THE STATISTICAL THEORY OF MESOSCOPIC NOISE

A short review

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Abstract. Microscopic theory of counting statistics of electrical noise is reviewed. We discuss a model of passive charge detector based on current fluctuations coupled to a spin, and its relation with the theory of photon counting in quantum optics. The statistics of tunneling current and, in particular, the properties of the third moment are studied in detail. The third moment is shown to be temperature-independent for tunneling in a generic many-body system. Then the statistics of mesoscopic transport is discussed. We consider applications of the functional determinant formula for the generating function of counting distribution to the DC and photo-assisted transport, and to mesoscopic pumping. A universal dependence of the noise in a mesoscopic pump on the pumping fields is obtained and shown to provide a method of measuring the quasiparticle charge in an open system without any fitting parameters.
Key words: counting statistics, third moment, photo-assisted transport, mesoscopic pumping

1. Introduction

The measurements performed by optical detectors, such as photon counters, are extended in the time domain, which makes them sensitive to temporal correlations of photons [1]. It has been known long ago in the theory of photodetection [2] that understanding photon counting is essentially a problem of many-particle statistics. Similar considerations apply to the electrical noise measurement, although it differs from photodetection in that the electrons, unlike photons, are not destroyed in the process of counting. The noise measurement, very much like photodetection, is a sensitive probe of temporal correlations between electrons.

Fermi correlations in the electron noise were originally studied by Leso-vik [3] (see also Ref. [4]) in a point contact, and then by Büttiker [5] in multiterminal systems, and by Beenakker and Büttiker [6] in mesoscopic conductors. Kane and Fisher proposed to employ the shot noise for detecting fractional quasiparticles in a Quantum Hall edge system [7]. Subsequent theoretical developments are summarized in a recent review [8].

Experimental studies of the shot noise, after first measurements in a point contact by Reznikov et al. [9] and Kumar et al. [10], focused on the quantum Hall regime. The fractional charges $e/3$ and $e/5$ were observed [11, 12, 13] at incompressible Landau level filling (see also recent work on noise at intermediate filling [14]). The shot noise in a mesoscopic conductor was observed by Steinbach et al. [15] and Schoelkopf et al. [16], who also studied noise in photo-assisted phase-coherent mesoscopic transport [17].

In this article we discuss counting statistics of electric noise and consider the probability distribution of charge transmitted in a fixed time interval [18, 19]. This distribution provides detailed information about current fluctuations. The counting statistics have been studied for the DC transport of free fermions [18, 20], the photo-assisted transport [21], the parametrically driven transport [19, 22], and in the mesoscopic regime [23] (also, see a review [24]). Nazarov developed Keldysh formalism for the counting statistics problem and applied it to mesoscopic transport in a weak localization regime [25] and, together with Bagrets, in a multiterminal geometry [26]. Charge doubling due to Andreev scattering in NS junctions was considered by Muzykantskii and Khmelnitskii [27], and in mesoscopic NS systems by Belzig and Nazarov [28]. Andreev and Kamenev [29] studied the problem of mesoscopic pumping in view of the results of Ref. [19]. Taddei and Fazio discussed counting statistics of entangled electron sources [30]. Statistics of transport in a Coulomb blockade regime was studied by Bagrets and Nazarov [31]. Photon statistics was considered by Beenakker and Schomerus [32] and Kindermann et al. [33]. The problem of back influence of a charge detector on current fluctuations in the context of counting statistics measurement was studied by Nazarov and Kindermann [37].

The possibility of measuring counting statistics using a fast charge integrator scheme was considered recently [34]. From the measured distribution all moments of charge fluctuations can be calculated and, conversely, the knowledge of all moments is in principle sufficient for recovering the full distribution. However, due to the central limit theorem, high moments are probably difficult to access experimentally. Therefore recent literature focused primarily on the third moment. It was found that the third moment obeys a generalized Schottky relation which holds in the tunneling regime at both high and low temperature, but involves a temperature-dependent Fano factor in the mesoscopic regime [22, 35]. Gutman and Gefen [35] stud-

ied the third moment using a sigma model approach, while Nagaev [36] demonstrated that all moments are correctly reproduced by an extension of the Boltzmann-Langevin kinetic equation.

In this article, after introducing the counting distribution (Sec.2), we review its microscopic definition based on a passive charge detector (Sec.2.1). In Sec.2.2 we study the statistics of tunneling in a generic many-body system. From the microscopic approach of Sec.2.1 we derive a bidirectional Poisson distribution for tunneling current, obtain a Schottky-like relation for the third moment and discuss its robustness. After that we discuss the relation of the counting statistics theory and the theory of photo-detection (Sec.2.3). Then we proceed to the problem of mesoscopic transport. In Sec.3.1 we review the results on the DC transport and the derivation of a functional determinant formula for the counting distribution generating function. In Sec.3.3 we review the work on the AC transport statistics, and then consider the problem of mesoscopic pumping (Sec.3.4). The counting statistics for generic pumping strategy at weak pumping is given by a bidirectional Poisson distribution. We show that the Fano factor varies between 0 and 1 as a function of the pumping fields phase difference.

2. General approach

The transmitted charge distribution can be characterized [18, 19] by electron counting probabilities p_n, usually accumulated in a generating function[1]

$$\chi(\lambda) = \sum e^{in\lambda} \, p_n \,. \tag{1}$$

The function $\chi(\lambda)$ is 2π-periodic in the *counting field* λ and has the property $\chi(0) = 1$ which follows from the probability normalization $\sum_n p_n = 1$. The term "counting field" will be motivated in Sec.2.1, where a microscopic definition of $\chi(\lambda)$ is discussed in which λ appears as a field that couples current fluctuations to a charge detector.

The generating function (1) is particularly well suited for characterizing statistics of the distribution p_n. The so-called *irreducible correlators* $\langle\!\langle \delta n^k \rangle\!\rangle$ (also known as *cummulants*) are expressed in terms of $\chi(\lambda)$ as

$$\ln \chi(\lambda) = \sum_{k=1}^{\infty} m_k \frac{(i\lambda)^k}{k!} \,, \qquad m_k \equiv \langle\!\langle \delta n^k \rangle\!\rangle \,. \tag{2}$$

The first two correlators in (2) give the mean and the variance:

$$m_1 = \overline{n}, \qquad m_2 = \overline{\delta n^2} = \overline{n^2} - \overline{n}^2, \tag{3}$$

[1]The function $\chi(\lambda)$ is also called a characteristic function [1].

where $\overline{f(n)}$ stands for $\sum_n f(n)p_n$. The third correlator[2]

$$m_3 = \langle\langle \delta n^3 \rangle\rangle \equiv \overline{\delta n^3} = \overline{(n - \overline{n})^3} \tag{4}$$

characterizes the asymmetry (or skewness) of the distribution p_n.

To illustrate the notion of a generating function, let us consider a Poisson process. It describes charge transport at very low transmission, with uncorrelated transmission events. For the Poisson distribution

$$p_k = \begin{cases} e^{-\bar{n}}\, \bar{n}^k/k! & k \geq 0 \\ 0 & k < 0 \end{cases} \quad \text{and} \quad \chi(\lambda) = \exp\left((e^{i\lambda} - 1)\bar{n} \right), \tag{5}$$

where $\bar{n} = It/e$ is the average number of particles transmitted during time t, with I the time-averaged current and e the elementary charge. Comparing (5) with (2), one finds that all cummulants of the Poisson distribution are identical: $m_k = \bar{n}$.

Another useful example is binomial statistics. A binomial distribution arises when a fixed number N of independent attempts to transmit particles is made, each attempt successful or unsuccessful with probabilities p and $q = 1 - p$. The probability to transmit k particles in this case is determined by the combinatorial number $C_N^k = N!/(N-k)!k!$ of k successful outcomes. The probability distribution and the generating function in this case are

$$p_k = C_N^k p^k q^{N-k} \quad \text{and} \quad \chi(\lambda) = \left(p e^{i\lambda} + q \right)^N. \tag{6}$$

The cummulants of the binomial distribution (6) can be found from (2) by expanding $\ln\chi$ in λ:

$$m_1 = pN, \quad m_2 = pqN, \quad m_3 = pq(q - p)N, \quad \dots \tag{7}$$

The binomial distribution (6) describes counting distribution of DC current noise for a single channel scatterer, such as point contact, at zero temperature (Sec.3.1).

