A few issues in turbulence and how to cope with them using computers

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CU Boulder, July 2011, pouquet@ucar.edu
Happy Bastille day
GENERAL OUTLINE for LECTURES

Physical complexity of flows on Earth and beyond
Vorticity and helicity dynamics
Kinematics of tensors and methodology
Exact laws, structures and different energy spectra in MHD?
Complexity of phenomenology: beyond Kolmogorov

Weak turbulence and beyond, towards strong turbulence with closures

II – Some results for MHD and for rotation

II - Modeling: why and how
II - The Lagrangian averaging model, for MHD and perhaps for fluids

II - Adaptive mesh refinement with spectral accuracy
II - Application to the dynamo problem at low magnetic Prandtl number
A Few Issues in Turbulence:

In search of a small parameter
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The theoretically solvable case of weak/wave turbulence
A Few Issues in Turbulence:

In search of a small parameter

The theoretically solvable case of weak/wave turbulence

But is it useful?
Conditions & methodology for weak/wave turbulence

\[ \partial_t u = \mathcal{L}_u u + \epsilon \mathcal{N}_u(u, u) \]

\[ \epsilon = 0 \implies \hat{u}(k, t) = \hat{u}_0(k) e^{-i\omega_k t} \quad \text{wave of frequency } \omega_k \]

\[ \epsilon \ll 1 \rightarrow \exists \text{ two different time scales:} \]

\[ \hat{u}(k, t) = a(k, t) e^{-i\omega_k t} \]

\[ \implies \begin{cases} 
\text{Fast variation in time of } e^{-i\omega_k t} \\
\text{Slow variation of } a(k, t) \text{ through wave coupling} \end{cases} \]

\[ \exists \text{ resonances } \rightarrow \text{“kinetic” equations} \]

\[ \rightarrow \text{rigorous closure of the statistical problem} \]
Conditions & methodology for weak/wave turbulence

- $\partial_t u = \mathcal{L}_x u + \epsilon N_x(u, u)$
  - $\epsilon = 0 \implies \hat{u}(k, t) = \hat{u}_0(k)e^{-i\omega_k t}$ wave of frequency $\omega_k$

- $\epsilon \ll 1 \rightarrow \exists$ two different time scales:
  - $\hat{u}(k, t) = a(k, t)e^{-i\omega_k t}$
    - Fast variation in time of $e^{-i\omega_k t}$
    - Slow variation of $a(k, t)$ through wave coupling

- $\exists$ resonances $\rightarrow$ “kinetic” equations

\[ \rightarrow \text{rigorous closure of the statistical problem} \]
\[ \frac{\partial u}{\partial t} = L_x u + \varepsilon N_x(u; u) \]

with \( \varepsilon \ll 1 \)

Fourier: \( A(k, t) = \int e^{i k \cdot x} u(x, t) \, dx \)

\( \omega(k) \) frequency associated with the linear operator

\( H(m, n) \) the Fourier representation of the non-linear operator:

\[ [\partial_t + i \omega(k)] A(k, t) = \varepsilon \int_{-\infty}^{\infty} H(m, n) A(m, t) A(n, t) \delta(k - m - n) \, dmdn \]

\( \delta_{kmn} = \delta(k - m - n) \) and \( d_{mn} = dmdn \) and \( H_{mn} = H(m, n), \omega_k = \omega(k) \)

\( A(k, t) = a_k e^{i \omega(k)t} \)

\[ \frac{\partial a_k}{\partial t} = \varepsilon \int H_{mn} a_m a_n e^{i(-\omega_k + \omega_m + \omega_n)t} \delta_{kmn} d_{mn} \]
Perform an $\epsilon$–expansion

$$a_\ell = a_{0,\ell} + \epsilon a_{1,\ell} + \epsilon^2 a_{2,\ell} + \ldots$$

and solve order by order iteratively.

Note that $a_{0,\ell}$ is constant.

$$a_{1,\ell} = \int H_{mn} a_{0,m} a_{0,n} \Delta(\omega_{\ell,mn}) \delta(\omega_{\ell,mn}) d_{mn} \quad .$$

with

$$\omega_{\ell,mn} = \omega(\ell) - \omega(m) - \omega(n)$$

$$\Delta(\omega_{\ell,mn}) = \int_0^t \exp \left[ it \omega_{\ell,mn} \right] dt = \frac{\exp \left[ i \omega_{\ell,mn} \right] - 1}{i \omega_{\ell,mn}}$$

Resonance occurs for $\omega_{\ell,mn} = 0$. 

