Chapter 1

Self-Consistent Theory of Anderson Localization: General Formalism and Applications

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The self-consistent theory of Anderson localization of quantum particles or classical waves in disordered media is reviewed. After presenting the basic concepts of the theory of Anderson localization in the case of electrons in disordered solids, the regimes of weak and strong localization are discussed. Then the scaling theory of the Anderson localization transition is reviewed. The renormalization group theory is introduced and results and consequences are presented. It is shown how scale-dependent terms in the renormalized perturbation theory of the inverse diffusion coefficient lead in a natural way to a self-consistent equation for the diffusion coefficient. The latter accounts quantitatively for the static and dynamic transport properties except for a region near the critical point. Several recent applications and extensions of the self-consistent theory, in particular for classical waves, are discussed.

1.1. Introduction to Anderson localization

1.1.1. *Brief historical review*

The localization of quantum particles by a static random potential, or of classical waves by random fluctuations of the medium, is one of the most intriguing phenomena in statistical physics. The key ingredient of localization, wave interference, was introduced in P. W. Anderson's seminal paper "Absence of diffusion in certain random lattices".[1] There it was shown that electrons may be localized by a random potential, so that diffusion is suppressed, even in a situation where classical particles would be delocalized. The fundamental reason for the localizing effect of a random potential on quantum particles or classical waves is the multiple interference of wave components scattered by randomly positioned scattering centers. The interference effect takes place, as long as the propagation is coherent.

It is interesting to note that the first application of the idea of localization concerned the spin diffusion D of electrons and not the electrical conductivity σ.

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Anderson considered a tight-binding model of electrons on a crystal lattice, with energy levels at each site chosen from a random distribution.[1] The traditional view had been, that scattering by the random potential causes the Bloch waves to lose well-defined momentum on the length scale of the mean-free path ℓ. Nevertheless, the wavefunction was thought to remain extended throughout the sample. Anderson pointed out that if the disorder is sufficiently strong, the particles may become localized, in that the envelope of the wave function $\psi(\boldsymbol{r})$ decays exponentially from some point $\boldsymbol{r}_0$ in space:

$$| \psi(\boldsymbol{r}) |\sim \exp(| \boldsymbol{r} - \boldsymbol{r}_0 | /\xi), \tag{1.1}$$

where ξ is the localization length.

There exist a number of review articles on the Anderson localization problem. The most complete account of the early work was presented by Lee and Ramakrishnan.[2] The seminal early work on interaction affects is presented in Ref.[3] A complete account of the early numerical work can be found in Ref.[4] A path integral formulation of weak localization is presented in Ref.[5] Several more review articles and books are cited along the way. In the following we will use units with Planck's constant $\hbar$ and Boltzmann's constant k_B equal to unity, unless stated otherwise.

1.1.2. *Electrons and classical waves in disordered systems*

The wavefunction $\psi(\boldsymbol{r})$ of a single electron of mass m in a random potential $V(\boldsymbol{r})$ obeys the stationary Schrödinger equation

$$\Big(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{r}) - E\Big)\psi(\boldsymbol{r}) = 0. \tag{1.2}$$

In the simplest case $V(\boldsymbol{r})$ may be assumed to obey Gaussian statistics with $\langle V(\boldsymbol{r})V(\boldsymbol{r'})\rangle = \langle V^2\rangle\delta(\boldsymbol{r} - \boldsymbol{r'})$, but many of the results presented below are valid for a much wider class of models. Electrons propagating in the random potential $V(\boldsymbol{r})$ will be scattered on average after a time τ. For weak random potential the scattering rate is given by

$$\frac{1}{\tau} = \pi N(E) \langle V^2\rangle \tag{1.3}$$

where $N(E)$ is the density of states at the energy E of the electron. In a metal the electrons carrying the charge current are those at the Fermi energy $E = E_F$. Within the time τ the electron travels a distance $\ell = v_F\tau$, where v_F is its velocity.

In close analogy the wave amplitude $\psi(\boldsymbol{r})$ of a classical monochromatic wave of frequency ω obeys the wave equation

$$\Big(\frac{\omega^2}{c^2(\boldsymbol{r})} + \boldsymbol{\nabla}^2\Big)\psi(\boldsymbol{r}) = 0. \tag{1.4}$$

Here $c(\boldsymbol{r})$ is the wave velocity at position $\boldsymbol{r}$ in an inhomogeneous medium, assumed to be a randomly fluctuating quantity. The main difference between the Schrödinger

equation and the wave equation is that in the wave equation the "random potential" $1/c^2(\boldsymbol{r})$ is multiplied by ω^2, so that disorder is suppressed in the limit $\omega \to 0$. By contrast, in the quantum case disorder will be dominant in the limit of low energy E. A further difference may arise if the wave amplitude is a vector quantity as, e.g., in the case of electromagnetic waves in $d = 3$ dimensions.

In real systems particles or wave packets are not independent, but interact. Electrons are coupled by the Coulomb interaction, leading to important effects that go much beyond the single particle model. Similarly, wave packets interact via nonlinear polarization of the medium. Apart from these complications, the physics of electronic wave packets and classical wave packets is quite similar. In the following we will present most of the discussion in the language of electronic wave packets.

1.1.3. *Weak localization*

The all-important effect of wave interference is most clearly seen in the limit of weak scattering, where it already may cause localization, but only in reduced dimensions. While it is difficult to observe full localization at finite temperature T, on account of the effect of interactions limiting the phase coherence, the dramatic signatures of localization are visible at finite T in the form of "weak localization".[6,7]

An electron or a wave packet moving through a disordered medium will be scattered by the random potential on the average after propagating a distance ℓ, the mean-free path. On larger length scales the propagation is diffusive. Weak localization is a consequence of destructive interference of two wave components starting at some point and returning to the same point after traversing time-reversed paths. Let the probability amplitudes for the wave packet to move from point $\boldsymbol{r}_0$ along some path C_1 back to $\boldsymbol{r}_0$ be A_1 and along a different path C_2 be A_2, then the transition probability for the particle to move either along C_1 or along C_2 will be

$$w =\mid A_1 + A_2 \mid^2 = w_{cl} + w_{int}, \tag{1.5}$$

where $w_{cl} =\mid A_1 \mid^2 + \mid A_2 \mid^2$ and $w_{int} = 2Re(A_1^* A_2)$. For any two paths the interference term w_{int} may be positive or negative, and thus averages to zero. However, if $A_2 = A_r$ is the amplitude of the time-reverse of path $A_1 = A$ and if time reversal holds, then $A = A_r$, i.e., the probability of return w is enhanced by a factor of two compared to the probability w_{cl} of a classical system:

$$w = 4 \mid A \mid^2 = 2w_{cl}. \tag{1.6}$$

In that case the probability for transmission is reduced, which leads to a reduced diffusion coefficient and a reduced conductivity. One may estimate the correction to the conductivity in the following qualitative way. The relative change of the conductivity σ by the above interference effect is equal to the probability of interference of two wave packets of extension λ, the wavelength, after returning to the starting point. The infinitesimal probability of return to the origin in time t of a particle diffusing in d dimension is given by $(4\pi Dt)^{-d/2}d^3r$ where D is the diffusion

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coefficient. Since the volume of interference in the time interval $[t, t+dt]$ is $\lambda^{d-1}vdt$, where v is the velocity of the wave packet, one finds the quantum correction to the conductivity $\delta\sigma$ as [6,7]

$$\frac{\delta\sigma}{\sigma_0} \approx -\int_\tau^{\tau_\phi} \frac{v\lambda^{d-1}dt}{(4\pi Dt)^{d/2}} = \begin{cases} -c_3\frac{\lambda^2}{\ell^2}(1-\frac{\tau}{\tau_\phi}), \ d=3 \\ -c_2\frac{\lambda}{\ell}\ln(\tau_\phi/\tau), \ d=2 \\ -c_1(\sqrt{\frac{\tau_\phi}{\tau}}-1), \ d=1. \end{cases} \qquad (1.7)$$