Statistically independent processes result in a generating function given by a product of generating functions for constituting processes: $\chi(\lambda) = \chi_1(\lambda)...\chi_k(\lambda)$. For example, consider *a biderectional Poisson distribution* defined as a mixture of two independent Poisson processes transmitting particles in opposite directions with the rates $\bar{n}$ and $\bar{n}'$. In this case,

$$\chi_{2P}(\lambda) = \exp\left((e^{i\lambda} - 1)\bar{n} \right) \cdot \exp\left((e^{-i\lambda} - 1)\bar{n}' \right). \tag{8}$$

[2]The relation between cummulants and correlators is generally more complicated than Eq.(4) for the third cummulant. For example, $m_4 = \langle\langle \delta q^4 \rangle\rangle = \overline{\delta q^4} - 3\left(\overline{\delta q^2}\right)^2$.

In Sec.2.2 we use the distribution (8) to describe statistics of tunneling current. In Sec.3.4 we show that it describes noise a mesoscopic pump.

2.1. A MICROSCOPIC REPRESENTATION OF $\chi(\lambda)$

Here we discuss a microscopic definition of counting statistics for a physical system. Adopting an inductive approach, we shall start with a specific model of current detector [20, 24]. We obtain the generating function $\chi(\lambda)$ for a particular current measurement scheme, and then argue that it describes generic measurement.

In a realistic noise measurement, e.g. in a mesoscopic wire or a point contact, the current fluctuations are not detected directly. Instead, the measurement is performed on the electromagnetic fluctuations (basically, voltage noise) induced by current fluctuations in the system. The electromagnetic fluctuations have to be amplified before being detected. The conversion of underlying microscopic fluctuations due to fermions (electrons, fractional charges, etc.) into fluctuations of bosons (photons) is crucial, since Bose fields can be amplified without compromising noise statistics, while Fermi statistics is not consistent with amplification.

Our goal is to clarify the microscopic picture of current fluctuations, rather than to describe realistic measurements. Thus we choose a *gedanken* measurement scheme well suited for that purpose. Consider a spin 1/2 placed near an electron system and magnetically coupled to the electric current. We restrict the coupling to the spin z component, so that the system in the presence of the spin is described by $\mathcal{H}(\mathbf{q}, \mathbf{p} - \mathbf{a}\sigma_3)$, where $\mathcal{H}(\mathbf{q}, \mathbf{p})$ is the electron Hamiltonian and $\mathbf{a}(r)$ is the spin vector potential scaled by e/c.

The scheme of current detection using such spin dynamics can be motivated quasiclassically. A spin coupled to a time-dependent classical current $I(t)$ by the interaction $\mathcal{H} = \frac{1}{2}\lambda\sigma_3 I(t)$ will precess at the rate proportional to current, which turns the spin into an analog galvanometer. Indeed, if the spin-current coupling is turned on at $t = 0$, the spin will start precessing around the z axis with the precession angle $\theta(t) = \lambda \int_0^t I(t')dt'$ equal to the transmitted charge times λ. The coupling constant λ, so far arbitrary, will be associated with counting field below.

In a fully quantum-mechanical problem, the spin evolution can be obtained from $i\dot{\sigma} = [\sigma, \mathcal{H}]$. Since the spin-current coupling Hamiltonian commutes with σ_3, the spin dynamics can be found explicitly. For that we consider the transverse spin components $\sigma_\pm \equiv \sigma_1 \pm i\sigma_2$ and write the evolution equation $i\dot{\sigma}_\pm = [\sigma_\pm, \mathcal{H}]$ in the form

$$i\dot{\sigma}_+ = \sigma_+ P_\downarrow \mathcal{H}(\mathbf{q}, \mathbf{p} + \mathbf{a}) - \mathcal{H}(\mathbf{q}, \mathbf{p} - \mathbf{a})P_\uparrow \sigma_+ , \qquad (9)$$

$$i\dot{\sigma}_- = \sigma_- P_\uparrow \mathcal{H}(\mathbf{q}, \mathbf{p} - \mathbf{a}) - \mathcal{H}(\mathbf{q}, \mathbf{p} + \mathbf{a})P_\downarrow \sigma_- , \qquad (10)$$

with $P_{\uparrow,\downarrow} = \frac{1}{2}(1 + \sigma_3)$ the up and down spin projectors. Here we used the raising/lowering properties of the operators $\sigma_\pm$ and replaced σ_3 by 1 to the left of σ_+ and by -1 to the right of σ_+ (and similarly for the σ_- equation).

We consider a measurement which is performed during time interval $0 < t' < t$, i.e. start with a free spin, $\mathbf{a}_{t<0} = 0$, couple it to the electron system at $t = 0$, maintain a finite coupling during $0 < t' < t$, and then turn it off. The expectation value of the transverse spin component at the time t, found by integrating Eqs. (9),(10), is

$$\langle \sigma_+(t) \rangle = \langle e^{i\mathcal{H}(\mathbf{q},\mathbf{p}-\mathbf{a})t} e^{-i\mathcal{H}(\mathbf{q},\mathbf{p}+\mathbf{a})t} \rangle_{\text{el}} \langle \sigma_+(0) \rangle_{\text{spin}}, \qquad (11)$$

while $\langle \sigma_-(t) \rangle = \langle \sigma_+(t) \rangle^*$. Note that the result of the coupled spin and current evolution factors into a product of quantities that depend separately on electron dynamics and on the initial state of the spin, as indicated by the subscripts.

The effect of current on spin precession is described by the dependence of the first term in Eq.(11) on the gauge field $\mathbf{a}$. To make contact with the quasiclassical discussion above, let us expand $\mathcal{H}$ in $\mathbf{a}$, assuming it to be small. The result is $\mathcal{H}(\mathbf{q}, \mathbf{p} \pm \mathbf{a}) = \mathcal{H}(\mathbf{q}, \mathbf{p}) \pm \mathbf{a}\mathbf{j}$, where $\mathbf{j}$ is electric current. Substituting this back in Eq.(11) and passing to the interaction representation with respect to the Hamiltonian of fermions uncoupled from the spin, we rewrite the average $\langle ... \rangle_{\text{el}}$ in (11) as

$$\left\langle \tilde{\text{T}}\text{exp}\left(i\int_t^0 \int \mathbf{a}\mathbf{j}(t')d^3r\,dt'\right) \text{Texp}\left(-i\int_0^t \int \mathbf{a}\mathbf{j}(t')d^3r\,dt'\right) \right\rangle_{\text{el}}. \qquad (12)$$

Let us consider a specific form of $\mathbf{a}$, taking it to be a pure gauge, $\oint \mathbf{a}d\mathbf{l} = 0$, within the electron system, and nonzero near a particular surface (e.g. a $\delta-$function on the surface). For a classical current, ignoring noncommutativity of current operators at different times, the expression (12) becomes

$$\left\langle e^{-i\theta(t)} \right\rangle_{\text{el}} \quad \text{with} \quad \theta(t) = \lambda \int_0^t I(t')dt'. \qquad (13)$$

Here $I(t)$ is the total current through the surface and $\lambda = -2\int \mathbf{a}d\mathbf{l}$, where the integral is taken across the surface. The form of Eq.(13) agrees with what one expects for the precession phase factor averaged over classical current fluctuations. For n electrons transmitted through the system during the measurement time, the precession angle is $\theta(t) = \lambda n$. This relates the average in (13) with the transmitted charge distribution:

$$\left\langle e^{-i\theta(t)} \right\rangle_{\text{el}} = \sum_n e^{i\lambda n} p_n. \qquad (14)$$

The relation with the spin precession in this case can be seen more clearly by combining the result (14) with Eq.(11),

$$\langle \sigma_+(t) \rangle = \sum_n p_n \left(e^{i\lambda n} \langle \sigma_+(0) \rangle_{\text{spin}} \right) ,$$ (15)

and recalling the transformation rule $\sigma'_+ = e^{i\theta}\sigma_+$ for spin rotation around the z-axis by an angle θ. This way of writing the result of spin evolution confirms the expected relationship between the charge counting probability distribution and the distribution of the spin precession angles.