Expand in small parameter, solve at lowest order and iterate.
The closure at second order

As is standard in similar computations for ODE’s (see for example, Bender & Orszag, 1978, Chapter II) the terms \( \delta q_0^{(N)} / \delta T_2 \) are chosen to remove secularities. The last step is to realize that since

\[
\varepsilon = \frac{\tau_\omega}{\tau_{NL}} \ll 1
\]

and since we wish to average over many wave periods, we have to evaluate integrals of the form

\[
\lim_{t \to \infty} \int f(k) \Delta(k, t) \, dk
\]

where

\[
\Delta(k) = \int_0^t e^{i\omega(k)t} \, dt
\]

contains the time-dependence, and \( \omega(k) \) (i.e. the dispersion relation) is the link with the (linearized) physical problem. We now use the Lemma:

\[
\lim_{t \to \infty} \int f(k) \Delta(k, t) \, dk = \pi f(0) + iP_C \int \frac{f(k)}{k} \, dk
\]

where \( P_C \) stands for the Cauchy Principal Value integral. In other words, what we are really doing here is to replace

\[
\lim_{t \to \infty} \int \frac{\sin \omega t}{\omega} \, dk
\]

by \( \pi \delta(\omega) \).

This allows you to perform the closure
Fundamental steps in the development

• A closure problem and the problem of cumulants:

\[ \partial_t \langle a_j a_{j'} \rangle = \langle a_j a_{j'} a_{j''} \rangle \]

\[ \partial_t \langle a_j a_{j'} a_{j''} a_{j'''(j''')} \rangle = \langle a_j a_{j'} a_{j''} a_{j'''(j''')} \rangle \]

\[ \equiv \sum \langle a_j a_{j'} \rangle \langle a_{j''} a_{j'''(j''')} \rangle + \langle a_j a_{j'} a_{j''} a_{j'''(j''')} \rangle_C \]

Cumulant

For small \( \epsilon \), one finds that there is no contribution at lowest order of the 4th order cumulants

\[ \rightarrow \text{ resulting equations “like” the random phase approximation} \]

• It is different from Eddy Damped Quasi Normal Markovian (EDQNM) models where

\[ \langle a_j a_{j'} a_{j''} a_{j'''(j''')} \rangle_C = -\mu_m \langle a_j a_{j'} a_{j''} \rangle \]

with \( \mu_m \) a characteristic rate

**Closure:**

No contribution at lowest order of 4th order cumulants
Traditional Closure schemes

\[ \langle a_j a_{j'} a_{j''} a_{j'''} \rangle_C = -\mu_m \langle a_j a_{j'} a_{j''} \rangle \]

\[ \mu_m = 0 \) (Quasi Normal approximation, Ogura; Millioshikov, mid 40's; Chandrasekhar, mid 50's) leads to negative energy spectra (lack of realisability)

\[ \mu_m = \mu_0, \neq 0 \quad \forall k \quad \rightarrow \text{energy spectrum } E(k) \sim k^{-2} \quad (\text{MRCM}) \quad (\text{and shocks ...}) \]

\[ \mu_m = \omega_{\text{int}}(k) \quad \rightarrow \text{energy spectrum } E(k) \sim k^{-5/3} \]

\[ \mu_m = \omega_{\text{int}}(k) + \omega_A(k) \rightarrow \text{energy spectrum } E(k) \sim k^{-3/2} \]

\[ \text{anisotropic DIA (Nakayama, 1999): } E(k) \sim k_{-3/2} \]

- Compute \( \mu_m \) from an auxiliary problem: Test Field Model (Kraichnan)
- Weak turbulence does it naturally
Traditional Closure schemes

\[
\langle a_j a_{j'} a_{j''} a_{j'''} \rangle_C = -\mu_m \langle a_j a_{j'} a_{j''} \rangle
\]

\[\mu_m = 0 \text{ (Quasi Normal approximation, Ogura; Millioshikov, mid 40’s; Chandrasekhar, mid 50’s) leads to negative energy spectra (lack of realisability)}\]

\[\mu_m = k^i \mu_0, \not= 0 \forall t \text{ (MRCM)} \rightarrow \text{energy spectrum } E(k) \sim k^{-2} \text{ for } i=1 \text{ (and shocks ...)}\]

\[\mu_m = \omega_{int}(k) \rightarrow \text{energy spectrum } E(k) \sim k^{-5/3}\]

\[\mu_m = \omega_{int}(k) + \omega_A(k) \rightarrow \text{energy spectrum } E(k) \sim k^{-3/2}\]

\[\text{anisotropic DIA (Nakayama, 1999): } E(k) \sim k_\perp^{-3/2}\]