Here $D = \frac{1}{d}v^2\tau$ is the diffusion constant, $\ell = v\tau$ is the mean-free path, τ is the mean time between successive elastic collisions, $\sigma_0 = e^2n\tau/m$ is the Drude conductivity with n as the particle density, and c_i are constants of order unity. The upper limit of the integral is the phase relaxation time τ_ϕ, i.e., the average time after which phase coherence is lost due to inelastic or other phase-shifting processes. For weak localization processes to exist at all, the inequality $\tau_\phi \gg \tau$ must hold. We note that the correction in three and two dimensions depends on the ratio of wavelength λ to mean-free path ℓ, and gets smaller in the limit of weak disorder, where $\lambda/\ell \ll 1$. In two and one dimension the correction grows large in the limit $\tau/\tau_\phi \to 0$ since one expects the phase relaxation rate $1/\tau_\phi$ for a system in thermodynamic equilibrium to go to zero for $T \to 0$. By contrast, in some cases a plateau behavior of $1/\tau_\phi$ as a function of temperature has been found experimentally, which gave rise to the speculation that the zero point fluctuations may cause decoherence. However, given a unique ground state, it is difficult to understand how a particle in the system may loose its phase coherence. Several physical mechanisms that may lead to a plateau of $1/\tau_\phi$ have been identified. For a recent discussion of these issues see Ref.[8]

With $\tau/\tau_\phi \to 0$ for $T \to 0$ the weak localization quantum correction will be large in any system in $d = 1, 2$, no matter how weak the disorder. As we will see, this behavior signals the fact that there are no extended states in $d = 1, 2$ dimensions. The characteristic length L_ϕ over which a wave packet retains phase coherence is related to τ_ϕ by the diffusion coefficient $L_\phi = \sqrt{D\tau_\phi}$. In systems of restricted dimension, e.g., films of thickness a or wires of diameter a, the effective dimensionality of the system with respect to localization is determined by the ratio L_ϕ/a. Namely, for $L_\phi \ll a$ the system is three-dimensional ($3d$), while for $L_\phi \gg a$ diffusion over time τ_ϕ takes place in the restricted geometry of the film or wire, and the effective dimension is therefore 2 or 1.

1.1.4. *Strong localization and the Anderson transition*

The appearance of localized states is easily understood in the limit of very strong disorder: localized orbitals will then exist at positions where the random potential forms a deep well. The admixture of adjacent orbitals by the hopping amplitudes will only cause a perturbation that does not delocalize the particle. The reason for this is that nearby orbitals will have sufficiently different energies so that the amount of admixture is small. On the other hand, orbitals close in energy will in

general be spatially far apart, so that their overlap is exponentially small. Thus, we can expect the wave functions in strongly disordered systems to be exponentially localized. Whether the particles become delocalized when the disorder strength is reduced, is a much more complex question. In one dimension it can be shown rigorously that all states are localized, no matter how weak the disorder.[9–11]

In three dimensions, the accepted view is that the particles are delocalized for weak disorder. In general, localized and extended states of the same energy do not coexist, since in a typical situation any small perturbation would lead to hybridization and thus to the delocalization of a localized state. We can therefore assume that the localized and extended states of a given energy are separated. For increasing disorder strength η there will then be a sharp transition from delocalized to localized states at a critical disorder strength η_c. A qualitative criterion as to when an Anderson transition is expected in $3d$-systems has been proposed by Ioffe and Regel.[12] It states that as the mean free path ℓ becomes shorter with increasing disorder, the Anderson transition occurs when ℓ is of the order of the wavelength λ of the particle (which amounts to the condition $k_F\ell \sim 1$ in metals, where k_F is the Fermi wave number). As we will see later, in $1d$ or $2d$ systems ℓ may be much longer than the wavelength and the particles are nonetheless localized. In fact, the relevant mean free path here is the one with respect to momentum transfer. A similar situation exists when we fix the disorder strength, but vary the energy E. Electrons in states near the bottom of the energy band are expected to be localized even by a weakly disordered potential, whereas electrons in states near the band center (in $d = 3$) will be delocalized, provided the disorder is not too strong. Thus there exists a critical energy E_c separating localized from delocalized states, the so-called mobility edge.[13,14] The electron mobility as a function of energy is identically zero on the localized side (at zero temperature), and increases continuously with energy separation $\mid E - E_c \mid$ in the delocalized, or metallic, phase. The continuous character of this quantum phase transition, termed *Anderson transition*, is a consequence of the scaling theory to be presented below.

Historically the continuous nature of the metal-insulator transition in disordered solids has been a point of controversy for many years. According to an earlier theory by Mott[13,14] the conductivity changes discontinuously at the transition, such that a "minimum metallic conductivity" exists on the metallic side of the transition. Numerical simulations[4] have shown beyond doubt that the transition is instead continuous, at least in the absence of interactions.

In the much more complex situation of interacting electrons one finds for the Hubbard model without disorder, using the Dynamical Mean-Field Theory (DMFT), that the Mott-Hubbard metal-insulator transition is discontinuous at finite temperatures, and that it becomes continuous in the limit $T \to 0$.[15,16] For the Hubbard model in the presence of disorder ("Anderson-Hubbard model") at $T = 0$ the situation is similar: the Mott-Hubbard metal-insulator transition is discontinuous for finite disorder and becomes continuous in the limit of vanishing

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disorder.[17,18]

1.2. Fundamental theoretical concepts of Anderson localization

The Anderson localization transition is a quantum phase transition, i.e., it is a transition at zero temperature tuned by a control parameter, e.g., the disorder strength, particle energy, or wave frequency. Unlike other quantum phase transitions, the Anderson transition does not have an obvious order parameter. Nonetheless, there exists a dynamically generated length scale, the localization or correlation length ξ, which tends to infinity as the transition is approached. Therefore, by drawing an analogy with magnetic phase transitions, Wegner early on proposed scaling properties.[19] Later, he formulated a field-theoretic description of the Anderson transition in the form of a non-linear sigma model (NLσM) of interacting matrices (rather than vectors, as for magnetic systems).[20] The NLσM was later formulated in the mathematically more tractable supersymmetric form.[21]

1.2.1. *Scaling theory of the conductance*

Wegner[19] argued that the Anderson localization transition should be described in the language of critical phenomena of continuous (quantum) phase transitions. This requires the assumption of a correlation length ξ diverging as a function of disorder strength η at the critical point

$$\xi(\eta) \sim \mid \eta - \eta_c \mid^{-\nu} . \tag{1.8}$$

The conductivity is then expected to obey the scaling law

$$\sigma(\eta) \sim \xi^{2-d} \sim (\eta_c - \eta)^s; \ \ \eta < \eta_c, \ \ d > 2. \tag{1.9}$$

This follows from the fact that σ, written in units of $e^2/(2\pi\hbar)$, has dimension $(1/\text{length})^{d-2}$, and the only characteristic length near the transition is the correlation length ξ. By comparing the conductivity exponent s with the exponent of ξ one finds

$$s = \nu(d - 2). \tag{1.10}$$

On the other hand, the conductance g of a d-dimensional cube of length L, which for a good metal of conductivity σ is given by $g(L) = \sigma L^{d-2}$, must obey the scaling property

$$g(\eta; L) = \Phi(L/\xi). \tag{1.11}$$

This means that g is a function of a single parameter L/ξ, so that each value of L/ξ corresponds to a value g.

1.2.2. *Renormalization group equation*

It then follows that $g(L)$ obeys the renormalization group (RG) equation

$$\frac{d \ln g}{d \ln L} = \beta(g), \tag{1.12}$$

where $\beta(g)$ is a function of g only, and does not depend on disorder. In a landmark paper, Abrahams, Anderson, Licciardello and Ramakrishnan[6] proposed the above equation and calculated the β-function in the limits of weak and strong disorder. A confirmation of the assumption of scaling was obtained from a calculation of the next-order term.[7]

At strong disorder one expects all states to be localized, with average localization length ξ. It then follows that $g(L)$ is an exponentially decreasing function of L:

$$g(L) \sim \exp(-L/\xi). \tag{1.13}$$

In comparison with the ohmic dependence $g \sim L^{d-2}$ this is a very non-ohmic behavior. The β-function is then given by

$$\beta(g) \sim \ln(g/g_c) < 0. \tag{1.14}$$

At weak disorder one finds from $g \sim L^{d-2}$ that

$$\beta(g) = d - 2. \tag{1.15}$$

The important question of whether the system is delocalized (metal) or localized (insulator) may be answered by integrating the RG equation from some starting point L_0, where $g(L_0)$ is known. Depending on whether $\beta(g)$ is positive or negative along the integration path, the conductance will scale to infinity or to zero, as L goes to infinity.

In $d = 3$ dimensions one has $\beta(g) > 0$ at large g, but $\beta(g) < 0$ at small g. Thus, there exists a critical point at $g = g_c$, where $\beta(g_c) = 0$, separating localized and delocalized behavior.

