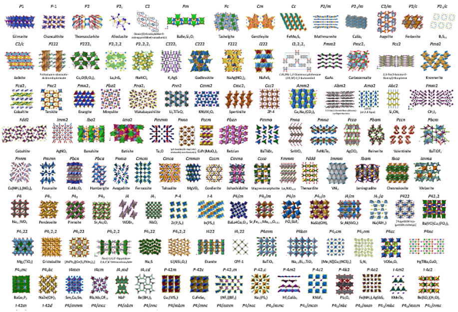
  4 modular extensions, 3 distinct phases:

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$5_1^{\zeta_2^1}$	8	$1 \times 4, 2$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}$	
9_1^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16}$	$3_{1/2}^B \boxtimes 3_{1/2}^B$
9_1^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{13}{16}, \frac{13}{16}, \frac{5}{16}, \frac{5}{16}$	$3_{-3/2}^B \boxtimes 3_{5/2}^B$
9_1^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{15}{16}, \frac{3}{16}, \frac{7}{16}, \frac{11}{16}$	$3_{-1/2}^B \boxtimes 3_{3/2}^B$
9_1^B	16	$1 \times 4, 2, \zeta_2^1 \times 4$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{3}{16}, \frac{15}{16}, \frac{11}{16}, \frac{7}{16}$	$3_{3/2}^B \boxtimes 3_{-1/2}^B$

Zoo of quantum phases of matter

- **230 crystals** from group theory





- **Infinity many topological orders in 2+1D** from category theory