- Compute \(\mu_m\) from an auxiliary problem: Test Field Model (Kraichnan)
- Weak turbulence does it naturally

Eddy
Damped
Quasi
Normal
Markovian
Model
Or
EDQNM
Closure ad hoc hypothesis vs. weak turbulence theory, & how they meet:

\[
\partial_t \langle a_j a_j' a_j'' \rangle = \langle a_j a_j' a_j'' a_j''' \rangle \quad \text{or} \quad \partial_t T_3 = Q_4 = \sum E_2^2 + Q_{4,C}
\]

\[
\equiv \sum \langle a_j a_j' \rangle \langle a_j'' a_j''' \rangle + \langle a_j a_j' a_j'' a_j''' \rangle_C
\]

Now, the EDQNM stipulates that the relaxation of triple correlation involves the characteristic times of the problem:

\[
Q_{4,C} = -\mu_m T_3
\]

with

\[
\mu_m = \tau_{NL}^{-1} + \tau_{wave}^{-1}
\]

And now take the limit of \( \tau_{wave} \to 0 \)

Henceforth, the fourth order cumulant in the limit of fast waves in the EDQNM goes to zero as well

\[
Q_{4,C} \to 0
\]

Thus, one may say that the EDQNM closure and the theory of weak turbulence are compatible in that limit.
Resulting equations of the simplified dynamics, in the weak turbulence regime, for MHD for all second-order moments, including helicity.

Note: equations are anisotropic, with expressions in terms of $k_{\text{perp}}$ and $k_{\parallel}$.
\[ \partial_t [k^2 I^s(k)] = \frac{\pi \varepsilon^2}{\beta_0} \int \left\{ \left[ L^3_{\perp} Z + \frac{k^2_{\perp}}{k^2_{\perp}} (Z^2 - X^2) \right] I^s(L) + \left( \frac{k^2_{\perp} Y^2}{2 L^2} - k^2_{\perp} k^2_{\parallel} + \frac{k^2_{\parallel} X^2}{2 L^2} \right) I^s(k) + \frac{k_{\parallel} X Y}{L^2} \left[ \Psi^s(k) + k^2 \Phi^s(k) \right] \right\} Q^s_{\perp}(\kappa) \delta(\kappa_{\parallel}) \delta_{\kappa \perp} \, d\kappa \, d\kappa_{\perp} \]

\[ - \frac{\varepsilon^2}{\beta_0} s R^s(k) \mathcal{O} \int \frac{1}{2 \kappa_{\perp} L^2} \left( (\kappa_{\parallel} Z - k_{\parallel} L^2_{\perp})^2 - k^2_{\perp} X^2 \right) Q^s_{\perp}(\kappa) \delta_{\kappa \perp} \, d\kappa \, d\kappa_{\perp} \]

(Simplified MHD, page 2)

\[ Q^s_{\perp}(\kappa) = k_m k_p q^s_{m-p}(\kappa) = X^2 \Psi^s(\kappa) + X (k_{\parallel} \kappa_{\parallel}^2 - k_{\parallel} Y) I^s(\kappa) + (k_{\parallel} Y - k_{\parallel} k_{\perp}^2) \phi^s(\kappa). \] (30)

Note that \( Q^s_{\perp} \) does not involve the spectral densities \( R^s(k) \), because of the symmetry properties of the equations. The geometrical coefficients appearing in the kinetic equations are

\[ X = (k_{\perp} \times \kappa_{\perp})_x = k_{\perp} \kappa_{\perp} \sin \theta, \] (31a)

\[ Y = k_{\perp} \cdot \kappa_{\perp} = k_{\perp} \kappa_{\perp} \cos \theta, \] (31b)

\[ Z = k_{\perp} \cdot L_{\perp} = k_{\perp}^2 - k_{\perp} \kappa_{\perp} \cos \theta \]
\[ = k_{\perp}^2 - Y, \] (31c)

\[ W = \kappa_{\perp} \cdot L_{\perp} = k_{\perp}^2 - L_{\perp}^2 - k_{\perp} \kappa_{\perp} \cos \theta \]
\[ = Z - L_{\perp}^2, \] (31d)

where \( \theta \) is the angle between \( k_{\perp} \) and \( \kappa_{\perp} \), and with

\[ d\kappa_{\perp} = \kappa_{\perp} d\kappa_{\perp} d\theta = \frac{L_{\perp}}{k_{\perp} \sin \theta} d\kappa_{\perp} dL_{\perp}, \] (32)

\[ \cos \theta = \frac{k_{\perp}^2 + k_{\perp}^2 - L_{\perp}^2}{2 k_{\perp} L_{\perp}}. \] (33)

In (27)–(29), \( \mathcal{O} \int \) means the Cauchy principal value of the integral in question.