On the other hand, in $d = 1$ dimension one has $\beta(g) < 0$ at large and small g, and by interpolation also for intermediate values of g, so that there is no transition in this case and all states are localized.

The dimension $d = 2$ apparently plays a special role, as in this case $\beta(g) \to 0$ for $g \to \infty$. In order to determine whether $\beta > 0$ or < 0 for large g one has to calculate the scale dependent (i.e., L-dependent) corrections to the Drude result at large g. This is precisely the weak localization correction already mentioned above. For a system of finite length $L < L_\phi$ we should replace $\frac{1}{\tau_\phi} = DL_\phi^{-2}$ in eq. (1.7) by DL^{-2}, leading to

$$g(L) = \sigma_0 - a \ \ln\left(\frac{L}{\ell}\right), \tag{1.16}$$

where a diagrammatic calculation[6] gives $a = 2/\pi$ and $\sigma_0 = \ell/\lambda_F$ (in units of $e^2/\hbar$; λ_F is the Fermi wave length) has been used. It follows that

$$\beta(g) = -\frac{a}{g}, \quad d = 2, \tag{1.17}$$

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so that we can expect $\beta(g) < 0$ for all g, implying that again all states are localized. This result is valid for the "usual" type of disorder, i.e., in case all symmetries, in particular time reversal symmetry (required for the weak localization correction to be present) are preserved. If time-reversal invariance is broken, e.g., by spin-flip scattering at magnetic impurities, the weak localization effect is somewhat reduced in dimensions $d = 2 + \epsilon$, $\epsilon << 1$, but is not completely removed. The first correction term in the β-function is then proportional to $-1/g^2$ (see, e.g., Ref.[21]) implying that all states are still localized (in $d = 3$ dimensions the leading correction term is again $\sim 1/g$; see Ref.[22]). In the presence of a magnetic field the situation is more complex, since the scaling of the Hall conductance is coupled to the scaling of g. As a result, one finds exactly one extended state per Landau energy level, which then gives rise to the quantum Hall effect.[23] On the other hand, if spin-rotation invariance is broken, but time-reversal invariance is preserved, as is the case of spin-orbit scattering, the correction term is proportional to $+1/g$, i.e., it is *anti-localizing*. In this case the β-function in $d = 2$ dimensions has a zero, implying the existence of an Anderson transition.[24]

1.2.3. *Critical exponents*

In the neighborhood of the critical point at $g = g_c$ in $d = 3$ we may expand the β-function as

$$\beta(g) = \frac{1}{y}\Big[\frac{g - g_c}{g_c}\Big], \quad \mid g - g_c \mid \ll g_c. \tag{1.18}$$

Integrating the RG equation for $g > g_c$ from $g(\ell) = g_0$ to $\beta \to 1$ at large L we find $g(L) = \sigma L$, where

$$\sigma \sim \frac{1}{\ell}(g(\ell) - g_c)^y. \tag{1.19}$$

Since $[g(\ell) - g_c] \propto (\eta_c - \eta)$, we conclude that the inverse of the slope of the β-function, y, is equal to the conductivity exponent $s = y$.

Similarly, one finds on the localized side $(g < g_c)$

$$g(L) \sim g_c \exp\Big[-c(g_c - g(\ell))^y L/\ell\Big] \sim g_c \exp(-L/\xi), \tag{1.20}$$

from which the localization length follows as

$$\xi \sim \ell|\eta - \eta_c|^{-y}. \tag{1.21}$$

The critical exponent ν governing the localization length is therefore $\nu = y = s$ in $d = 3$ dimensions.

Since the critical conductance $g_c = O(1)$ in $d = 3$, there exist no analytical methods to calculate the β-function in the critical region in a quantitative way. A perturbative expansion in $2 + \epsilon$ dimensions, where $g_c \gg 1$, is possible, but the expansion in ϵ is not well-behaved, so that it cannot be used to obtain quantitative results for s and ν in $d = 3$. There exist, however, reliable results on ν from numerical studies, according to which $s = \nu = 1.58 \pm 0.02$.[4,25]

1.2.4. *Dynamical scaling*

The dynamical conductivity $\sigma(\omega)$, i.e., the a.c. conductivity at frequency ω, in the thermodynamic limit in $d = 3$ obeys the scaling law[26,27]

$$\sigma(\omega; \eta) = \frac{1}{\xi} \Phi(L_\omega/\xi), \tag{1.22}$$

where the scaling function Φ has been introduced in eq. (1.11). Here L_ω is the typical length which a wave packet travels in the time of one cycle, $1/\omega$. Since the motion is diffusive it obeys $L_\omega = \sqrt{D(\omega)/\omega}$. It is important to note that the diffusion coefficient $D(\omega)$ is energy scale dependent and is related to the conductivity via the Einstein relation

$$\sigma(\omega) = \hbar N(E) D(\omega), \tag{1.23}$$

where $N(E)$ is the density of states at the particle energy E.

At the Anderson transition, where $\xi \to \infty$, we expect $\sigma(\omega)$ to be finite. It follows that $\lim_{\xi \to \infty} \Phi(L_\omega/\xi) \sim \xi/L_\omega$ and consequently

$$\sigma(\omega; \eta) \sim \frac{1}{L_\omega} \quad , \quad \eta = \eta_c. \tag{1.24}$$

This is a self-consistent equation for $\sigma(\omega)$, with solution

$$\sigma(\omega) \sim \omega^{1/3} \quad , \quad \eta = \eta_c. \tag{1.25}$$

More precisely, in the above expressions ω should be replaced by the imaginary frequency $-i\omega$, such that $\sigma(\omega)$ is a complex-valued quantity.

In a more general notation, introducing the dynamical critical exponent z by $\sigma(\omega) \sim \omega^{1/z}$, we conclude that $z = 3$. The dynamical scaling is valid in a wide neighborhood of the critical point, defined by $\omega > \frac{1}{\tau}(\xi/\ell)^{-z} \sim | \eta - \eta_c |^{\nu z}$, where $\nu z \approx 4.8$. This scaling regime is accessible in experiment, not only by measuring the dynamical conductivity directly, but also by observing that at finite temperature the scaling in ω is cut off by the phase relaxation rate $1/\tau_\phi$.[27] Therefore, assuming a single temperature power law $1/\tau_\phi \sim T^p$, one finds the following scaling law for the temperature dependent d.c. conductivity

$$\sigma(T; \eta) \sim T^{p/3} \Phi_T(\xi T^{p/3}). \tag{1.26}$$

Using this scaling law one may in principle determine the critical exponent ν from the temperature dependence of the conductivity in the vicinity of the critical point. In the case of disordered metals or semiconductors, where studies of this type have been performed, the effect of electron-electron interaction has to be taken into account. One major modification in the above is that the Einstein relation is changed. Namely, the single-particle density of states (which is not critical) is replaced by the compressibility $\partial n/\partial \mu$, with n as the density and μ as the chemical potential, which in the presence of the long-range Coulomb interaction is expected to vanish at the transition, i.e., the system becomes incompressible. Another change is that

the frequency cutoff is given by the temperature. The critical exponents determined from experiment vary widely, from $s = 0.5$ (Ref.[28]) and $s = 1$ (Ref.[29]) to $s = 1.6$ (Ref.[30,31]), and from $z = 2$ (Ref.[30]) to $z = 2.94$ (Ref.[31]).

1.3. Renormalized perturbation theory of quantum transport in disordered media

The field-theoretic description in terms of the nonlinear σ model (NLσM) mentioned in the beginning of Sec. 1.2 is believed to be an exact framework within which the critical properties of the Anderson transition may be, in principle, calculated exactly. The mapping of the initial microscopic model onto the NLσM requires a number of simplifications, so that the noncritical properties like the critical disorder η_c, the behavior in anisotropic systems, or systems of finite extension are no longer well represented by this model. In addition, it is not known how to solve the NLσM in cases of major interest, such as in $d = 3$ dimensions.

It is therefore useful to consider approximation schemes, which on one hand keep the information about the specific properties of the system and on the other hand account approximately for the critical properties at the transition. Such a scheme, the self-consistent theory of Anderson localization, is available at least for the orthogonal ensemble (in which both, time reversal and spin rotation symmetry are conserved). This approach has been developed by us in Refs.[32,33] and was reviewed in Ref.[34] It may be termed "self-consistent one-loop approximation" in the language of renormalization group theory but has, in fact, been derived following a somewhat different logic as will be discussed below.