This discussion clarifies the meaning of the quantity $\langle ... \rangle_{\text{el}}$ in Eq.(11), linking it to the counting distribution generating function. Motivated by this, we use Eq.(11) to give a microscopic definition of counting statistics. We rewrite the quantity $\langle ... \rangle_{\text{el}}$ as a Keldysh partition function

$$\chi(\lambda) = \left\langle \mathrm{T_K} \exp \left(-i \int_{C_{0,t}} \hat{\mathcal{H}}_\lambda(t') dt' \right) \right\rangle_{\text{el}}$$ (16)

with the integral taken over the Keldysh time contour $C_{0,t} \equiv [0 \to t \to 0]$, first forward and then backward in time. The counting field λ is related to the gauge field $\mathbf{a}$ via $\lambda = \mp\frac{1}{2} \int \mathbf{a}d\mathbf{l}$, where the integration path goes across the region where scattering takes place and noise is generated (e.g. across the barrier in the point contact). The sign $\mp$ indicates that the field $\mathbf{a}$ is antisymmetric on the upper and lower parts of the Keldysh contour. Because of that, even though $\mathbf{a}$ resembles in many ways an ordinary electromagnetic gauge field (allowing for gauge transformations, etc.), it has no such meaning. We emphasize that $\mathbf{a}$ is really an auxiliary field describing coupling with a *virtual* measurement device, such as the spin 1/2 above.

The microscopic formula (16), originating from the analysis of a coupling with spin 1/2, is in fact adequate for any ideal "passive charge detector" without internal dynamics. We shall use this formula below to obtain counting statistics for several physical situations of interest, including tunneling and mesoscopic transport.

This still leaves some questions about universality and limitations of Eq.(16). Nazarov and Kindermann [37] considered a more general scheme of charge detection and recovered the expression (16). Although this is reassuring, Ref. [37] concludes that the detector back action is inevitable. Thus it is still desirable to study more realistic models of noise detection that include conversion of microscopic current noise into electromagnetic field (photons) as well as an amplifier. Some aspects of electron-to-photon noise conversion were studied by Beenakker and Schomerus [32].

It is also of interest to compare the back action effects in different models. We argue that the above scheme is likely to describe noise measurement

with the least back action, since coupling to the precessing spin $1/2$ affects only the phase of electron forward scattering amplitude, without changing scattering probabilities.

2.2. STATISTICS OF THE TUNNELING CURRENT

The problem of the tunneling current noise provides a simple test for the microscopic formula (16). The starting point of our analysis will be the tunneling Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2 + \hat{V}$, where $\hat{\mathcal{H}}_{1,2}$ describe the leads and $\hat{V} = \hat{J}_{12} + \hat{J}_{21}$ is the tunneling operator. The specific form of the operators $\hat{J}_{12}$, $\hat{J}_{21}$ that describe tunneling of a quasiparticle between the leads will not be important for the most of our discussion. Both the discussion and the results for the tunneling current statistics obtained in this section are valid for a generic interacting many-body system.

The counting field **a** in this case can be taken localized on the barrier, entering the Hamiltonian through the phase factors $\exp(\pm i \int \mathbf{a} d\mathbf{l}) = \exp(\pm i\lambda/2)$ of the operators $\hat{J}_{12}$, $\hat{J}_{21}$. The tunneling operator then is

$$\hat{V}_\lambda = e^{\frac{i}{2}\lambda(t)} \hat{J}_{12}(t) + e^{-\frac{i}{2}\lambda(t)} \hat{J}_{21}(t) \,. \tag{17}$$

Here $\lambda(t) = \pm\lambda$ is antisymmetric on the Keldysh contour $C_{0,t}$.

In what follows we compute $\chi(\lambda)$ and establish a relation with the Kubo theorem for the tunneling current [39]. For that, we perform the usual gauge transformation turning the bias voltage into the tunneling operator phase factor as $\hat{J}_{12} \to \hat{J}_{12} e^{-ieVt}$, $\hat{J}_{21} \to \hat{J}_{21} e^{ieVt}$. Passing to the Keldysh interaction representation, we write

$$\chi(\lambda) = \left\langle \mathrm{T_K} \exp\left(-i \int_{C_{0,t}} \hat{V}_{\lambda(t')}(t') dt' \right) \right\rangle . \tag{18}$$

Diagrammatically, the partition function (18) is a sum of linked cluster diagrams with appropriate combinatorial factors. To the lowest order in the tunneling operators $\hat{J}_{12}$, $\hat{J}_{21}$ we only need to consider linked clusters of the second order. This gives

$$\chi(\lambda) = e^{W(\lambda)}, \quad W(\lambda) = -\frac{1}{2} \iint_{C_{0,t}} \left\langle \mathrm{T_K} \hat{V}_{\lambda(t')}(t') \hat{V}_{\lambda(t'')}(t'') \right\rangle dt' dt''. \tag{19}$$

This result is correct for the measurement time t much larger than the correlation time in the contacts that determines the characteristic time separation $t' - t''$ at which the correlator in (19) decays.

There are several different contributions to the integral in (19), arising from t' and t'' taken on the forward or backward parts of the contour $C_{0,t}$.

Evaluating them separately, we obtain

$$W(\lambda) = \int_0^t \int_0^t \left\langle \hat{V}_{-\lambda}(t')\hat{V}_\lambda(t'') \right\rangle dt''dt' \tag{20}$$

$$- \int_0^t \int_0^{t'} \left\langle \hat{V}_\lambda(t')\hat{V}_\lambda(t'') \right\rangle dt''dt' - \int_0^t \int_{t'}^t \left\langle \hat{V}_{-\lambda}(t')\hat{V}_{-\lambda}(t'') \right\rangle dt''dt'.$$

We substitute the expression (17) in Eq.(20) and average by pairing $\hat{J}_{12}$ with $\hat{J}_{21}$. This gives

$$W(\lambda) = (e^{i\lambda}-1)N_{1\to2}(t) + (e^{-i\lambda}-1)N_{2\to1}(t) \tag{21}$$

with

$$N_{j\to k} = \int_0^t \int_0^t \langle \hat{J}_{kj}(t')\hat{J}_{jk}(t'') \rangle\, dt'dt'' = g_{jk}t \tag{22}$$

the mean charge transmitted from the contact j to the contact k in a time t. Exponentiating Eq.(21) gives nothing but the bidirectional Poisson distribution $\chi_{2P}(\lambda)$ defined by Eq.(8) with the transition rates given by $\bar{n} = N_{1\to2} = g_{12}t$, $\bar{n}' = N_{2\to1} = g_{21}t$, respectively.

Eq.(8) yields interesting relations between different statistics of the distribution. The cummulants $\langle\langle \delta n^k \rangle\rangle$, obtained by expanding $\ln \chi_{2P}(\lambda)$ in λ, are

$$m_k = \langle\langle \delta n^k \rangle\rangle = \begin{cases} (g_{12} - g_{21})t, & k \text{ odd;} \\ (g_{12} + g_{21})t, & k \text{ even.} \end{cases} \tag{23}$$

Setting $k = 1, 2$ we express $g_{12} \pm g_{21}$ through the time-averaged current and the low frequency noise spectral density[3]:

$$g_{12} - g_{21} = I/q_0, \quad g_{12} + g_{21} = \mathcal{S}_0/2q_0^2, \tag{24}$$

with q_0 the tunneling charge. Of special interest for us will be the third correlator of the transmitted charge

$$\langle\langle \delta q^3 \rangle\rangle \equiv \overline{\left(\delta q - \overline{\delta q} \right)^3}. \tag{25}$$

For this correlator Eq. (23) gives $\langle\langle \delta q^3 \rangle\rangle = C_3 t$ with the coefficient C_3 (the spectral density of the third correlator at $\omega = 0$) related to current as

$$C_3 \equiv \langle\langle \delta q^3 \rangle\rangle/t = q_0^2 I. \tag{26}$$

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[3]The spectral density of the noise is defined through the symmetrized current correlator as $\mathcal{S}_\omega = \int \left\{ \overline{\delta I(t), \delta I(0)} \right\}_+ e^{i\omega t}dt$. At $\omega = 0$, one can write $\mathcal{S}_0$ in terms of the variance of charge $q(\tau) = \int_t^{t+\tau} I(t')dt'$ transmitted during a long time τ as $\mathcal{S}_0 = \frac{2}{\tau}\overline{\delta q^2(\tau)}$.

We note that the relation (26) holds for the distribution (8) at any ratio $(g_{12} - g_{21})/(g_{12} + g_{21})$ of the mean transmitted charge to the variance.