Question:

Why can one call this set of equations a "simplified" dynamics for MHD?

Further simplification: 2D MHD limit

\[ s = \pm 1, \ E_{\pm} = v^2 + b^2 \pm 2v \cdot b \]

Resulting equation for energy spectrum \( E^s(\mathbf{k}) = (k_1^2/k_2^2)q^s(\mathbf{k}) \), with \( \mathcal{D}_{k\kappa L} = \delta(k_\parallel) \delta_{k_\perp \kappa_\perp} dk d\mathbf{L} \)

\[
\partial_t E^s(\mathbf{k}) = \frac{\pi\epsilon^2}{b_0} \int \frac{(k_\perp \cdot L_\perp)^2 (\mathbf{k} \times \kappa)^2}{k_\perp^2 L_\perp^2 \kappa_\perp^2} \left[ E^{-s}(\kappa) - E^s(\mathbf{L}) \right] \mathcal{D}_{k\kappa L}
\]
Simplified version (2D MHD)

\[ s = \pm 1, \quad E_\pm = v^2 + b^2 \pm 2v \cdot b \]

Resulting equation for energy spectrum \( E^s(k) = (k_\perp^2 / k_\parallel^2) q^s(k) \), with \( D_{k\kappa L} = \delta(k_\parallel) \delta_{k,\kappa L} d\kappa dL \)

\[
\partial_t E^s(k) = \frac{\pi \varepsilon^2}{b_0} \int \frac{\left( k_\perp \cdot L_\perp \right)^2 (k \times \kappa)^2}{k_\perp^2 L_\perp^2 \kappa^2} \frac{E^{-s}(\kappa)}{[E^s(L) - E^s(k)]} D_{k\kappa L}
\]

Geometrical & physical coefficient, with \( k = \kappa + L \)
Simplified version (2D MHD)

\[ s = \pm 1, \quad E_\pm = v^2 + b^2 \pm 2v \cdot b \]

Resulting equation for energy spectrum \( E^s(k) = (k_\perp^2 / k_\parallel^2)q^s(k) \), with \( \mathcal{D}_{k\kappa L} = \delta(\kappa_\parallel) \delta_{k,\kappa L} dkL \)

\[
\partial_t E^s(k) = \frac{\pi \varepsilon^2}{b_0} \int \frac{\left( k_\perp \cdot L_\perp \right)^2 (k \times \kappa)_\parallel^2}{k_\perp^2 L_\perp^2 k_\parallel^2} E^{-s}(\kappa) \nonumber [E^s(L) - E^s(k)] \mathcal{D}_{k\kappa L}
\]

Geometrical & physical coefficient, with \( k = \kappa + L \)

Emission term (eddy noise source)
Simplified version (2D MHD)

\[ s = \pm 1, \quad E_{\pm} = v^2 + b^2 \pm 2v.b \]

Resulting equation for energy spectrum \( E^s(k) = \frac{k_\parallel^2}{k_\perp^2} q^s(k) \), with \( \mathcal{D}_{k\kappa L} = \delta(\kappa_\parallel) \delta_{k_\kappa L} dk dL \)

\[
\partial_t E^s(k) = \frac{\pi \varepsilon^2}{b_0} j \frac{(k_\perp \cdot L_\perp)^2 (k \times \kappa)_\parallel^2}{k_\perp^2 L_\perp^2 \kappa_\parallel^2} E^{-s}(\kappa) \\
\quad \left[ E^s(L) - E^s(k) \right] \mathcal{D}_{k\kappa L}
\]

Geometrical & physical coefficient, with \( k = \kappa + L \)

``Emission''

(\textit{eddy noise})

``Absorption''

(\textit{eddy viscosity})
Simplified version (2D MHD)

\[ s = \pm 1, \quad E_{\pm} = v^2 + b^2 \pm 2v.b \]

Resulting equation for energy spectrum \( E^s(\kappa) = \left(\frac{k_{\perp}^2}{k_{||}^2}\right) q^s(\kappa) \), with \( \mathcal{D}_{k\kappa L} = \delta(\kappa_{||}) \delta_{k,k_{||}} \delta_{L,L_{||}} d\kappa dL \)

\[
\partial_t E^s(\kappa) = \frac{\pi \varepsilon^2}{b_0} \int \frac{(k_{\perp} \cdot L_{\perp})^2 (k \times \kappa)^2_{\parallel}}{k_{\perp}^2 L_{\perp}^2 \kappa_{\perp}^2} \left[ E^s(L) - E^s(\kappa) \right] \mathcal{D}_{k\kappa L}
\]