The appropriate language to formulate a microscopic theory of quantum transport or wave transport in disordered media is a renormalized perturbation theory in the disorder potential. The building blocks of this theory for the model defined by eq. (2) are (i) the renormalized one-particle retarded (advanced) Green's functions averaged over disorder

$$G_{\boldsymbol{k}}^{R,A}(E) = \left[E - k^2/2m - \Sigma_{\boldsymbol{k}}^{R,A}(E) \right]^{-1}, \tag{1.27}$$

where $\Sigma_{\boldsymbol{k}}^{R}(E) = (\Sigma_{\boldsymbol{k}}^{A}(E))^*$ is the self-energy, and (ii) the random potential correlator $\langle V^2 \rangle$. The self-energy Σ is a non-critical quantity and can be approximated by $\Sigma_{\boldsymbol{k}}^{R}(E) \simeq -i/2\tau$, where $1/\tau$ is the momentum relaxation rate entering the Drude formula of the conductivity and isotropic scattering is assumed.

The quantity of central interest here is the diffusion coefficient D. It follows from very general considerations[35] that the density-response function describing the change in density caused by an external space and time dependent chemical potential is given by

$$\chi(\boldsymbol{q}, \omega) = \frac{D(\boldsymbol{q}, \omega)q^2}{-i\omega + D(\boldsymbol{q}, \omega)q^2} \chi_0, \tag{1.28}$$

where $D(\boldsymbol{q}, \omega)$ is a generalized diffusion coefficient. The static susceptibility (which is non-critical in the model of non-interacting particles) is given by $\chi_0 = N_F$, where N_F is the density of states at the Fermi level. The form of χ is dictated by particle number conservation and may be expressed in terms of $G^{R,A}$ as

$$\chi(\boldsymbol{q}, \omega) = -\frac{\omega}{2\pi i} \sum_{\boldsymbol{k}, \boldsymbol{k}'} \Phi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega) + \chi_0. \tag{1.29}$$

The two-particle quantity

$$\Phi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega) = \left\langle G^R_{k_+, k'_+} G^A_{k_-, k'_-} \right\rangle, \tag{1.30}$$

where $G^{R,A}_{k,k'}$ are non-averaged single-particle Green's functions, $k_\pm = (\boldsymbol{k} \pm \boldsymbol{q}/2, E \pm \omega/2)$, and the angular brackets denote averaging over disorder, may be written in terms of the irreducible vertex function U as

$$\Phi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega) = G^R_{k_+} G^A_{k_-} \left[\delta_{\boldsymbol{k}, \boldsymbol{k}'} + \sum_{\boldsymbol{k}''} U_{\boldsymbol{k}\boldsymbol{k}''}(\boldsymbol{q}, \omega) \Phi_{\boldsymbol{k}''\boldsymbol{k}'}(\boldsymbol{q}, \omega) \right]. \tag{1.31}$$

In a diagrammatic formulation the vertex function U is given by the sum of all particle-hole irreducible diagrams of the four-point vertex function. By expressing $G^R G^A$ as

$$G^R_{k_+} G^A_{k_-} = \frac{\Delta G_{\boldsymbol{k}}}{\omega - \boldsymbol{k} \cdot \boldsymbol{q}/m - \Delta \Sigma_{\boldsymbol{k}}}, \tag{1.32}$$

where $\Delta G_{\boldsymbol{k}} = G^R_{k_+} - G^A_{k_-}$ and $\Delta \Sigma_{\boldsymbol{k}} = \Sigma^R_{k_+} - \Sigma^A_{k_-}$ one may rewrite eq. (1.31) in the form of a kinetic equation

$$\left(\omega - \frac{\boldsymbol{k} \cdot \boldsymbol{q}}{m} - \Delta \Sigma_{\boldsymbol{k}} \right) \Phi_{\boldsymbol{k}\boldsymbol{k}'} = -\Delta G_{\boldsymbol{k}} \left[\delta_{\boldsymbol{k}\boldsymbol{k}'} + \sum_{\boldsymbol{k}''} U_{\boldsymbol{k}\boldsymbol{k}''} \Phi_{\boldsymbol{k}''\boldsymbol{k}'} \right]. \tag{1.33}$$

By summing eq. (1.33) over $\boldsymbol{k}, \boldsymbol{k}'$ one finds the continuity equation

$$\omega \Phi(\boldsymbol{q}, \omega) - q\Phi_j(\boldsymbol{q}, \omega) = 2\pi i N_F \tag{1.34}$$

with the density-relaxation function

$$\Phi(\boldsymbol{q}, \omega) = \sum_{\boldsymbol{k}, \boldsymbol{k}'} \Phi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega), \tag{1.35}$$

and the current-density relaxation function

$$\Phi_j(\boldsymbol{q}, \omega) = \sum_{\boldsymbol{k}, \boldsymbol{k}'} \frac{\boldsymbol{k} \cdot \hat{\boldsymbol{q}}}{m} \Phi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega), \tag{1.36}$$

where $\hat{\boldsymbol{q}} = \boldsymbol{q} / | \boldsymbol{q} |$. Here the Ward identity $\Delta \Sigma_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} U_{\boldsymbol{k}\boldsymbol{k}'} \Delta G_{\boldsymbol{k}'} = \sum_{\boldsymbol{k}'} U_{\boldsymbol{k}'\boldsymbol{k}} \Delta G_{\boldsymbol{k}'}$ has been used.[33] Since the Ward identity plays a central role in the derivation of the self-consistent equation, we provide a short proof which does not rely on the perturbation expansion employed in Ref.[33] Instead the proof follows the derivation of a similar Ward identity in the case of wave propagation in disordered media.[36]

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Starting from the equations of motion of the single particle Green's function before impurity averaging

$$\Big[E + \frac{\omega}{2} + i0 + \frac{1}{2m}\nabla^2_{\boldsymbol{r}_1} - V(\boldsymbol{r}_1)\Big]G^R(\boldsymbol{r}_1, \boldsymbol{r}_2; E + \frac{\omega}{2}) = \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2), \qquad (1.37)$$

$$\Big[E - \frac{\omega}{2} - i0 + \frac{1}{2m}\nabla^2_{\boldsymbol{r}_3} - V(\boldsymbol{r}_3)\Big]G^A(\boldsymbol{r}_3, \boldsymbol{r}_4; E - \frac{\omega}{2}) = \delta(\boldsymbol{r}_3 - \boldsymbol{r}_4) \qquad (1.38)$$

we multiply the first of these equations by $G^A(\boldsymbol{r}_3, \boldsymbol{r}_4; E - \frac{\omega}{2})$ and the second by $G^R(\boldsymbol{r}_1, \boldsymbol{r}_2; E + \frac{\omega}{2})$ and take the difference. We now perform the limit $\boldsymbol{r}_1 \to \boldsymbol{r}_3$, upon which the terms containing the disorder potential $V(\boldsymbol{r}_i)$, $i = 1, 3$, cancels out. Finally, the disorder average is taken and the result is Fourier transformed into momentum space, with the result

$$\sum_{\boldsymbol{k}}(\omega - \frac{\boldsymbol{k}\cdot\hat{\boldsymbol{q}}}{m})\Phi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega) = G^R_{k'_+} - G^A_{k'_-}. \qquad (1.39)$$

Comparing with eq. (1.33) it is seen that the Ward identity indeed holds.

In the hydrodynamic limit, i.e., $\omega\tau \ll 1$, $q\ell \ll 1$, the current density is proportional to the gradient of the density, which is expressed in Fourier space by

$$\Phi_j + iqD(\boldsymbol{q}, \omega)\Phi = 0. \qquad (1.40)$$

In fact, multiplying eq. (1.33) by $\boldsymbol{k}\cdot\hat{\boldsymbol{q}}/m$ and summing over $\boldsymbol{k}$ and $\boldsymbol{k}'$, one may derive relation (1.40) and by comparison finds

$$D_0/D(\boldsymbol{q}, \omega) = 1 - \eta\frac{2E}{mn}\sum_{\boldsymbol{k},\boldsymbol{k}'}(\boldsymbol{k}\cdot\hat{\boldsymbol{q}})G^R_{k_+}G^A_{k_-}U_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega)G^R_{k'_+}G^A_{k'_-}(\boldsymbol{k}'\cdot\hat{\boldsymbol{q}}), \qquad (1.41)$$

where $\eta = \pi N_F\langle V^2\rangle = \frac{1}{2\pi E\tau}$ is the disorder parameter, and $D_0 = \frac{1}{d}v^2\tau$ is the bare diffusion constant.