The quantities (22) have several general properties. First, by writing the expectation values (22) in a basis of exact microscopic states and using the detailed balance relation, we obtain

$$N_{1\to 2}/N_{2\to 1} \equiv g_{12}/g_{21} = \exp(eV/k_{\mathrm{B}}T)\,, \qquad (27)$$

where V is the voltage applied to the contacts. Using this result to calculate the ratio of the first and second cummulants, Eq.(24), we have $(g_{12} - g_{21})/(g_{12} + g_{21}) = \tanh(eV/k_{\mathrm{B}}T)$. This gives the noise-current relation

$$\mathcal{S}_0 = 2q_0 \, \coth(eV/k_{\mathrm{B}}T) \, I \qquad (28)$$

that holds for arbitrary $eV/k_{\mathrm{B}}T$. This relation was pointed out by Sukhorukov and Loss [38].

Also, one can establish a relation of the quantities (22) with the Kubo theorem. We consider the tunneling current operator

$$\hat{\mathcal{I}}(t) = -iq_0 \left(\hat{J}_{12}(t) - \hat{J}_{21}(t) \right). \qquad (29)$$

From the Kubo theorem for the tunneling current [39], the mean integrated current $\int_0^t \langle \hat{\mathcal{I}}(t') \rangle dt'$ is

$$q_0 \int_0^t \int_0^t \left\langle \left[\hat{J}_{21}(t'), \hat{J}_{12}(t'') \right] \right\rangle dt' dt'' = q_0 \left(N_{1\to 2} - N_{2\to 1} \right). \qquad (30)$$

By writing $N_{j\to k} = g_{jk}t$, we confirm the first relation (24). To obtain the second relation (24) we consider the variance of the charge transmitted in time t, given by $\langle\langle \delta q^2 \rangle\rangle = q_0^2 \int_0^t\int_0^t \langle \{\hat{\mathcal{I}}(t'), \hat{\mathcal{I}}(t'')\}_+ \rangle \, dt' dt''$. This integral can be rewritten as

$$\int_0^t \int_0^t \left\langle \left\{ \hat{J}_{12}(t'), \hat{J}_{21}(t'') \right\}_+ \right\rangle dt' dt'' = N_{1\to 2} + N_{2\to 1}\,, \qquad (31)$$

which immediately leads to the second relation (24).

We conclude that the tunneling current statistics, described by Eq.(8), are simpler than in a generic system. The current-noise relation, typically known in a generic system only at equilibrium (Nyquist) and in the fully out-of-equilibrium (Schottky) regimes, for the tunneling current is given by Eq.(28) at arbitrary $eV/k_{\mathrm{B}}T$.

In contrast, the relation (26) obeyed by the third correlator (4) is completely insensitive to the crossover between the Nyquist and Schottky noise regimes. The meaning of Eq.(26) is similar to that of the Schottky formula

$\mathcal{S}_0 = 2\langle\!\langle \delta q^2 \rangle\!\rangle = 2q_0 I$. However, the Schottky current-noise relation is valid only when charge flow is unidirectional, i.e. at low temperatures $k_B T \ll eV$, since $g_{12}/g_{21} = \exp(eV/k_B T)$, while Eq.(26) holds at any $eV/k_B T$.

In experiment, when the current-noise relation is used to determine the tunneling quasiparticle charge q_0 from the tunneling current noise, it is crucial to maintain low temperature $k_B T \ll eV$. The requirement of a cold sample at a relatively high bias voltage is the origin of a well known difficulty in the noise measurement. In contrast, the relation (26) is not constrained by any requirement on sample temperature.

This property of the third moment, if confirmed experimentally, may prove to be quite useful for measuring quasiparticle charge. In particular, this applies to the situations when the $I - V$ characteristic is strongly nonlinear, when it is usually difficult to unambiguously interpret the noise *versus* current dependence as a shot noise effect or as a result of thermal noise generated by non-linear conductance. This appears to be a completely general problem pertinent to any interacting system. Namely, in the systems such as Luttinger liquids, the $I - V$ nonlinearities arise at $eV \geq k_B T$. However, it is exactly this voltage that has to be applied for measuring the shot noise in the Schottky regime.

Finally, we note that the universality of the third moment is specific for the tunneling problem. In other situations, such as a point contact or a mesoscopic system, the third moment is temperature-dependent [34, 35, 36].

2.3. A RELATION TO THE THEORY OF PHOTODETECTION

The statistics of tunneling particles was not specified in the above discussion, since everything said so far is good for both bosons and fermions. To illustrate this, here we discuss the relation of the present approach to the theory of photon counting [1, 2]. A system of photons interacting with atoms in a photon detector can be accounted for by a Hamiltonian of the form $\mathcal{H} = \mathcal{H}_p + \mathcal{H}_a + V$, where $\mathcal{H}_p$ describes free electromagnetic field, $\mathcal{H}_a$ is the Hamiltonian of atoms in the detector, and

$$\hat{V} = \sum_{j,k} \left(u_{j,k} e^{\frac{i}{2}\lambda} b_j^\dagger a_k + u_{j,k}^* e^{-\frac{i}{2}\lambda} a_k^\dagger b_j \right) \tag{32}$$

describes the interaction of photons with the atoms, i.e. the process of photon absorption and atom excitation. Here a_k are the canonical Bose operators of photon modes, labeled by k, and b_j are the operators describing excitation of the atoms. Since the operator $\hat{V}$ transfers excitations between the field and the atom systems, it can be interpreted as a "tunneling operator." (The only difference is in the unidirectional character of the "current" induced by $\hat{V}$, since photons can be only absorbed in the detector but not

created.) This analogy allows one to use the formalism of Sec.2.1 to study photon counting, and for that purpose we added a counting field in (32) (compare to Eq.(17)).

Given all that, the generating function for photons has the form (18) which we rewrite to show an explicit dependence on the measurement time:

$$\chi_t(\lambda) = \left\langle \mathcal{U}_{-\lambda}^{-1}(t)\mathcal{U}_\lambda(t) \right\rangle, \qquad \mathcal{U}_\lambda(t) = \text{Texp}\left(-i \int_0^t \hat{V}_\lambda(t')dt' \right), \qquad (33)$$

The task of evaluating the partition function (33) is simplified by the weakness of the photon-atom coupling. (Each atom is excited during the counting time t with a very small probability.) The expression (33) can thus be evaluated by taking into account the interaction of a photon with each of the atoms only to the lowest order. This is also similar to the tunneling problem.

However, at this stage the similarity with tunneling ends, since photon coherence time can be much longer than the measurement time t. The method of Sec.2.2, based on the linked cluster expansion of $\ln \chi$, should be modified to account for the long coherence times. Another complication is that the photon density matrix is not specified, since we are not limiting the discussion to thermal photon sources.

We handle the partition function (33) by averaging over atoms, while keeping the photon variables free. As explained above, only pairwise averages of atoms' operators are needed. We write them as $\langle b_j^\dagger(t)b_{j'}(t') \rangle = 0$, $\langle b_j(t)b_{j'}^\dagger(t') \rangle = \tau_j \delta(t - t')\delta_{jj'}$, where τ_j is a constant of the order of the excitation time of an atom, and the δ-function is actually a function of the width $\simeq \tau_j$. (Typically, τ_j is a very short, microscopic time.)

Turning to the calculation, let us consider the difference $\chi_{t+\Delta}(\lambda) - \chi_t(\lambda)$, with the time increment Δ large compared to τ_j, but much smaller than the characteristic photon coherence time. Expanding $\mathcal{U}_\lambda(t + \Delta)$ to the second order in Δ, we write it as

$$\left(1 - i \int_t^{t+\Delta} \hat{V}_\lambda(t')dt' - \frac{1}{2} \int_t^{t+\Delta} \int_t^{t'} \hat{V}_\lambda(t')\hat{V}_\lambda(t'')dt'dt'' \right) \mathcal{U}_\lambda(t). \qquad (34)$$

Substituting this in Eq.(33) along with a similar expression for $\mathcal{U}_{-\lambda}^{-1}(t+\Delta)$, and averaging over the atoms as described above, we obtain

$$\partial_t \chi_t(\lambda) = (\chi_{t+\Delta}(\lambda) - \chi_t(\lambda))/\Delta = \sum_k \eta_k (e^{i\lambda} - 1) \left\langle \mathcal{U}_{-\lambda}^{-1}(t)a_k^\dagger a_k \mathcal{U}_\lambda(t) \right\rangle \quad (35)$$

with $\eta_k = \sum_j \tau_j |u_{j,k}|^2$ the detector efficiency parameters. The solution of Eq.(35) has the form well known in optics [1, 2]:

$$\chi_t(\lambda) = \prod_k \chi_t^{(k)}(\lambda), \quad \chi_t^{(k)}(\lambda) = \left\langle : \exp\left(\eta_k t(e^{i\lambda} - 1)a_k^\dagger a_k \right) : \right\rangle_k, \qquad (36)$$

where $: ... :$ is the normal ordering symbol and $\langle ... \rangle_k$ is the average over photon density matrix. The product rule in (36) indicates that the counting distributions for different electromagnetic modes are statistically independent. The physical meaning of normal ordering is that each photon, after having been detected, is absorbed in the detector and destroyed.

