Selected References:

→: FIRST OF ALL: (!)

THIS WORK WAS DONE WITH MY COLLABORATORS:

SHINSEI RYU
ANDREAS SCHNYDER
AKIRA FURUSAKI

AND

JOEL MOORE

go to arXiv reference: click "doi" link, takes you to Journal website

→: SOME REFERENCES

-: MOST RECENT OVERVIEW:

A W W LUDWIG: arXiv: 1512.08882
Phys. Scr. T168 (2016) 014001
[Publ. 31 Dec. 2015] (Nobel Symposium)

-: Schnyder, Ryu, Furusaki, Ludwig: PRB 78, 195125 (2008)

-: " " " " : Proc. L. D. Landau Memorial Conf. "Advances in Theoretical Physics", ed V Lebedev and M Feigelman vol 1134 p 10

-: Ryu, Schnyder, Furusaki, Ludwig: New J. Phys. 12 065010 (2010)

-: Ryu, Moore, Ludwig: PRB 85 045104 (2012)

-: A Kitaev, Proc. L. D. Landau Memorial Conf. "Advances in Theoretical Physics" ed V Lebedev and M Feigelman, Vol 1134 p 22

—: Chiu, Teo, Schnyder, Ryu: arXiv 1505.03535

—: Zirnbauer: J. Math. Phys. 37 4986 (1996)

—: Altland, Zirnbauer: PRB 55 1142 (1997)

—: Heinzner, Huckleberry, Zirnbauer: Commun. Math. Phys 257 725 (2005)

—: Ryu: Phys. Scr. T164 014009 (2015)

—: Witten: arXiv: 1508.04715

There are two important types of Topological Phases of Matter:

(i): <u>Topol. Phases with intrinsic topol. order</u>
(examples: Fractional Quantum Hall States, ..)
They have
* ground state degeneracies on topologically non-trivial position spaces (e.g. Tori)
* Anyonic excitations which may have fractional quantum numbers and non-trivial Braiding properties.

(ii): <u>Symmetry Protected Topological (SPT) Phases</u>
* which have <u>none</u> of the above
* but their ground states cannot be continuously deformed into a direct product state (without crossing a quantum phase transition at which the gap closes), as long as the symmetry protecting the SPT is preserved.

→: Topological Insulators and Superconductors of non-interacting Fermions provide the simplest (and the first) examples of SPT phases.

They can be completely classified in any dimension of space.

Here I will review different approaches to this classification.

→: More general, interacting Fermionic SPT phases the next step of generalization of non-interacting Fermionic Top. Ins. + Supercond.s.

These are not fully understood today, but existing results build on the "template" of non-interacting Top. Ins. + Supercond.s.

→: The simplest and most fundamental classification scheme for topol. insul.s + supercond.s applies to systems for which

no symmetry that is unitarily realized on the 1^{st}-quantized Hamiltonian is required to protect the topological phase,

in the sense to be explained below.

This is the scheme I will be talking about here.

→: unitarily realized symmetries include

- translational invariance
- symmetries of an underlying crystal lattice
- spatial reflection "inversion", parity symmetry
- $SU(2)$ spin-rotation symmetry

$\vdots$

→: It is of course possible to ask what classification of topol. insul.s + supercond.s emerges when certain unitarily realized symmetries are required to protect a topological phase.

The answer to this question will however clearly not be as universal, because it will depend on the specific unitarily realized symmetry group (required to protect the top. phase).

For Bosonic SPTs the group cohomology approach aims to address this question. There exists also a generalization of this to Fermionic SPTs (= "group super cohomology") which is currently being explored.

→: For this reason we ask here the most universal question possible:

> What are the possible topol. phases of non-interacting Fermions, when <u>no</u> unitarily realized symmetry is required to protect the topol. phase?

→: In order to obtain an exhaustive classification (that incorporates all possible cases), one first needs to have a framework within which to describe all possible Hamiltonians.

This framework is the so-called "Ten-Fold Way", which I will now review.