Geometrical & physical coefficient, with \( k = \kappa + L \)

Emission

Absorption

(editable noise)

(editable viscosity)

Galtier et al., Astrophys. J. 2002
Evidence of weak MHD turbulence with a $k_{\text{perp}}^{-2}$ spectrum

- In Reduced MHD computations (Dmitruk et al., PoP 10, 2003)

- In the Jovian magnetosphere (Saur et al., Astron. Astrophys. 386, 2002)

Compensated spectrum
Isotropic phenomenology of turbulence with waves

- Assumption: \( \hat{i} = \frac{T_W}{T_{NL}} << 1 \); transfer time \( T_{tr} \) evaluated as
  \[
  T_{tr} = \frac{T_{NL}}{\hat{i}} = T_{NL} \ast \left( \frac{T_{NL}}{T_W} \right)
  \]
  with \( T_{NL} = l/u_l \) and \( T_W = 1/\Omega \)

- Constant energy flux: \( \varepsilon = \frac{DE}{Dt} \sim k^*E(k) / T_{tr} \)
  
  \[ E(k) \sim \left[ \varepsilon \Omega \right]^{1/2} k^{-2} \]  
  (Dubrulle & Valdetarro, 1992; Zhou, 1995)

Structure functions: \( \langle \delta u(l)^p \rangle \sim l^{\zeta_p} \), \( \zeta_p = p/2 \)
Isotropic phenomenology of turbulence with waves

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\[ E(k) \sim [\epsilon \Omega]^{1/2} k^{-2} \]

(Dubrulle & Valdetarro, 1992; Zhou, 1995)

Structure functions: \( <\delta u(l)^p> \sim l^{\zeta_p} \), \( \zeta_p = p/2 \)

Exercise: MHD: \( T_W = l/B_0 \), \( E(k) \sim [\epsilon B_0]^{1/2} k^{-3/2} \), \( \zeta_p = p/4 \)
Isotropic phenomenology of turbulence with waves

• Assumption: \( \hat{\tau} = \tau_W / \tau_{NL} \ll 1 \); transfer time \( T_{tr} \) evaluated as

\[
T_{tr} = \frac{T_{NL}}{\hat{\tau}} = T_{NL}^* \left( \frac{T_{NL}}{T_W} \right)
\]

with \( T_{NL} = l/u_l \) and \( T_W = 1/\Omega \)

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Anisotropic case: \( T_W = l_{para}/B_0 \), \( E(k_{\text{perp}}, k_{\text{para}}) \sim \left[ \varepsilon B_0 \right]^{1/2} k_{\text{perp}}^{-2} k_{\text{para}}^{-1/2} \)
The scaling of the energy spectrum at high enough rotation rate can differ from the classical Kolmogorov spectrum, i.e. $E(k) \neq k^{-5/3}$ (Morize et al., 2005)
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Is it stopping at $\sim 2.3$?

But it does not stop at $k^{-2}$ …
Initial conditions: fully developed non rotating Kolmogorov flow, $1536^3$ grid
$T=0$ to $T=30$, going through dark blue, green, mauve, red, pink, pale blue
**GHOST**

- **Geophysical High Order Suite for Turbulence** *(Gomez & Mininni)*

- *Community code*
  - Pseudo spectral, incompressible Navier-Stokes (including rotation and passive scalar), and magnetic fields (MHD, with or w/o Hall term); it also includes some LES (the alpha model; a helical spectral model)

- The code parallelizes linearly up to 2,000 processors using MPI, and now up to 40,000 processors using hybrid Open-MP / MPI *(Mininni et al. 2011, see arxiv:1003.432)*

- **Community Data** *(2048^3 forced Navier-Stokes turbulence with and without helicity; 1536^3 and 3072^3 helically forced rotating turbulence; 1536^3 decaying turbulence with a magnetic field, 2048^3 MHD with symmetries) [3D visualization with VAPOR freeware]*
Top view & side view of (left) relative helicity (positive or negative) & (right) vorticity

Taylor-Green non-helical forcing, $k_0=4$, $512^3$, $Ro=0.35$
Scaling of structure functions in rotating turbulence

Simand et al., ‘00; Baroud et al., ‘02

DNS

As time evolves

Mininni+AP, PRE 79
From Taylor-Green forcing (globally non helical)

to ABC forcing (Beltrami flow, fully helical)

for rotating turbulence
With helicity, strong coherent structures form that are organized.