From Eq.(36), the counting probability of m photons in one mode is

$$p_m^{(k)} = \frac{(\eta_k t)^m}{m!} \langle\, :(a_k^\dagger a_k)^m e^{-\eta_k t a_k^\dagger a_k} :\, \rangle_k .\tag{37}$$

Eqs.(36),(37) are central to the theory of photon counting [1]. Particularly interesting is the case of a coherent photon state $|z\rangle$, $a|z\rangle = z\,|z\rangle$, with a complex z, corresponding to the radiation field of an ideal laser. In this case Eq.(37) yields the Poisson distribution $p_m = e^{-Jt}(Jt)^m/m!$, $J = \eta|z|^2$, which describes the so-called minimally bunched light sources.

3. Counting statistics of mesoscopic transport

Here we consider the problem of counting statistics in a mesoscopic transport. From now on we adopt the noninteracting particle approximation and use the scattering approach [40], in which the system is characterized by a single particle scattering matrix. Depending on the nature of the problem, the matrix can be stationary or time-dependent. Even for noninteracting particles the problem of counting statistics remains nontrivial due to correlations between different particles arising from Fermi statistics.

Counting statistics can be analyzed using the microscopic formula (16). However, there is a more efficient way of handling the noninteracting problem. One can obtain a formula for the generating function $\chi(\lambda)$ in terms of a functional determinant that involves the scattering matrix and the density matrix of reservoirs. Then for each particular problem one has to analyze and evaluate an appropriate determinant. Although functional determinants can be nontrivial to deal with, this approach is still much simpler than the one based directly on Eq.(16).

3.1. STATISTICS OF THE DC TRANSPORT

He we discuss the problem of time-independent scattering. We consider a conductor with m scattering channels describing states within one or several current leads. The scattering is elastic and will be characterized by a $m \times m$ matrix S. Although in applications so far the 2×2 matrices (i.e. the problems with two scattering channels) have been more common than larger matrices, the general determinant structure of $\chi(\lambda)$ will be revealed only for matrices of arbitrary size m.

For elastic scattering one can obtain $\chi(\lambda)$ from a quasiclassical argument. In this case, particles with different energies contribute to counting statistics independently, and thus one can "symbolically" write

$$\chi(\lambda) = \prod_\epsilon \chi_\epsilon(\lambda)\,, \quad \text{i.e.} \quad \chi(\lambda) = \exp\left(t \int \ln \chi_\epsilon(\lambda) \frac{d\epsilon}{2\pi\hbar} \right), \qquad (38)$$

where $\chi_\epsilon(\lambda)$ is the contribution of particles with energy ϵ. The factor $2\pi\hbar$ is written based on the quasiclassical phase space volume normalization, $d\mathcal{V} = d\epsilon dt/2\pi\hbar$. The quantity $\chi_\epsilon(\lambda)$ depends on the scattering matrix S and on the energy distribution $n_i(\epsilon)$ in the channels.

To obtain $\chi_\epsilon(\lambda)$ we introduce a vector of counting fields λ_j, $j = 1, ..., m$, one for each channel, and consider all possible multi-particle scattering processes at fixed energy. The processes can involve any number $k \leq m$ of particles each coming out of one of the m channels and being scattered into another channel. Since the particles are indistinguishable fermions, no two particles can share an incoming or outgoing channel. One can then write $\chi_\epsilon(\lambda)$ as a sum over all different multiparticle scattering processes:

$$\chi_\epsilon(\lambda) = \sum_{i_1,...,i_k,j_1,...,j_k} e^{\frac{i}{2}(\lambda_{i_1}+...+\lambda_{i_k}-\lambda_{j_1}-...-\lambda_{j_k})} P_{i_1,...,i_k \,|\, j_1,...,j_k}\,, \qquad (39)$$

where the rate of k particles transition from channels $i_1, ..., i_k$ into channels $j_1, ..., j_k$ is given by

$$P_{i_1,...,i_k \,|\, j_1,...,j_k} = \left| S^{j_1,...,j_k}_{i_1,...,i_k} \right|^2 \prod_{i \neq i_\alpha} (1 - n_i(\epsilon)) \prod_{i = i_\alpha} n_i(\epsilon)\,. \qquad (40)$$

Here $S^{j_1,...,j_k}_{i_1,...,i_k}$ is an antisymmetrized product of k single particle amplitudes, which is nothing but the minor of the matrix S with rows $j_1, ..., j_k$ and columns $i_1, ..., i_k$. The product of n_i and $1 - n_i$ gives the probability to have k particles come out of the channels $i_1, ..., i_k$.

An important insight in the structure of the expression (39) can be obtained by noting that it has a form of a determinant:

$$\chi_\epsilon(\lambda) = \det\left(\hat{1} - \hat{n}_\epsilon + \hat{n}_\epsilon S^{-1}_{-\lambda} S_\lambda \right), \qquad (S_\lambda)_{ij} = e^{\frac{i}{4}(\lambda_i - \lambda_j)} S_{ij}. \qquad (41)$$

Here $\hat{n}_\epsilon$ is a diagonal $m \times m$ matrix of channel occupancy at the energy ϵ and the counting field λ_j enters in the phase factors of the matrix S_λ. To demonstrate that the expressions (39) and (41) are identical one has to expand the determinant (41) and go through a bit of matrix algebra. The formula (41) is particularly useful because, as we shall see below, it can be generalized to time-dependent problems.

Let us now focus on the simplest case $m = 2$ which describes transport in a point contact, with the two channels corresponding to current leads. The 2×2 scattering matrix S contains reflection and transmission amplitudes. In this case, since there are only six terms in Eq.(39), the determinant formula (41) is not necessary. From Eq.(39) we obtain

$$\chi_\epsilon(\lambda) = (1 - n_1)(1 - n_2) + (|S_{11}|^2 + e^{\frac{i}{2}(\lambda_2 - \lambda_1)}|S_{21}|^2)n_1(1 - n_2)$$
$$+ (|S_{22}|^2 + e^{\frac{i}{2}(\lambda_1 - \lambda_2)}|S_{12}|^2)n_2(1 - n_1) + |\det S|^2 n_1 n_2 , \quad (42)$$

where the energy dependence of $n_j(\epsilon)$ is suppressed. By using the unitarity relations $|S_{1i}|^2 + |S_{2i}|^2 = 1$, $|\det S| = 1$, Eq.(42) can be simplified:

$$\chi_\epsilon(\lambda) = 1 + p(e^{i\lambda} - 1)n_1(1 - n_2) + p(e^{-i\lambda} - 1)n_2(1 - n_1) . \quad (43)$$

Here $p = |S_{21}|^2 = |S_{12}|^2$ is the transmission coefficient and $\lambda = \lambda_2 - \lambda_1$. (We denote the transmission probability by p instead of a more traditional t to avoid confusion with the measurement time.)

To obtain the full counting statistics integrated over all energies, one has to specify the energy distribution in the leads and use Eq.(38). We consider a barrier with energy-independent transmission and the leads at temperature T biased by voltage V. Then $n_{1,2} = n_F(\epsilon \mp eV/2)$ with n_F the Fermi function.

At $T = 0$, since $n_F(\epsilon)$ takes values 0 and 1, for $V > 0$ we have

$$\chi_\epsilon(\lambda) = \begin{cases} e^{i\lambda}p + 1 - p , & |\epsilon| < \frac{1}{2}eV; \\ 1 , & |\epsilon| > \frac{1}{2}eV. \end{cases} \quad (44)$$

After doing the integral in Eq.(38) we obtain the binomial distribution (6), $\chi_N(\lambda) = (e^{i\lambda}p + 1 - p)^{N(t)}$, with the number of attempts $N(t) = eVt/2\pi\hbar$.