As the Anderson transition is approached the left-hand-side of eq. (1.41) will diverge for $q, \omega \to 0$, and therefore the irreducible vertex U has to diverge, too. The leading divergent contribution to U is given by the set of diagrams obtained by using the following property of the full vertex function Γ (the sum of all four-point vertex diagrams) in the presence of time-reversal symmetry:[32,33]

$$\Gamma_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{q}, \omega) = \Gamma_{(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q})/2, (\boldsymbol{k}'-\boldsymbol{k}+\boldsymbol{q})/2}(\boldsymbol{k} + \boldsymbol{k}', \omega). \qquad (1.42)$$

This relation follows if one twists the particle-hole (p-h) diagrams of Γ such that the lower line has its direction reversed, i.e., the diagram becomes a particle-particle (p-p) diagram. Now, if time-reversal symmetry holds, one may reverse the arrow on the lower Green's function lines if one lets $\boldsymbol{k} \to -\boldsymbol{k}$ at the same time. This operation transforms p-p-diagrams back into p-h diagrams, so that an identity is established relating each diagram of Γ to its transformed diagram Γ^T, which yields the above relation.

The leading singular diagrams of Γ give rise to the diffusion pole

$$\Gamma_D = \frac{1}{2\pi N_F \tau^2} \; \frac{1}{-i\omega + Dq^2}, \tag{1.43}$$

where D is the renormalized diffusion coefficient. These diagrams are of the ladder-type and therefore reducible. Their transformed counterparts Γ_D^T are, however, irreducible and thus contribute to U. We may therefore approximate the singular part of U by

$$U_{\boldsymbol{kk'}}^{\mathrm{sing}} = \frac{1}{2\pi N_F \tau^2} \; \frac{1}{-i\omega + D(\boldsymbol{k} + \boldsymbol{k'})^2}. \tag{1.44}$$

In low-order perturbation theory U^{sing} is given by the "maximally crossed diagrams", which when summed up give a result $U^{\mathrm{sing},0}$ similar to eq. (1.44), with D replaced by the diffusion constant D_0. When $U^{\mathrm{sing},0}$ is substituted as a vertex correction into the conductivity diagram, the result is exactly the weak-localization correction discussed in Sec. 1.1.3. The structure of the kernel $U_{\boldsymbol{kk'}}$ has been analyzed from a general viewpoint in Ref.[37] The importance of the diffusion pole for the Anderson localization problem was discussed in Ref.[38,39] in connection with the derivation of mean-field theories for disordered systems in the limit of high spatial dimensions.

1.4. Self-consistent theory of Anderson localization

It follows from eq. (1.41) that for $d \leq 2$ even the lowest-order correction in the disorder parameter η to the inverse diffusion coefficient (obtained by replacing $U_{\boldsymbol{kk'}}$ by $U^{\mathrm{sing},0}$) yields a contribution which, in principle, diverges in the limit $\omega \to 0$. This infrared divergence depends crucially on the dimension d and leads to a breakdown of perturbation theory in dimensions $d \leq 2$. In higher dimensions the divergence takes place at finite disorder strength. Since the fundamental reason for the divergence of $D_0/D(0,0)$, eq. (1.41), is the presence of diffusion poles in the kernel $U_{\boldsymbol{kk'}}$, and since these diffusion poles depend on the renormalized diffusion coefficient, Vollhardt and Wölfle[32,33] interpreted eq. (1.41) as a self-consistent equation for the diffusion coefficient.

By construction eq. (1.41) is in agreement with perturbation theory. An earlier attempt to set up a self-consistent equation in the spirit of mode-mode coupling theory[40] failed to reproduce the weak localization results, as it did not account for quantum interference effects. A later *ad hoc* modification of the latter theory led to a self-consistency scheme[41,42] which is in partial agreement with the one presented here, the main difference being that an additional classical (i.e., not interference related) mechanism of localization is included.

When U^{sing} from eq. (1.44) is substituted for U, eq. (1.41) for the diffusion coefficient $D(\omega)$ (i.e., in the limit $q \to 0$) leads to the following self-consistent

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equation for the frequency-dependent diffusion coefficient $D(\omega)$:[32,33]

$$\frac{D_0}{D(\omega)} = 1 + \frac{k_F^{2-d}}{\pi m} \int_0^{1/\ell} dQ \frac{Q^{d-1}}{-i\omega + D(\omega)Q^2}. \tag{1.45}$$

Here we assumed that a finite limit $\lim_{\boldsymbol{q}\to 0} D(\boldsymbol{q},\omega) = D(\omega)$ exists, and that Q is limited to $1/\ell$ in the diffusive regime.

Eq. (1.45) may be re-expressed as

$$\frac{D(\omega)}{D_0} = 1 - \eta d k_F^{2-d} \int_0^{1/\ell} dQ \frac{Q^{d-1}}{-i\omega/D(\omega) + Q^2}. \tag{1.46}$$

1.4.1. *Results of the self-consistent theory of Anderson localization*

In $d = 3$ eq. (3.14) has a solution in the limit $\omega \to 0$ up to a critical disorder strength η_c

$$D = D_0(1 - \frac{\eta}{\eta_c}), \quad \eta < \eta_c = \frac{1}{\sqrt{3}\pi}, \tag{1.47}$$

which implies the critical exponent of the conductivity $s = 1$. The ω-dependence of $D(\omega)$ at the critical point is obtained as[43]

$$D(\omega) = D_0(\omega\tau)^{1/3}, \quad \eta = \eta_c, \tag{1.48}$$

implying a dynamical critical exponent $z = 3$ in agreement with the exact result of Wegner.[19]

At stronger disorder, $\eta > \eta_c$, all states are found to be localized. The localization length ξ, defined by $\xi^{-2} = \lim_{\omega\to 0}(-i\omega/D(\omega))$, is found as

$$\xi = \frac{\sqrt{\pi}}{2}\ell\left|1 - \frac{\eta}{\eta_c}\right|^{-1}, \tag{1.49}$$

i.e., the exponent is $\nu = 1$. For general d in the interval $2 < d < 4$ one finds Wegner scaling, $s = \nu(d - 2)$. An extension of the self-consistent theory with respect to the momentum dependence of the renormalized diffusion coefficient near the Anderson transition has been proposed in Ref.[44] It leads to a modified critical exponent of the localization exponent, $\nu = 1/(d-2) + 1/2$, which is in much better agreement with numerical results in $d = 3$. The conductivity exponent is found to be unchanged ($s = 1$), i.e., Wegner scaling is no longer obeyed.

In dimensions $d \leq 2$, there is no metallic-type solution. The localization length is found as

$$\xi = \ell\left[\exp\frac{1}{\eta} - 1\right]^{1/2}, \quad d = 2$$
$$\xi \cong c_1\ell, \quad d = 1 \tag{1.50}$$

where the coefficient $c_1 \approx 2.6$, while the exact result is $c_1 = 4$.[11]

The β-function has been derived from the self-consistent equation for the length-dependent diffusion coefficient, where a lower cutoff $1/L$ has been applied to the
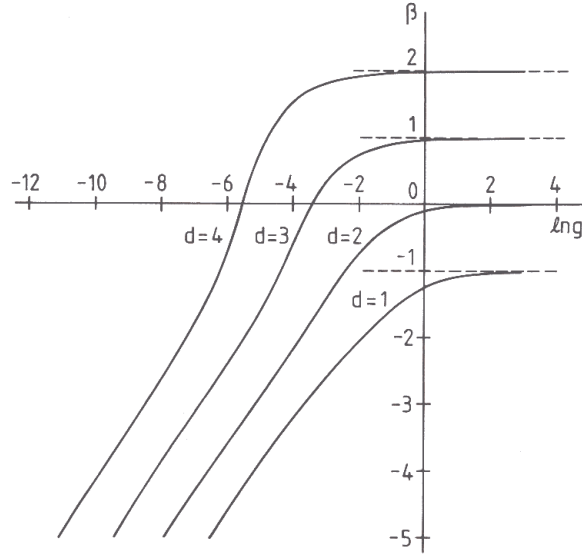
Fig. 1.1.   Renormalization group β-function in dimensions $d = 1, 2, 3$ for the orthogonal ensemble, as obtained from the self-consistent theory.[45]

Q-integral in eq. (1.46). The result[45] for $d = 3$ dimensions in the metallic regime is given by

$$\beta(g) = \frac{g - g_c}{g}, \quad g > g_c = \frac{1}{\pi^2},$$
(1.51)

and in the localized regime by

$$\beta(g) = 1 - \frac{1}{\pi^2 g} \frac{1 + x}{1 + x^2} e^{-x} - \frac{x^2}{1 + x} \quad , \quad g < g_c.$$
(1.52)

Here $x = x(g)$ is the inverse function of

$$g = \frac{1}{\pi^2} (1 + x) e^{-x} (1 - x \arctan \frac{1}{x}).$$
(1.53)

The β-functions in $d = 1, 2, 3$ obtained in this way are shown in Fig. 1.1.