Beltrami Core Vortices

Non-helical case

**FIG. 9:** From top to bottom and from left to right, slices of the energy density, vorticity intensity, $z$ component of the velocity, and helicity density, in run B at $t \approx 30$. 
Zoom on a Beltrami core vortex amidst a tangle of smaller-scale vortex filaments.

Together with particle trajectories.

$1536^3$ grid, $k_f=7$, $Re=5100$, $Ro=0.06,$
ZOOM on Vorticity:

Beltrami core vortices

Helical forcing at $k_F=7$

DNS on $1536^3$ grid points

$Re=5100$, $Ro=0.06$

Updrafts, with $H>0$

From Taylor-Green forcing (globally non helical) to ABC forcing (Beltrami flow, fully helical) for rotating flows

Weakened intermittency in the **direct** energy cascade

Scaling exponents of structure functions

\[ \langle \delta f = f(x+r) - f(x) \rangle^p \sim r^{\xi_p} \]

of velocity and helicity

\[ \langle \delta u^2(l) \rangle \sim l^{1.4} \]

The energy in the **direct cascade is self-similar** for strong rotation, whereas helicity displays some modicum of intermittency

\[ \xi_p = p/2 \text{ for the non-helical case (Simand + '00; Baroud + '02; Mininni & AP '09) not observed here} \]
Scaling exponents of structure functions

\[ \zeta_p \sim \frac{3p}{4} \text{ at high } \Omega \]

experimentally as well

van Bhokhoven et al. 2009
So, what’s happening?

New spectral law for energy and helicity at high rotation
Isotropic phenomenology of turbulence with waves:

- Small parameter: \( \hat{\tau} = \tau_w / \tau_{NL} \); transfer time \( T_{tr} \) evaluated as:
  \[
  T_{tr} = T_{NL} / \hat{\tau} = T_{NL} * (T_{NL} / T_W) \quad \text{with } T_{NL} = l/u_l \text{ and } T_W = 1/\Omega
  \]

- Constant helicity flux: \( \varepsilon^\sim = DH/Dt \sim k*H(k) / T_{tr} \)
- Assume \( E(k) \sim k^{-e}, H(k) \sim k^{-h} \)

\[ e + h = 4 \] in the helical case with rotation

Assuming maximal helicity \([H(k)=kE(k)]\) leads to \( e = 5/2 \) and structure functions:
\[
<\delta u(l)^p> \sim l^{\zeta_p}, \quad \zeta_p = 3p/4 \quad \text{(Mininni & AP, 2009)}
\]

But is maximal helicity a reachable solution?
\[ E(k) = E_{\Omega} + E_K \sim \varepsilon^a \tilde{\varepsilon}^b \Omega^f k^{-e} + \varepsilon^{2/3} k^{-5/3} \]

\[ H(k) = H_{\Omega} + H_K \sim \varepsilon^c \tilde{\varepsilon}^d \Omega^g k^{-h} + \tilde{\varepsilon}^{-1/3} k^{-5/3} \]

\[ \varepsilon = \frac{dE}{dt}, \tilde{\varepsilon} = \frac{dH}{dt}, \mathcal{F}(a, b, c, d, e, f, g) = 0 \]

Zeman wavenumber at which \( \tau_W = \tau_{NL} \): \( k_\Omega \sim \varepsilon^a \tilde{\varepsilon}^b \Omega^y \)

Wavenumbers \( k_e, h \) at which \( E_{\Omega}, H_{\Omega} = E_K, H_K \)

\[ k_e \sim \varepsilon^\delta \tilde{\varepsilon}^\phi \Omega^{\frac{3(3a+3e-7)}{3e-5}}, k_h \sim \varepsilon^\psi \tilde{\varepsilon}^\xi \Omega^{\frac{-3(3a+3e-8)}{3e-7}} \]

\[ k_\Omega = k_E \rightarrow k_\Omega \sim \varepsilon^{-1/2} \Omega^{3/2} \forall e \text{ and } \]

\[ E_{\Omega} \sim \varepsilon^{\frac{3-e}{2}} \Omega^{\frac{3e-5}{2}} k^{-e} \quad (1) \]

\[ H_{\Omega} \sim \varepsilon^{\frac{-3-e}{2}} \tilde{\varepsilon}^{\frac{7-3e}{2}} k^{-(4-e)} \quad (2) \]

\[ \rightarrow \boxed{e \leq 7/3} \text{ if } k_{e,h,\Omega} \rightarrow \infty \text{ for } \Omega \rightarrow \infty \]
\( k^x \) - **Compensated spectra for energy (x=e) & helicity (x=h)**