This means that, in agreement with intuition, in the energy window eV the transport is just the single particle transmission and reflection, while the states with energies in the Fermi sea, populated in both reservoirs, are noiseless. (At $V < 0$ the result is similar, with $e^{i\lambda}$ replaced by $e^{-i\lambda}$, which corresponds to the DC current sign reversal.)

We note that the noninteger number of attempts $N(t) = eVt/2\pi\hbar$ is an artifact of a quasiclassical calculation. In a more careful analysis the number of attempts is characterized by a narrow distribution P_N peaked at $\overline{N} = N(t)$, and the generating function is a weighted sum $\sum_N P_N \chi_N(\lambda)$. Since the peak width is a sublinear function of the measurement time t (in fact, $\overline{\delta N^2} \propto \ln t$), the statistics to the leading order in t are correctly described by the binomial distribution.

One can also consider the problem at arbitrary $k_B T/eV$ [20]. The integral in Eq.(38), although less trivial, can be carried out, giving

$$\chi(\lambda) = \exp(-u_+ u_- N_T) , \quad N_T = t \, k_B T/2\pi\hbar, \quad (45)$$

where

$$u_\pm = v \pm \cosh^{-1}(p\cosh(v+i\lambda) + (1-p)\cosh v), \quad v = eV/2k_BT. \quad (46)$$

At low temperature $k_BT \ll eV$, Eq.(45) reproduces the binomial statistics. At low voltage $eV \ll k_BT$ (or high temperature) Eq.(45) gives the counting statistics of the equilibrium Nyquist noise:

$$\chi(\lambda) = e^{-\lambda_*^2 N_T}, \qquad \sin(\lambda_*/2) = p^{1/2}\sin(\lambda/2). \quad (47)$$

Remarkably, even at equilibrium the noise is non-gaussian, except for special case of full transmission, $p=1$, $\lambda_* = \lambda$, when it becomes gaussian.

3.2. STATISTICS OF TIME-DEPENDENT SCATTERING

The time-dependent scattering problem describes photon-assisted transport. There are two groups of practically interesting problems: the AC-driven systems with static scattering potential, such as tunneling barriers or point contacts in the presence of a microwave field [17, 41], and the electron pumps with time-dependent scattering potential controlled externally, e.g. by gate voltages [42].

Typically, the time of individual particle transit through the scattering region is much shorter than the period at which the system is driven. This situation is described, in the instantaneous scattering approximation, by a time-dependent scattering matrix $S(t)$ that characterizes single particle scattering at time t. The question of interest is how Fermi statistics of many-body scattering states affects the counting statistics.

One can construct a theory of counting statistics of time-dependent scattering [19] by generalizing the results of Sec.3.1 for the statistics of a generic time-independent scattering. In particular, as we discuss below, the determinant formula (41), along with Eq.(38), allows a straightforward extension to time-dependent problems. In that, the generating function $\chi(\lambda)$ acquires a form of a functional determinant.

Let us consider a scattering matrix $S(t)$ varying periodically in time with frequency Ω. The analysis is most simple in the frequency representation [19], in which the scattering operator S has off-diagonal matrix elements $S_{\omega',\omega}$ with discrete frequency change $\omega' - \omega = n\Omega$. In this approach the energy axis is divided into intervals $n\Omega < \omega < (n+1)\Omega$ and each such interval is treated as a separate conduction channel. In doing so it is convenient (and in some cases necessary) to assign a separate counting field λ_n to each frequency channel, giving the counting field a frequency channel index in addition to the conduction channel dependence displayed in Eq.(41).

Since the scattering operator conserves energy modulo multiple of $\hbar\Omega$, the scattering can be viewed as *elastic* in the extended channel representation, which allows to employ the method of Sec.3.1. We note also that

the form of the determinant in Eq.(41) is not particularly sensitive to the size of the scattering matrix. Thus one can use it even when the number of channels is infinite, provided that the determinant remains well defined. This procedure brings (41) to the form of a determinant of an infinite size matrix. Determinant regularization can be accomplished by truncating this matrix at very high and low frequencies, thereby eliminating empty states and the states deep in the Fermi sea which do not contribute to noise.

Finally, we note that the product rule (38) for $\chi(\lambda)$ is consistent with the determinant structure, since scattering processes at the energies different modulo $n\Omega$ are decoupled. This allows to keep the answer for $\chi(\lambda)$ in the form of the determinant (41), where now the scattering operator S is considered in the entire frequency domain, rather than at the discrete frequencies $\omega + n\Omega$. The resulting functional determinant has a simple form in the time representation:

$$\chi(\lambda) = \det\left(\hat{1} + \hat{n}(t,t')\left(\hat{T}_\lambda(t) - \hat{1}\right)\right), \quad \hat{T}_\lambda(t) = S^\dagger_{-\lambda}(t)S_\lambda(t), \qquad (48)$$

where $(S_\lambda)_{jj'} = e^{\frac{i}{4}(\lambda_j - \lambda_{j'})}S_{jj'}$ as above, and $\hat{n}$ is the density matrix of reservoirs. The operator $\hat{n}$, diagonal in the channel index, is given by

$$n_{jj'}(t,t') = \delta_{jj'}\int n_j(\hbar\omega)\,e^{i\omega(t'-t)}\,d\omega/2\pi \,. \qquad (49)$$

In general $\hat{n}(t,t')$ depends on the energy distribution parameters, such as temperature and chemical potential. In equilibrium, by taking Fourier transform of the Fermi function $n_F(\epsilon - \mu)$, one obtains

$$n_j(t,t') = \frac{e^{-i\mu_j(t-t')/\hbar}}{2\beta\sinh(\pi(t - t' + i\delta)/\beta)}, \quad \beta = \hbar/k_B T\,. \qquad (50)$$

Finite bias voltage V s described by $\mu_1 - \mu_2 = eV$. At $T = 0$ and $V = 0$, Eq.(50) gives $n(t,t') = i/(2\pi(t - t' + i\delta))$. The result (48) holds for an arbitrary (even nonequilibrium) energy distribution in reservoirs.

The functional determinant of an infinite matrix (48) should be handled carefully. One can show that, in a mathematical sense, the quantity (48) is well defined. For the states with energies deep in the Fermi sea, $\hat{n} = 1$ and, since $\det\left(\hat{T}_\lambda(t)\right) = 1$ due to unitarity of S, these states do not contribute to the determinant (48). Similarly, since $\hat{n} = 0$ for the states with very high energy, these states also do not affect the determinant. Effectively, the determinant is controlled by a group of states near the Fermi level, in agreement with intuition about transport in a driven system. The absence of ultraviolet divergences allows one to go freely between different representations, e.g. to switch from the frequency domain to the time domain, which facilitates calculations [24, 21].

The above derivation of the formula (48) based on a generalization of the result (41) for time-independent scattering might seem not entirely rigorous. A more mathematically sound derivation that starts directly from the microscopic expression (16) was proposed recently by Klich [43].

3.3. CASE STUDIES

Here we briefly review the time-dependent scattering problems for which the counting statistics have been studied. From several examples for which $\chi(\lambda)$ has been obtained it appears that the problem does not allow a general solution. Instead, the problem can be handled only for suitably chosen form of the time dependence $S(t)$.

In Ref. [19] a two channel problem was considered with $S(t)$ of the form

$$S(\tau) \equiv \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} = \begin{pmatrix} B + be^{-i\Omega\tau} & \bar{A} + \bar{a}e^{i\Omega\tau} \\ A + ae^{-i\Omega\tau} & -\bar{B} - \bar{b}e^{i\Omega\tau} \end{pmatrix}, \tag{51}$$

which is unitary for any t provided $|A|^2 + |a|^2 + |B|^2 + |b|^2 = 1$, $A\bar{a} + B\bar{b} = 0$. The problem was solved by using the extended channel representation in the frequency domain, in which each frequency interval $n\Omega < \omega < (n+1)\Omega$ is treated as a separate scattering channel, as discussed above.

For the reservoirs at zero temperature and without bias voltage the charge distribution for m pumping cycles is described by

$$\chi(\lambda) = \left(1 + p_1(e^{i\lambda} - 1) + p_2(e^{-i\lambda} - 1)\right)^m \tag{52}$$

with $p_1 = |a|^4/(|a|^2 + |b|^2)$ and $p_2 = |b|^4/(|a|^2 + |b|^2)$. This result means that at each pumping cycle one electron is pumped in one direction with probability p_1, or in the opposite direction with probability p_2, or no charge is pumped with probability $1 - p_1 - p_2$. The multiplicative dependence of $\chi(\lambda)$ on the number of pumping cycles m indicates that the outcomes of different cycles are statistically independent. One can thus view (52) as a generalization of the binomial distribution (6).