The phase boundary separating localized and extended states in a disordered three-dimensional system may be determined approximately by a variety of methods. For electrons on a cubic lattice with nearest-neighbor hopping and one orbital per site with random energy ϵ_i chosen from a box distribution in the interval $[-W/2, W/2]$, the phase diagram has been determined by numerical simulations[46] as shown in Fig. 1.2.

Also shown is the result of an analytic expression obtained from the self-consistent theory[47] applied to a tight-binding model, where the Coherent Potential

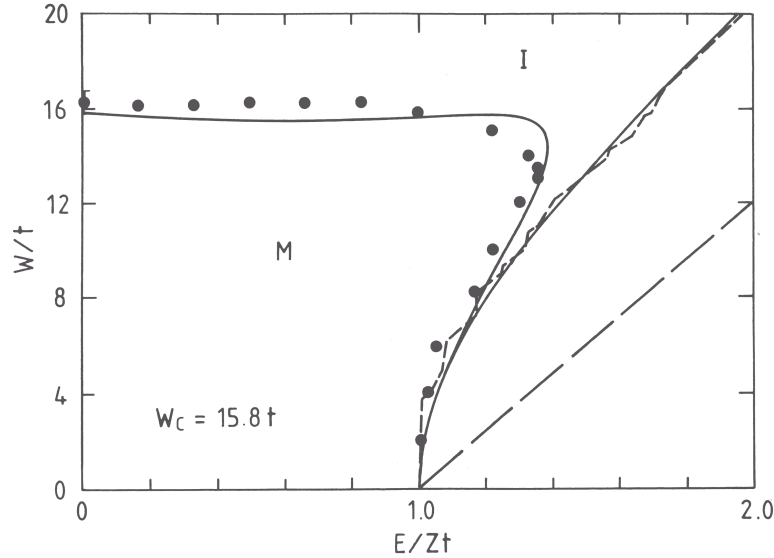*P. Wölfle and D. Vollhardt*

Fig. 1.2.   Phase diagram showing metallic (M) and insulating (I) regions of the tight-binding model with site-diagonal disorder (box distribution of width W). Dots: numerical study;[4] solid line: self-consistent theory.[47] The remaining lines are bounds on the energy spectrum; see Ref.[34]

Approximation (CPA) was used to evaluate the single-particle properties; no adjustable parameters enter. The agreement is seen to be very good.

1.5. Applications of the self-consistent theory of Anderson localization

The self-consistent theory of Anderson localization proposed by us in 1980[32,33] was applied and extended to account for many of the salient features of disordered systems. Here we briefly review the more recent developments, not yet described in our review.[34] While initially the main interest had focussed on disordered electronic systems, in recent years the interest shifted to localization of classical waves and even more recently, to ultracold atom systems. We first review an extension of the self-consistent theory to the case of weak applied magnetic and electric fields.

1.5.1. *Effect of static magnetic and electric fields*

1.5.1.1. *Magnetic fields*

One of the limitations of the self-consistent theory has been the difficulty to treat scale dependent contributions to the conductivity in the presence of a magnetic field in perturbation theory. As explained above, a magnetic field induces a "mass"

in the Cooperon propagator and therefore removes the localizing interference effect leading to localization of all states in $d \leq 2$ dimensions. On the other hand studies of the nonlinear σ-model show that in higher (two-loop) order scale dependent terms appear which are generated solely by diffusion propagators (diffusons). There is, however, a general theorem of perturbation theory, related to gauge invariance, stating that the singular contribution of any diagram with one diffuson and an arbitrary decoration with additional impurity lines cancels within a group of related diagrams.[33] The way out of this apparent contradiction has only been found very recently.[48]

A satisfactory generalized self-consistent theory for the case of unitary symmetry, including the two-loop and higher contributions has not been formulated yet. Nonetheless there is a parameter regime of weak magnetic field B ($\omega_c = eB/mc$) and moderately strong disorder, $\omega_c\tau << 1/\epsilon_F\tau \lesssim 1$ in which the one-loop contributions still dominate over the two-loop contributions and a generalized self-consistent theory may be formulated. The most complete discussion of this approach was given by Bryksin and Kleinert,[49] who proposed a set of two coupled self-consistent equations for the diffusion coefficients D_{ph} in the particle-hole channel (diffuson) and D_{pp} in the particle-particle channel (Cooperon) of a $2d$ system:

$$\frac{D_{pp}}{D_0} = 1 - g\left[\psi(\frac{1}{2} + \ell_B^2\kappa^2 + \frac{\ell_B^2}{4\tau_\phi D_{ph}}) - \psi(\frac{1}{2} + \frac{\ell_B^2}{4\tau_\phi D_{ph}})\right], \tag{1.54}$$

$$\frac{D_{ph}}{D_0} = 1 - g\ln(1 + \frac{\tau_\phi}{\tau}\frac{D_{pp}}{D_0}). \tag{1.55}$$

Here $\psi(z)$ is the digamma function, $g = 2/(\pi k_F\ell)$ is the coupling constant, $\ell_B = (c/eB)^{1/2}$ is the magnetic length, and $\kappa = 1/(\sqrt{2}\ell)$. The solution of these equations allows one to extend the results of weak localization theory, e.g., for the negative magnetoresistance, to the regime of moderately strong disorder, leading to renormalized values of the parameters of weak localization theory. Good agreement has been found with experimental data in that range.[49]

1.5.1.2. *Electric fields*

An applied static electric field $\boldsymbol{E}$ affects the localization physics in the following way: electrons drifting under the influence of $\boldsymbol{E}$ experience a reduced probability of return, weakening the localization effect provided by interference of return paths. This effect is incorporated into the Cooperon dynamics, leading to a new term in the diffusion pole

$$\Gamma_D = \frac{1}{2\pi N_F\tau^2}\frac{1}{-i\omega + Dq^2 + i\mu_d\boldsymbol{q}\cdot\boldsymbol{E}}, \tag{1.56}$$

where $\mu_d = e/(m\tau)$ is the mobility. The electric field term leads to the appearance of a localization transition even in dimension $d = 1, 2$. Near the transition in $d = 1$

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the diffusion coefficient is found as[49]

$$D(E) = \begin{cases} D_0(1 - E_0/E), & \text{for } E > E_0 \\ 0 & \text{for } E_0 < E, \end{cases} \tag{1.57}$$

in agreement with the exact result in Ref.,[50] where $E_0 = (2n/(e\pi N_F^2 D_0)$ is the threshold field. In $d = 2$ dimensions the behavior above threshold is logarithmic:

$$D(E) = \begin{cases} D_0 \ln(E/E_0), & \text{for } E > E_0 \\ 0 & \text{for } E_0 < E, \end{cases} \tag{1.58}$$

where $E_0 = (4\epsilon_F/e\kappa)\exp(-\pi k_F\ell/2)$. The relaxation of the charge current following a sudden switch on of the electric field has been considered in Ref.[51] There it was found that the current has a long time tail $\propto t^{-1/2}$ as a consequence of the infrared singular behavior of the Cooperon pole.