**1536^3 run**

- \( k_F = 7 \)
- \( \text{Re} = 5100 \)
- \( \text{Ro} = 0.06 \)

Compensated spectra for the new spectral law:

\[ k_{\text{perp}}^4 E(k) H(k) / k_F \]

Mininni & AP, Phys. Fluids 2010
$k^x$ - **Compensated spectra for energy** ($x=e$) & **helicity** ($x=h$)

1536$^3$ run
- $k_F=7$
- $Re=5100$
- $Ro=0.06$

Solid: $e=2.1$
Dash: $h=1.9$

**Dash-dot:** $h = 2.1 = e$

Fluxes of energy (solid)
And helicity (dash)

Mininni & AP,
Phys. Fluids 2010
$k^x$ - **Compensated** spectra for energy ($x=e$) & helicity ($x=h$)

**1536^3 run**
- $k_F=7$
- Re=5100
- Ro=0.06

Dash-dot: $h = 2.1 = e$
Solid: $e=2.1$
Dash: $h=1.9$

Fluxes of energy (solid)
And helicity (dash)

Mininni & AP,
Phys. Fluids 2010
NORMAlIZED RATIO OF HELICITY TO ENERGY TO SMALL SCALES

as a function of rotation

Mininni & AP, Phys. Fluids 2010
• Does the clear self-similarity of the direct cascade of energy in this quasi-2D flow imply conformal invariance, à la Bernard et al. (2006), which these authors found in 2D NS in the inverse energy cascade?

Thalabard et al., PRL to appear, arXiv:1104.1658

Zero vorticity paths in z-averaged field
• Does the clear self-similarity of the direct cascade of energy in this quasi-2D flow imply conformal invariance, à la Bernard et al. (2006), which these authors found in 2D NS in the inverse energy cascade?

Yes, with $\kappa = 3.6 \pm 0.1$, $\kappa \neq 6$ (2D NS inverse cascade)

*Thalabard et al., PRL to appear, arXiv:1104.1658*
Going beyond, at higher resolution

- What about recovery of isotropy at small scale beyond what we call the Zeman scale $l_\Omega$ at which $T_W = T_{NL} \rightarrow l_\Omega = [\varepsilon/\Omega^3]^{1/2}$

- Large run to resolve the inverse cascade, the wave-modulated anisotropic inertial range, the isotropic inertial range and the dissipation range

- $3072^3$ grid points, Tera-grid allocation of 21 million hours on ~30,000 proc (i.e., 700 hours of clock time, or 6 weeks)
Return to isotropy in the small scales.

$3072^3$ grid

$Ro \sim 0.07$

$Re \sim 24000$

*NSF Tera-grid*
Return to isotropy in the small scales.

$3072^3$ grid

$\text{Ro} \sim 0.07$

$\text{Re} \sim 24000$

$\text{NSF Tera-grid}$
Return to isotropy in the small scales, angular dependence of spectra

$3072^3$ grid
Ro $\sim 0.07$
Re $\sim 24000$

*NSF Tera-grid*
Summary of results

• In the presence of helicity and rotation, the direct transfer to small scales is dominated by the helicity cascade and the energy cascade to small scales is quenched because of the inverse cascade.

• This provides a "small" parameter for the problem (the normalized ratio of energy to helicity fluxes), besides the small Rossby number.

• The direct energy cascade is non-intermittent and conformal invariant (when properly averaged in the vertical direction). It is also (presumably) different from (i) the non-helical case, and (ii) the (presumably) self-similar inverse cascade of energy to large scales.

• There is a change of inertial index in the small scales from a Kolmogorov law to a law steeper than what is predicted by a wave-induced non-helical model, with a possible breaking of universality and with a possible $e \leq 7/3$, $h \geq 5/3$ limit.

• Isotropy recovers at small scale provided the Zeman scale is resolved.

• The flow produces strong organized long-lived columnar helical structures, Beltrami Core Vortices, at scales slightly smaller than the injection scale, with also a growth of structures at large scales.
Some questions

• Can helicity help in interpreting laboratory experiments or atmospheric data?

• Is there experimental evidence for this e+h=4 law?
  • Is there experimental evidence for Beltrami Core Vortices?

• What about the large Reynolds number limit?