The problem (51) was also studied in Ref. [19] at a finite bias voltage, when the counting distribution is not as simple as (52). To describe the result, for a given bias voltage V we find an integer n such that $nf < \frac{e}{\hbar}V \leq (n+1)f$, where $f = \Omega/2\pi$ is the cyclic frequency in (51). Then for a long measurement time $t \gg \Omega^{-1}$ the counting distribution is

$$\chi(\lambda) = \chi_n^{N_>}(\lambda) \cdot \chi_{n+1}^{N_<}(\lambda), \tag{53}$$

where $N_< = \left(\frac{e}{\hbar}V - nf\right)t$, $N_> = \left((n+1)f - \frac{e}{\hbar}V\right)t$, and the functions $\chi_n(\lambda)$ are finite degree polynomials in $e^{\pm i\lambda}$. The form of $\chi_n(\lambda)$ depends on A, B, a, and b (we refer to Ref. [19] for details).

The product rule (53) means that the cummulants of the distribution $\chi(\lambda)$ depend on V in a piecewise linear way, $m_k(V) = N_> m_k^{(n)} + N_< m_k^{(n+1)}$, with cusp-like singularities at $eV = nhf \equiv n\hbar\Omega$. These singularities are generic for the noise in photo-assisted phase-coherent transport [44, 17, 41].

Another time-dependent problem for which solution can be obtained in a closed form is mesoscopic transport in the presence of an AC voltage [24]. The scatterer in this case is a time-independent 2×2 matrix, while the voltage $V(t)$ enters in the phase factors of the density matrix in (48):

$$n_{1,2}(t,t') = e^{\pm \frac{i}{2}(\varphi(t') - \varphi(t))} n_{1,2}^{(0)}(t,t'), \quad \dot{\varphi}(t) = \frac{e}{\hbar} V(t) \tag{54}$$

(compare this with the formula (50) for constant bias voltage).

The counting distribution (48) for a family of such problems has been studied in Ref. [21]. It was noted earlier [45] that noise is minimized at fixed transmitted charge for a special form of time-dependent voltage:

$$V(t) = \frac{h}{e} \sum_{k=1,\dots,m} \frac{2\tau_k}{(t - t_k)^2 + \tau_k^2} \cdot \tag{55}$$

Each of the Lorentzian voltage pulses (55) corresponds to a 2π phase change in $\varphi(t)$. Interestingly, the noise-minimizing pulses (55) have large degeneracy: they produce noise which is insensitive to the pulses' widths τ_k and peak positions t_k. This calls for an interpretation of the pulses (55) as independent attempts to transmit charge. Not surprisingly, the counting statistics for such pulses was found to be binomial:

$$\chi(\lambda) = (1 + t(e^{i\lambda} - 1))^m \tag{56}$$

with t the transmission constant. The lowest possible noise for a current pumped by voltage pulses is thus equal to that of a DC current with the same transmitted charge.

The method of Ref. [21] also allows to find the distribution for an arbitrary sum of the pulses (55) with alternating signs. For example, two opposite pulses

$$V(t) = \frac{h}{e} \left(\frac{2\tau_1}{(t - t_1)^2 + \tau_1^2} - \frac{2\tau_2}{(t - t_2)^2 + \tau_2^2} \right) \tag{57}$$

give rise to the counting distribution

$$\chi(\lambda) = 1 - 2F + F(e^{i\lambda} + e^{-i\lambda}), \quad F = t(1 - t) \left| \frac{z_1^* - z_2}{z_1 - z_2} \right|^2 \tag{58}$$

with $z_{1,2} = t_{1,2} + i\tau_{1,2}$. The quantity $|\dots|^2$ is a measure of pulses' overlap in time, varying between 0 for a full overlap and 1 for no overlap. For nonoverlapping pulses, $\chi(\lambda)$ factors into $(t e^{i\lambda} + 1 - t)(t e^{-i\lambda} + 1 - t)$, in agreement with the interpretation of a binomial distribution for independent attempts.

3.4. MESOSCOPIC PUMPING

A DC current in a mesoscopic system, such as an open quantum dot, can be induced by pumping, i.e. by modulating its area, shape, or other parameters [46, 48, 47]. After pumping was demonstrated experimentally [42], it came into the focus of mesoscopic literature (for references see [47, 49]). In particular, Brouwer made an interesting observation that the time-averaged pumped current is a purely geometric property of the path in the scattering matrix parameter space, insensitive to path parameterization.

Transport through a mesoscopic system is described [40] by a scattering matrix S which depends on externally driven parameters and varies cyclically with time. The matrix $S(t)$ defines a path in the space of all scattering matrices. For a system with m scattering channels, the matrix space is the group $U(m) = SU(m) \times U(1)$. In an experiment one can, in principle, realize any path in the space of scattering matrices.

Counting distribution for a parametrically driven open system was discussed by Andreev and Kamenev who adapted the results [19]

obtained for specific pumping cycles [Eqs.(51),(52)]. However, since the relation between the path in the scattering matrix space and the external pumping parameters is generally unknown, only the results valid for generic paths are of interest in this problem.

Here we consider the weak pumping regime, when the path $S(t)$ is a sufficiently small, but otherwise arbitrary loop, and show that in this case the counting distribution is universal [22], taking the form of bidirectional Poisson distribution (8). From that, we obtain the dependence of the noise on the amplitude and relative phase of the voltages driving the pump.

Before turning to the calculation, we discuss general dependence of counting statistics on the path in matrix space. Different paths $S(t)$, in principle, give rise to different current and noise. However, there is a remarkable property of invariance with respect to group shifts. Any two paths,

$$S(t) \quad \text{and} \quad S'(t) = S(t)S_0, \tag{59}$$

where S_0 is a time-independent matrix in $U(m)$, give rise to the same counting statistics at zero temperature. We note that only the right shifts of the form (59) leave counting statistics invariant, whereas the left shifts generally change it. One can explain the result (59) qualitatively as follows. The change of scattering matrix, $S(t) \to S'(t) = S(t)S_0$, is equivalent to replacing states in the *incoming* scattering channels by their superpositions $\psi^\alpha = S_{0\beta}^\alpha \psi^\beta$. At zero temperature, however, Fermi reservoirs are noiseless and also such are any their superpositions. Correlation between superposition states of noiseless reservoirs is negligible, while current fluctuations arise only due to the time-dependent scattering. Therefore, noise statistics remain unchanged. A simple formal proof of the result (59) is given below.

For a weak pumping field it is sufficient to evaluate (48) in the time domain by expanding $\ln\det(...)$ in powers of δS and keeping non-vanishing terms of lowest order. In doing so, however, we preserve full functional dependence on λ which gives all moments of counting statistics. We write $S(t) = e^{A(t)}S^{(0)}$ with antihermitian $A(t)$ representing small perturbation, $\mathrm{tr}A^\dagger A \ll 1$. Here $S^{(0)}$ is scattering matrix of the system in the absence of pumping. Substituting this into (48) one obtains

$$\hat{T}_\lambda(t) \equiv \hat{T}_\lambda^{(0)} + \delta T_\lambda(t) = S_{-\lambda}^{(0)\dagger}e^{-A_{-\lambda}(t)}e^{A_\lambda(t)}S_\lambda^{(0)} \qquad (60)$$

with $\hat{T}_\lambda^{(0)} = S_{-\lambda}^{(0)\dagger}S_\lambda^{(0)}$ and $A_\lambda(t) = e^{i\frac{\lambda}{4}\sigma_3}A(t)e^{-i\frac{\lambda}{4}\sigma_3}$ (here σ_3 equals $+1$ for the left and -1 for the right channel). Now, we expand (48):

$$\ln\chi(\lambda) = \ln\det Q_0 + \mathrm{tr}R - \frac{1}{2}\mathrm{tr}R^2 + \frac{1}{3}\mathrm{tr}R^3 - ..., \qquad (61)$$

where $Q_0 = 1 + \hat{n}(\hat{T}_\lambda^{(0)} - 1)$ and $R = Q_0^{-1}\hat{n}\delta T_\lambda$. At zero temperature, from $\hat{n}^2 = \hat{n}$ it follows that $\det Q_0 = 1$ and $R = S_\lambda^{(0)\,-1}\hat{n}\left(e^{-A_{-\lambda}(t)}e^{A_\lambda(t)} - 1\right)S_\lambda^{(0)}$. Therefore,

$$\ln\chi(\lambda) = \mathrm{tr}\ \hat{n}\hat{M} - \frac{1}{2}\mathrm{tr}(\hat{n}\hat{M})^2 + \frac{1}{3}\mathrm{tr}(\hat{n}\hat{M})^3 - ..., \qquad (62)$$

where $\hat{M} = e^{-A_{-\lambda}(t)}e^{A_\lambda(t)} - 1$. Note that at this stage there is no dependence left on the constant matrix $S^{(0)}$, which proves the invariance under the group shifts (59).