- Zirnbauer
- Altland + Zirnbauer
- Schnyder, Ryu, Furusaki, Ludwig

→: Of course, even when the unitarily realized symmetry group is not required to protect a topol. phase, it may still be a symmetry of the Hamiltonian (i.e. may commute with it).

But, we may always choose a basis in which the Hamiltonian takes on block-diagonal form, and the blocks possess no invariance properties with respect to the unitarily realized symmetry. (Hence, there are no constraints on any of those blocks arising from the unitarily realized symmetry.)

– we will be somewhat more explicit below. –

It will thus have to be the properties of these block-Hamiltonians that are responsible for the topol. properties of the system.

These block-Hamiltonians can have very little specific structure. It turns out that the only properties the blocks can possess are certain reality conditions (block Hamiltonian is real or complex in a certain sense).

These reality conditions have a very transparent physical meaning: they turn out to reflect the properties of the Hamiltonian under time-reversal and charge-conjugation (= particle-hole) symmetry.

Time-reversal and charge-conjugation symmetries are fundamentally different from ordinary symmetry operations in that they are not realized by unitary (linear), but by anti-unitary (anti-linear) operators on the Hilbert space of the 1st quantized Hamiltonian.

As we will now review, there are only 10 ways a bulk Hamiltonian can respond to time-reversal and charge-conjugation (particle-hole) ~~of~~ symmetries.
[Zirnbauer (1996), Altland + Zirnbauer (1997)].

→ Let us be more specific:

–: First consider non-superconducting systems.

2^{nd} quantized Hamiltonian

$$\boxed{\hat{H} = \sum_{A,B} \hat{\psi}^{+}_{A}\, H_{A,B}\, \hat{\psi}_{B} = \hat{\psi}^{+} H \hat{\psi}}$$

[canonical creation annihilation operators]

–: "Regularize" system on a lattice. Then

(for spinless Fermions) A, B = labels lattice sites $= i, j = 1, \ldots, N$

or:

(Fermions with Pauli spin $\sigma = \pm 1$) $A = (i, \sigma)$

$A, B = 1, \ldots, N$ = twice # of lattice sites

–: $\boxed{H = \{H_{A,B}\} = N \times N \text{ matrix of numbers} = 1^{st} \text{ quantized (single-particle) Hamiltonian}}$

–: We are interested in the thermodynamic limit $N \gg 1$.

–: $$\boxed{\hat{\psi} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \vdots \\ \hat{\psi}_N \end{pmatrix}, \quad \hat{\psi}^{+} = (\hat{\psi}^{+}_1, \hat{\psi}^{+}_2, \ldots, \hat{\psi}^{+}_N)}$$

→: unitarily realized symmetries

—: Assume H invariant under a group G_0 of symmetries, unitarily realized on single-particle Hilbert space:

$\exists$ $N \times N$ matrices $\mathcal{U}$ [$= \mathcal{U}_g$ = a linear (unitary) representation of G_0, $g \in G_0$]

which commute with H (= 1st quantized Hamilt.)

$$\boxed{\mathcal{U} H \mathcal{U}^{+} = H}$$

—: In 2nd quantized language, this corresponds to operators $\hat{\mathcal{U}}$ acting on the Fermion Fock space via

$$\boxed{\hat{\mathcal{U}}\, \hat{\psi}_A\, \hat{\mathcal{U}}^{-1} = \sum_B (\mathcal{U}^{+})_{AB}\, \hat{\psi}_B\,, \quad \hat{\mathcal{U}}\, \hat{\psi}^{+}_A\, \hat{\mathcal{U}}^{-1} = \sum_B \hat{\psi}^{+}_B\, \mathcal{U}_{BA}}$$

and commuting with the 2nd quantized Hamiltonian

$$\boxed{\hat{\mathcal{U}}\, \hat{H}\, \hat{\mathcal{U}}^{-1} = \hat{H}}$$

→: In this situation the 1st quantized Hamiltonian H (= a $N \times N$ matrix) has a block-diagonal structure:

The N-dimensional single-particle Hilbert space V spanned by the single-particle states

$$\boxed{|A\rangle = \psi_A^{\dagger} |0\rangle, \quad A = 1, \ldots, N}$$

($|0\rangle$ is the Fock vacuum $\psi_A |0\rangle = 0$)

decomposes into a direct sum of vector spaces V_λ associated with certain irreducible representations (irreps) λ of G_0:

$$\boxed{V = \bigoplus_{\lambda}^{\text{certain irreps}} V_\lambda} \qquad \text{(Ten-7.1)}$$

In each vector space V_λ one can choose a (orthonormal) basis of the form

G_0 acts only on $|w^{(\lambda)}\rangle_k$

H acts only on $|v^{(\lambda)}\rangle_\alpha$

$$|v^{(\lambda)}\rangle_\alpha \otimes |w^{(\lambda)}\rangle_k$$

$k = 1, \ldots, d_\lambda$ = dimension of the irrep. λ

$\alpha = 1, \ldots, m_\lambda$ = multiplicity with which λ occurs in the vector space V

Thus each irrep. λ occurring in the decomposition of V [eq. (Ten-7.1)] defines a block Hamiltonian $H^{(\lambda)}$ which is a $m^{(\lambda)} \times m^{(\lambda)}$ - matrix with matrix elements

$$\boxed{H^{(\lambda)}_{\alpha,\beta} = {}_{\alpha}\langle V^{(\lambda)} | H | V^{(\lambda)} \rangle_{\beta}} \qquad \alpha,\beta = 1,\dots, m_{\lambda} = \text{(multiplicity)}$$

→: Let us ask the following question:

Fix a symmetry Group G_0 and consider all possible single-particle Hamiltonians H which commute with all symmetry operations in G_0 which are unitarily realized on the single-particle Hilbert space.

As we run through the set of all these Hamiltonians, what sets of matrices does one obtain for the blocks $H^{(\lambda)}$?

Answer:

The resulting set of block Hamiltonians $H^{(\lambda)}$ is independent of the symmetry group, and essentially independent of the irrep. λ.

Remarkable fact:

It turns out that there are only 10 possible such sets of matrices $H^{(\lambda)}$:

A complete list of the corresponding possible time-evolution operators

$$\boxed{U^{(\lambda)}(t) = \exp\{it\,H^{(\lambda)}\}}$$

are listed in Table "Ten Fold Way".

Why is this useful?

This makes the problem of listing all Hamiltonians (= all blocks) into finite problem, no matter what the symmetry group G_0 is, and no matter what the irrep.

What is behind this result?

- : Any symmetry in quantum mechanics must be realized by a unitary (= linear) or anti-unitary (= anti-linear) operator in Hilbert space [Wigner-von Neuman Theorem]

- : Because we have already exhausted the properties of the Hamiltonian following from any unitarily realized symmetries (in G_0), the only properties the block Hamiltonians can depend on are the behavior under anti-unitarily realized symmetries.

As we will review now, there are only very few anti-unitarily realized symmetry operations (modulo unitarily realized ones).

This is the reason why there are only 10 possibilities for the blocks $H^{(\lambda)}$.

Time-reversal symmetry:

$\hat{J}$ = implements time-reversal on the Fermion Fock space of 2^{nd} quantization:

DEF:

$$\hat{J}\,\hat{\psi}_A\,\hat{J}^{-1} = \sum_B (u_T^+)_{AB}\,\hat{\psi}_B \quad ; \quad \hat{J}\,\hat{\psi}_A^+\,\hat{J} = \sum_B \hat{\psi}_B^+\,(u_T)_{BA}$$

$$\hat{J}\, i\, \hat{J} = (-i)\,, \quad \text{(anti-unitary)}$$

where u_T = unitary $N \times N$-matrix (to preserve the canonical commutation relations)

2^{nd} quantized Hamiltonian $\hat{H}$ $\hat{J}$-invariant, if and only if

$$\boxed{\hat{J}\,\hat{H}\,\hat{J}^{-1} = \hat{H}} \quad (*)$$

One immediately checks that this is equivalent to the condition

$$\boxed{u_T\, H^*\, u_T^+ = H} \quad (**)$$

on the 1^{st} quantized Hamiltonian H.