The way in which electric and magnetic fields affect transport near the localization transition in anisotropic systems was studied in Refs.[52,53]

1.5.2. *Anisotropic systems, films and wires*

The question of how the scaling properties of the conductance are modified in anisotropic systems was first addressed in Ref.[54] There it was established that even in the presence of an anisotropic electronic band structure and an anisotropic impurity-scattering cross section the one-parameter scaling theory holds. The ratios of the components of the conductivity tensor are invariant under scaling, implying that the geometric mean of the conductivity components plays the role of the scaling quantity. This feature is preserved by the self-consistent theory. Numerical studies of anisotropic systems[55,56] appeared to cast doubts on the one-parameter scaling hypothesis. However, a later more careful study of the problem in $d = 2$ dimensions showed that indeed one-parameter scaling is obeyed:[57] the ratio of the localization lengths (in the direction of the principal axes) turns out to be proportional to the square root of the ratio of the conductivities. A comparison with the self-consistent theory in the somewhat simpler form of the "potential-well analogy"[58] showed again qualitative agreement. Localization in anisotropic systems has also been considered in a model with anisotropic random potential correlations, and the phase diagram has been mapped out within an extension of the self-consistent theory.[59] The same authors explored the consequences of finite-range correlations of the random potential within a generalization of the self-consistent theory.[60]

A somewhat different but related question is the behavior of the conductance of a film of finite thickness, or a wire of finite diameter. There is no doubt that in the thermodynamic limit these systems behave like true $2d$ or $1d$ systems. It is, however, interesting to understand how this behavior is approached. Numerical studies of metallic disordered films as a function of film thickness seemed to indicate a localization transition as a function of thickness,[61] in contradiction to the results of the self-consistent theory applied to this system. A further study by the same

authors[62] on systems of finite thickness in a magnetic field explored the possibility of a delocalization transition controlled by both the thickness and the magnetic field. The transitions obtained are pseudo-transitions marking a crossover from strong to weak localization, as confirmed in a later more accurate numerical study.[63]

1.5.3. *Anderson localization of classical waves*

The concept of the self-consistent theory of localization can be carried over to the case of propagation of classical waves in disordered media. Here we sketch the formulation following the presentation of Kroha, Soukoulis, and Wölfle.[64] For scalar waves propagating in a medium of randomly positioned point scatterers of density $n_I = a^{-3}$, modelled by spheres of volume V_s, the average phase velocity is given by $c_{ph} = c_0[1 + (V_s/a^3)\Delta\epsilon]^{-1/2}$, where c_0 is the bare phase velocity and $\Delta\epsilon$ characterizes the strength of the scattering ("dielectric contrast"). The Green's function of the wave equation is defined as

$$G_{\boldsymbol{k}}(\omega) = [G_0^{-1}(\omega) - \Sigma_{\boldsymbol{k}}(\omega)], \qquad (1.59)$$

where $G_0^{-1}(\omega) = \omega^2 - c_0^2 k^2$. The self-energy Σ may be determined within the Coherent Potential Approximation (CPA) (see, e.g., Ref.[65]) provided it is independent of k. Then the bare diffusion constant is found as

$$D_0 = 2c(\omega)\frac{c_0}{\omega}G_0^{-1}(\omega)\sum_{\boldsymbol{k}}(\boldsymbol{k}\cdot\widehat{\boldsymbol{q}})^2(\mathrm{Im}G_{\boldsymbol{k}}^A)^2. \qquad (1.60)$$

The renormalized diffusion coefficient may be shown to satisfy the self-consistency equation

$$D(\Omega) = D_0 - 2[c(\omega)\frac{c_0}{\omega}]^2\frac{\mathrm{Im}\Sigma}{(\mathrm{Im}G_0)^2}\frac{D(\Omega)}{D_0}\sum_{\boldsymbol{k},\boldsymbol{k}'}(\boldsymbol{k}\cdot\widehat{\boldsymbol{q}})\frac{\mathrm{Im}G_{\boldsymbol{k}}(\mathrm{Im}G_{\boldsymbol{k}'})^2}{-i\Omega + D(\Omega)(\boldsymbol{k}+\boldsymbol{k}')^2}(\boldsymbol{k}'\cdot\widehat{\boldsymbol{q}}).$$
$$(1.61)$$

Here Ω is the external frequency while ω is the frequency of the waves which enter in one-particle quantities. This equation can be solved in the limit $\Omega \to 0$ to obtain the diffusion coefficient in the delocalized phase and the localization length $\xi = \lim_{\Omega\to 0}[D(\Omega)/(-i\Omega)]^{1/2}$ in the localized phase. One finds that it is much harder to localize classical waves as compared to electrons, and there is only a narrow region of the phase diagram (at reasonable contrast $\Delta\epsilon$) where localization is found.[64]

A more realistic theory of the propagation of electromagnetic waves in disordered materials with loss or gain mechanisms keeping the vector character of the fields has been worked out by Lubatsch, Kroha, and Busch.[66] We briefly sketch the main results here. The electric field amplitude $\boldsymbol{E}_\omega(\boldsymbol{r})$ of an electromagnetic wave of frequency ω in a medium with random dielectric constant $\epsilon(\boldsymbol{r};\omega) = \overline{\epsilon}(\omega) + \Delta\epsilon(\boldsymbol{r};\omega)$, obeys the wave equation

$$\nabla \times (\nabla \times \boldsymbol{E}_\omega(\boldsymbol{r})) - \frac{\omega^2}{c^2}\epsilon(\boldsymbol{r};\omega)\boldsymbol{E}_\omega(\boldsymbol{r}) = \omega\boldsymbol{J}_\omega(\boldsymbol{r}), \qquad (1.62)$$

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where $\langle \Delta\epsilon(\boldsymbol{r};\omega) \rangle = 0$. In the following the random part of the dielectric function will be modelled as $\Delta\epsilon(\boldsymbol{r};\omega) = -(c^2/\omega^2)h(\omega)V(\boldsymbol{r})$. The Green's functions of the wave equation, after disorder averaging, are defined as

$$\boldsymbol{G}_{\boldsymbol{k}}^{R,A}(\omega) = [(\frac{\omega^2}{c^2}\overline{\epsilon} - k^2)\boldsymbol{P} - \boldsymbol{\Sigma}_{\boldsymbol{k}}^{R,A}(\omega)]^{-1}. \tag{1.63}$$

Here $\boldsymbol{G}$ and the self-energy $\boldsymbol{\Sigma}$ are (3×3) tensors and $\boldsymbol{P} = \boldsymbol{1} - \widehat{\boldsymbol{k}} \otimes \widehat{\boldsymbol{k}}$ is the projector onto the transverse subspace (here and in the following the hat symbol denotes a unit vector). Transport properties are contained in the two-particle correlation function (a tensor of rank four)

$$\boldsymbol{\Phi}_{\boldsymbol{kk'}}(\boldsymbol{q},\Omega) = \left\langle \boldsymbol{G}_{k_+,k'_+}^R \otimes \boldsymbol{G}_{k_-,k'_-}^A \right\rangle, \tag{1.64}$$

where $k_{\pm} = (\boldsymbol{k} \pm \boldsymbol{q}/2, \omega \pm \Omega/2)$, etc., which obeys the Bethe-Salpeter equation

$$\boldsymbol{\Phi}_{\boldsymbol{kk'}}(\boldsymbol{q},\Omega) = \boldsymbol{G}_{k_+}^R \otimes \boldsymbol{G}_{k_-}^A \Big[\delta_{\boldsymbol{kk'}} + \sum_{\boldsymbol{k''}} \boldsymbol{U}_{\boldsymbol{kk''}}(\boldsymbol{q},\Omega)\boldsymbol{\Phi}_{\boldsymbol{k''k'}}(\boldsymbol{q},\Omega) \Big]. \tag{1.65}$$

As in the case of electrons in a random potential considered above, the Bethe-Salpeter equation may be converted into a kinetic equation for the integrated intensity correlation tensor $\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{q},\Omega) = \sum_{\boldsymbol{k'}} \boldsymbol{\Phi}_{\boldsymbol{kk'}}(\boldsymbol{q},\Omega)$ of the form

$$\left(\Delta\boldsymbol{G}_{\boldsymbol{k},0}^{-1}(\omega) - \Delta\boldsymbol{\Sigma}_{\boldsymbol{k}}\right)\boldsymbol{\Phi}_{\boldsymbol{k}} = -\Delta\boldsymbol{G}_{\boldsymbol{k}}\Big[\boldsymbol{1} \otimes \boldsymbol{1} + \sum_{\boldsymbol{k''}} \boldsymbol{U}_{\boldsymbol{kk''}}\boldsymbol{\Phi}_{\boldsymbol{k''}}\Big], \tag{1.66}$$

where $\Delta\boldsymbol{G}_{\boldsymbol{k},0}^{-1}(\omega) = [\boldsymbol{G}_{k_+,0}^R]^{-1} \otimes \boldsymbol{1} - \boldsymbol{1} \otimes [\boldsymbol{G}_{k_-,0}^A]^{-1}$, $\Delta\boldsymbol{\Sigma}_{\boldsymbol{k}} = \boldsymbol{\Sigma}_{k_+}^R \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{\Sigma}_{k_-}^A$, and $\Delta\boldsymbol{G}_{\boldsymbol{k}} = \boldsymbol{G}_{k_+}^R \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{G}_{k_-}^A$.