We need to expand (62) in powers of the pumping field, which amounts to taking the lowest order terms of the expansion in powers of the matrix $A(t)$. One can check that the two $\mathcal{O}(A)$ terms arising from the first term in (62) vanish. The $\mathcal{O}(A^2)$ terms arise from the first and second term in (62) and have the form

$$\ln\chi = \frac{1}{2}\mathrm{tr}\left(\hat{n}\left(A_{-\lambda}^2 + A_\lambda^2 - 2A_{-\lambda}A_\lambda\right)\right) - \frac{1}{2}\mathrm{tr}(\hat{n}B_\lambda)^2 \qquad (63)$$

with $B_\lambda(t) = A_\lambda(t) - A_{-\lambda}(t)$. At zero temperature, by using $\hat{n}^2 = \hat{n}$, one can bring (63) to the form

$$\frac{1}{2}\,\mathrm{tr}\left(\hat{n}\left[A_\lambda, A_{-\lambda}\right]\right) + \frac{1}{2}\left(\mathrm{tr}\left(\hat{n}^2 B_\lambda^2\right) - \mathrm{tr}(\hat{n}B_\lambda)^2\right). \qquad (64)$$

The first term of (64) has to be regularized in the Schwinger anomaly fashion, by splitting points, $t', t'' = t \pm \eta/2$, which gives

$$\frac{1}{2}\oint n(t', t'')\mathrm{tr}\left(A_{-\lambda}(t'')A_\lambda(t') - A_\lambda(t'')A_{-\lambda}(t')\right)dt\,. \qquad (65)$$

Averaging over small η can be achieved either by inserting in (65) additional integrals over t', t'', or simply by replacing $A_\lambda(t) \to \frac{1}{2}\left(A_\lambda(t) + A_\lambda(t')\right)$, etc. After taking the limit $\eta \to 0$, Eq.(65) becomes

$$\frac{i}{8\pi} \oint \text{tr}\left(A_{-\lambda}\partial_t A_\lambda - A_\lambda\partial_t A_{-\lambda}\right) dt\,. \tag{66}$$

The second term of (64) can be written as

$$\frac{1}{4(2\pi)^2} \oint\oint \frac{\text{tr}\left(B_\lambda(t) - B_\lambda(t')\right)^2}{(t-t')^2} dt dt'\,. \tag{67}$$

We decompose $A = a_0 + z + z^\dagger$ with respect to the right and left channels, so that $[\sigma_3, a_0] = 0$, $[\sigma_3, z] = -2z$, $\left[\sigma_3, z^\dagger\right] = 2z^\dagger$. Then $A_\lambda \equiv e^{-i\frac{\lambda}{4}\sigma_3} A e^{i\frac{\lambda}{4}\sigma_3} = a_0 + e^{i\frac{\lambda}{2}}z^\dagger + e^{-i\frac{\lambda}{2}}z$, $B_\lambda = \left(e^{i\frac{\lambda}{2}} - e^{-i\frac{\lambda}{2}}\right)W$, $W \equiv z^\dagger - z$. Substituting this into (66) and (67) one rewrites Eqs.(66),(67) in terms of $W(t)$:[4]

$$\frac{\sin\lambda}{8\pi} \oint \text{tr}\left([\sigma_3, W]\,\partial_t W\right) dt \tag{68}$$

and

$$\frac{(1-\cos\lambda)}{2(2\pi)^2} \oint\oint \frac{\text{tr}\left(W(t) - W(t')\right)^2}{(t-t')^2} dt dt'\,. \tag{69}$$

From that we obtain the counting distribution for one pumping cycle:

$$\chi(\lambda) = \exp\left(u(e^{i\lambda} - 1) + v(e^{-i\lambda} - 1)\right) \tag{70}$$

with the transmitted charge average $I = e(u-v)$ and variance $J = e^2(u+v)$, related to the noise spectral density by $\mathcal{S}_0 = J\Omega/\pi$ (see Sec.2.2).

The parameters u and v in (70) can be expressed through $z(t)$ and $z^\dagger(t)$ as follows. Let us write $z(t)$ as $z_+(t) + z_-(t)$, where $z_+(t)$ and $z_-(t)$ contain only positive or negative Fourier harmonics, respectively. Then

$$u = \frac{i}{4\pi} \oint \text{tr}\left(z_-^\dagger\partial_t z_+ - z_+\partial_t z_-^\dagger\right) dt = \sum_{\omega>0} \omega\,\text{tr}z_{-\omega}^\dagger z_\omega\,, \tag{71}$$

$$v = \frac{i}{4\pi} \oint \text{tr}\left(z_-\partial_t z_+^\dagger - z_+^\dagger\partial_t z_-\right) dt = -\sum_{\omega<0} \omega\,\text{tr}z_{-\omega}^\dagger z_\omega\,. \tag{72}$$

This demonstartes that $u \geq 0$ and $v \geq 0$. It is straightforward to show that (68) equals $i\sin\lambda(u-v)$, whereas (69) equals $(\cos\lambda - 1)(u+v)$.

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[4]Eq.(68) is essentially identical to the result obtained by Brouwer for the average pumped current [48]. The integral in (68) is invariant under reparameterization, and thus has a purely geometric character determined by the contour $S(t)$ in $U(m)$.

Now we consider a single channel pump, $S(t) \in U(2)$. In this case, $z = z(t)\sigma_-$ and $z^\dagger = z^*(t)\sigma_+$. For a harmonic driving signal, without loss of generality, one can write

$$z(t) = z_1 V_1 \cos(\Omega t + \theta) + z_2 V_2 \cos(\Omega t), \tag{73}$$

where $V_{1,2}$ are pumping signal amplitudes, and complex parameters $z_{1,2}$ depend on microscopic details. From (71) we find the rates

$$u = \frac{1}{4}\left|z_1 V_1 e^{i\theta} + z_2 V_2\right|^2, \quad v = \frac{1}{4}\left|z_1 V_1 e^{-i\theta} + z_2 V_2\right|^2. \tag{74}$$

Both u and v can vanish at a particular amplitude ratio V_1/V_2 and phase θ. When this happens, the two Poisson processes (70) are reduced to one, and the current-to-noise ratio gives elementary charge, $I/J = \pm e^{-1}$. This happens at the extrema of I/J as a function of $w = (V_1/V_2)e^{i\theta}$, for (74) reached at $w = -z_2/z_1, \; -\bar{z}_2/\bar{z}_1$.

Reducing the counting statistics (70) to purely poissonian by varying pumping parameters is possible, in principle, for any number of channels n. However, since the number of parameters to be tuned is $2n^2$, this method is practical perhaps only for small channel numbers. Although the method is demonstrated for non-interacting fermions, we argue that it can be applied to interacting systems as well. Poisson statistics results from the absence of correlations of transmitted particles, which must be the case in a generic system, interacting or noninteracting, at small pumping current. Using the dependence of the rates u, v on the driving signal to maximize I/J, i.e. to eliminate one of the two Poisson processes (70), one could then obtain the charge quantum in the standard way as $e = J/I$.

To summarize the results of this section, in the weak pumping regime the distribution of charge transmitted per cycle is of bidirectional Poisson form (8), i.e., it is fully characterized by only two parameters, average current and noise. The current to noise ratio I/J, scaled by elementary charge, varies between 1 and -1, depending on the relation between driving signals phases and amplitudes. Thus the quantity $\max(|I|/J)$ gives the inverse of elementary charge *without any fitting parameters*. Polianski et al. [49] recently studied the dependence of I/J on the mesoscopic scattering ensemble parameters, and found that, within the random matrix theory, the nearly extremal values close to ± 1 can be reached with finite probability. This may permit to use the noise in a pump to measure quasiparticle charge in open systems, such as Luttinger liquids.

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