→: Convenient to introduce

$$\boxed{T := \hat{J}\Big|_{1^{st}\text{ quantized}}} \quad \text{(Ten-12.1)}$$

Then (**) (p. Ten-11) can be written as

$$\boxed{T H T^{-1} = H, \quad \text{where } T = u_T \cdot K} \quad \text{(Ten-12.2)}$$

Here K = anti-unitary complex conjugation operator

$$\boxed{K H K^{-1} = H^*} \quad \text{(Ten-12.3)}$$

(H^*: all matrix-elements complex conjugated)

and

$$\boxed{K^2 = \mathbb{1}} \quad \text{(Ten-12.4)}$$

→: One immediately checks that $\hat{J}^2$, which is a unitary operator $\mathcal{U}$ [as on p. (Ten-6)], has an ~~where the~~ associated unitary matrix

$$\boxed{\mathcal{U} = u_T\, u_T^*} \quad \text{(Ten-12.5)}$$

[This follows also from Eq. (Ten-12.2) above:

$$T^2 = u_T K u_T K = u_T (K u_T K^{-1}) = u_T u_T^* \; .$$
]

→: It follows from Eq ⊛ / p. Ten-11 :

$$\boxed{\hat{J}^2 \hat{H} \hat{J}^{-2} = \hat{J}(\hat{J}\hat{H}\hat{J}^{-1})\hat{J}^{-1} = \hat{H}}$$

This implies the corresponding statement for the 1^{st}-quantized operators:

$$\boxed{(u_T u_T^*) H = H (u_T u_T^*)} \Leftarrow$$

since: $\hat{J}^2 \leftrightarrow u_T u_T^*$ (1^{st} quant)

Since H will run over an irreducible space of Hamiltonians (as we will see later), we must have [by Schur's Lemma]:

$$\boxed{u_T u_T^* = e^{i\gamma} \cdot \mathbb{1}}$$

[Alternatively, on physical grounds, $(\hat{J})^2$ should give back the same state, up to possibly a phase.]

→: Now consider

$$u_T u_T^* u_T = \underbrace{(u_T u_T^*) u_T = u_T (u_T^* u_T)}_{\Downarrow}$$

$$e^{i\gamma} \cdot u_T = u_T \cdot e^{-i\gamma}$$

$$\Rightarrow: \quad e^{2i\gamma} u_T = u_T$$

$$\Rightarrow \quad e^{2i\gamma} = 1$$

$$\Rightarrow \quad \boxed{e^{i\gamma} = \pm 1}$$

so that

$$\boxed{T^2 = u_T u_T^* = \pm 1}$$

Conclusion:

There are three ways a Hamiltonian can respond to time-reversal symmetry:

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$$\mathcal{T}^2 \left(= \mathcal{T} \right) = \begin{cases} 0 & H \text{ is not time-rev. inv.} \\ +1 & H \text{ is time-rev. inv. and } \mathcal{T}^2 = +1 \\ -1 & \text{" " " " } \mathcal{T}^2 = -1 \end{cases}$$

sometime we write for short "$\mathcal{T}$"

When $\mathcal{T}^2 = -1$,

$$\boxed{\hat{\mathcal{J}}^2 \vec{\psi}_A \hat{\mathcal{J}}^{-2} = (-1) \vec{\psi}_A, \quad \hat{\mathcal{J}}^2 \vec{\psi}^{\dagger}_A \hat{\mathcal{J}}^{-2} = -\hat{\psi}^{\dagger}_A}$$

(since $\mathcal{U} = u_T u_T^* = -1$)

Since a state in the Fock space with q fermions is created from the Fock space vacuum by applying q fermion creation operators, we have

$$\boxed{\begin{aligned} \hat{\mathcal{J}}^2 &= (-1)^{\hat{Q}} = \text{Fermion number parity operator} \\ \hat{Q} &= \sum_A \vec{\psi}^{\dagger}_A \vec{\psi}_A \end{aligned}}$$