The kinetic equation serves to derive the energy conservation equation and the equivalent of Fick's law:

$$[\Omega + \frac{i}{\tau_L(\Omega)}]P_E(\boldsymbol{q},\Omega) + \boldsymbol{q} \cdot \boldsymbol{J}_E(\boldsymbol{q},\Omega) = S(\boldsymbol{q},\Omega) \tag{1.67}$$

$$\boldsymbol{J}_E(\boldsymbol{q},\Omega) = iP_E(\boldsymbol{q},\Omega)\boldsymbol{D}(\Omega) \cdot \boldsymbol{q}. \tag{1.68}$$

Here

$$P_E(\boldsymbol{q},\Omega) = (\omega/c_p)^2 \sum_{\boldsymbol{k}} \boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{q},\Omega) \tag{1.69}$$

is the energy-density relaxation function, with c_p as the renormalized phase velocity, and

$$\boldsymbol{J}_E(\boldsymbol{q},\Omega) = (\omega/c_p)\boldsymbol{v}_E(\omega) \sum_{\boldsymbol{k}} (\boldsymbol{k} \cdot \widehat{\boldsymbol{q}})\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{q},\Omega) \tag{1.70}$$

is the energy-current density relaxation function, with $\boldsymbol{v}_E(\omega)$ as the energy transport velocity; for the definitions of c_p and $\boldsymbol{v}_E(\omega)$ we refer the reader to Ref.[66] When energy absorption by the medium is taken into account (as expressed by the imaginary part of the dielectric function), or conversely, if a medium with gain is

considered, energy is not conserved, as expressed by the loss/gain rate $\frac{1}{\tau_L(\Omega)}$. The energy-diffusion coefficient tensor $\boldsymbol{D}(\Omega)$ is found as

$$\boldsymbol{D}(\Omega) = \frac{1}{3} v_E(\omega) \boldsymbol{l}_T, \tag{1.71}$$

where the tensor of transport mean free path is given by

$$\boldsymbol{l}_T = \frac{c_p}{\omega}(A + \kappa)\boldsymbol{l}. \tag{1.72}$$

Here the main contribution to $\boldsymbol{l}$ has a form which is analogous to eq. (1.41):

$$\boldsymbol{l}^{-1} = a_1^{-1} \sum_{\boldsymbol{k},\boldsymbol{k}'} (\boldsymbol{k} \cdot \widehat{\boldsymbol{q}}) \Delta \boldsymbol{G}_{\boldsymbol{k}} \boldsymbol{U}_{\boldsymbol{k}\boldsymbol{k}''}(0, \Omega) \Delta \boldsymbol{G}_{\boldsymbol{k}'} (\boldsymbol{k}' \cdot \widehat{\boldsymbol{q}}), \tag{1.73}$$

and a_1 and A are defined in Ref.[66] The quantity κ describes scattering caused by a mismatch of absorption/gain between the scattering objects and the medium.

The energy density propagator $P_E(\boldsymbol{q}, \Omega)$ in the limit of small $\boldsymbol{q}, \Omega$ follows from eqs. (1.67), (1.68) as

$$P_E(\boldsymbol{q}, \Omega) = [\Omega + \frac{i}{\tau_L(\Omega)} + i\boldsymbol{q} \cdot \boldsymbol{D}(\Omega) \cdot \boldsymbol{q}]^{-1} S(\boldsymbol{q}, \Omega). \tag{1.74}$$

Replacing $\boldsymbol{U}_{\boldsymbol{k}\boldsymbol{k}''}(0, \Omega)$ by its singular part proportional to the diffusion propagator $P_E(\boldsymbol{q}, \Omega)$ one arrives at a self-consistent equation for the diffusion coefficient tensor. The latter provides a framework for the description of the interplay between localization and stimulated emission in materials with gain, i.e., the problem of the random laser.[67]

The predictions of the self-consistent theory have also been probed by comparison with numerical results[68] for transmission of waves in unbounded $1d$ and $2d$ systems and through strips of finite width. Good overall agreement is found.

The localization of phonons and the ultrasound attenuation in layered crystals with random impurities has been studied within the self-consistent theory in Ref.[69]

1.5.4. *Transport through open interfaces*

Most of the discussion so far considered transport in infinitely extended systems, with the exception of the scaling theory for systems of length L. In some cases, however, transport through plate-shaped systems in the direction perpendicular to the plate surface is of interest. As pointed out by van Tiggelen and collaborators,[70,71] the weak localization physics changes near an open boundary, as the finite probability of escape through the interface diminishes the return probability necessary for interference. In the framework of the self-consistent theory this effect may be taken into account quantitatively. To this end it is useful to express eq. (1.45) in position-energy space as

$$\frac{D_0}{D(\omega)} = 1 + 2\pi \frac{k_F^{2-d}}{m} C(\boldsymbol{r}, \boldsymbol{r}'), \tag{1.75}$$

where $C(\boldsymbol{r}, \boldsymbol{r'})$ is a solution of the diffusion equation, and a cut-off $Q < 1/\ell$ to the momentum was applied to the spectrum of the Q-modes in eq. (1.45)

$$[-i\omega + D(\omega)\nabla^2]C(\boldsymbol{r}, \boldsymbol{r'}) = \delta(\boldsymbol{r} - \boldsymbol{r'}). \tag{1.76}$$

The above formulation now allows one to describe position-dependent diffusion processes, as they appear near the sample surface in a confined geometry, e.g., transmission through a slab. In that case the diffusion coefficient may be assumed to be position dependent, $D = D(\boldsymbol{r}, \omega)$. Then $C(\boldsymbol{r}, \boldsymbol{r'})$ obeys the modified diffusion equation[72]

$$[-i\omega + \nabla D(\boldsymbol{r}, \omega)\nabla]C(\boldsymbol{r}, \boldsymbol{r'}) = \delta(\boldsymbol{r} - \boldsymbol{r'}). \tag{1.77}$$

The solution is subject to an appropriate boundary condition at the surface of the sample. A microscopic derivation of the above equation in diagrammatic language was given in Ref.[73] Further confirmation of the theory was obtained in,[74] where the above equations was derived within the nonlinear σ-model framework in the weak coupling limit. The theory accounts very well for the localization properties of accoustic waves transmitted through a strongly scattering plate.[72]

It is natural to ask whether a position dependent diffusion coefficient will change the critical behavior obtained from the scaling properties of the conductance of finite size samples. This question was addressed in Ref.[75] with the result that the critical exponents are unchanged and the β-function is hardly modified by the improved approximation. The scaling of the transmission coefficient for classical waves through a disordered madium near the Anderson transition was considered within the position dependent self-consistent theory in Ref.[76]

The transmission of microwave pulses through quasi-one-dimensional samples has been measured recently and was analyzed in terms of the self-consistent theory.[77] It was found that while the self-consistent theory can account very well for the propagation at intermediate times, it fails at longer times when the transport occurs by hopping between localized regions.

Anderson localization of atoms in a Bose-Einstein condensate released from a trap and subject to a random potential has been considered in the framework of the self-consistent theory in Ref.[78] The authors show that the scaling properties govern the dynamical behavior of the expanding atom cloud, so that the critical exponents determine the power law in time obeyed by the expanding cloud size.

1.6. Conclusion

Anderson localization in disordered systems continues to be a very lively field of research. Current investigations do not concentrate so much on disordered electrons but on classical waves (light, electromagnetic microwaves, acoustic waves), or ultracold atoms in the presence of disorder. Although the fundamental concepts of Anderson localization are well understood by now, there still remain a number of

open questions. Some of them are related to the analytical theory of critical properties near the Anderson transition. Others concern the quantitative description of realistic materials, e.g., the question under which conditions light or acoustic waves become localized. The self-consistent theory of Anderson localization has been, and will continue to be, a versatile tool for the investigation of these problems. It allows one to incorporate the detailed characteristics of the system such as the energy dispersion relation, the particular form of disorder, the shape of the sample, and loss or gain mechanisms in an efficient way. The self-consistent theory is not only applicable to stationary transport problems, but also to dynamical situations such as pulse propagation or the behavior after a sudden switch-on.

As Anderson localization is a wave-interference phenomenon, the limitations of phase coherence are an important subject of study in this context. By now Anderson localization has been observed in many different systems beyond doubt. On the other hand, the observation of the Anderson transition itself is a much more challenging task. Here the recent investigations of classical waves and atomic matter waves offer fascinating, new perspectives which will undoubtedly lead to a deeper understanding of the localization phenomenon.

Acknowledgements

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