• How does the dynamics change in terms of the relative alignment between the velocity and the vorticity [relative helicity \( \rho(k)=H(k)/kE(k) \)]?
Some questions

Does the nature of the imposed forcing at large scale play a role? (helical or not: yes; random vs. deterministic? 2D vs 3D?)

- What happens locally in space? What structures transfer to small vs. large scales? What are the Beltrami Core Vortex structures made of? How do they evolve and interact to lead to both a direct and an inverse cascade?

- Universality?
- Modeling: isotropic vs. anisotropic?
- Need/nature of helical contribution?

- What happens when helicity is neither zero nor maximal globally?
A Few Issues in Turbulence

II – What do we mean by modeling?
Kolmogorov-compensated energy spectra: $k^{5/3} E(k)$

Navier-Stokes, ABC forcing

Small Kolmogorov $k^{-5/3}$ law
(flatt part of the spectrum)

K41 scaling increases in range, as
the Reynolds number increases

- **Bottleneck** at dissipation scale

**Solid:** $2048^3$, $R_v = 10^4$, $R_\infty \sim 1200$

**Dash:** $1024^3$, $R_v = 4000$

Large effort

- Linear number of modes \( N \sim \text{Reynolds number } R_e \)
  \( (N \sim L_0/\ell_{\text{diss}} \sim R_e^{3/4} \text{ for a Kolmogorov spectrum}) \)

- 1D FFT cost is \( N \log N \)

- Time of computation \( \sim T_{\text{diss}}/T_{\text{NL}} \sim R_e \)

- Cost of three-dimensional computation \( \sim R_e^4 \)

*Moore’s law: doubling of resolution every 6 years …*

4096^3 Navier-Stokes on Earth Simulator: 16 Teraflops, 10 TeraBytes

12288^3 (NSF plan): 2 Petaflops, 200 Terabytes, 20MW, $200M, 10^{5+}\, \text{CPUs}

*Data output, analysis, visualization and storage*
There are several ways out

- **Zero**-dimensional models: phenomenology, shell (scalar) models, SOC, …

- **One** dimensional: Burgers equation and its extension to fully compressible flows, MHD flows, kinetic effects (the Hada equation), solitons, …

- **Two**-dimensional problem with either 2 or 3 components (2D2C, 2D3C) and “thick” 2D

- Implementing **symmetries** in 3D at all times

- Adaptive mesh refinement (**AMR**)

- Quasi - direct numerical modeling, with **filtering** really

- Eddy viscosity and Large-Eddy Simulations (**LES**)

And combining them
Reversal of the Earth’s magnetic field over the last 2Myrs

(Valet, Nature, 2005)

Temporal asymmetry and chaos in reversal processes
Surface (1 bar) radial magnetic fields for

**Jupiter, Saturn, & Earth versus Uranus & Neptune**

(16-degree truncation, *Sabine Stanley, 2006*)

Axially dipolar

Quadrupole ~ dipole

- The resulting flow shares similarities with the Cadarache von Kárman dynamo experiment in liquid sodium or gallium (at $P_M \sim 10^6$) (Marié et al., *MHD*, 38, 163, 2002).

- The flow is highly turbulent.


\[
F = \begin{bmatrix}
\sin(k_0x)\cos(k_0y)\cos(k_0z) \\
-\cos(k_0x)\sin(k_0y)\cos(k_0z) \\
0
\end{bmatrix}
\]
Is there a lack of universality in MHD turbulence?

- Tool: a code which enforces symmetries
- Two runs, $512^3$ grids, one enforcing the Taylor-Green symmetries, one a generic pseudo-spectral code

Pouquet et al. GAFD 104, 2010
\( B_0 = 0, \quad F = 0 \)

\( E_v(t=0) = 1 \)

\( E_M(t=0) = 1 \)

\( H_M(t=0) = 0 \)

\( H_C(t=0) \sim 0 \)

Symmetric MHD Taylor-Green 2048\(^3\) equivalent res. \( R_{\text{lam}} \sim 1300 \), \textit{Lee et al. 2010}
$k^{5/3} E_T(k)$

Energy spectra for 3 different runs

$B_0 = 0$, $F = 0$

$E_v(t=0) =$

$E_M(t=0) = 1$

$H_M(t=0) = 0$

$H_C(t=0) \sim 0$

Symmetric MHD Taylor-Green $2048^3$ equivalent res. $R_{\text{lam}} \sim 1300$, Lee et al. 2010
Symmetric MHD Taylor-Green $2048^3$ equivalent res. $R_{\text{lam}} \sim 1300$, Lee et al. 2010

$B_0 = 0, F=0$

Energy spectra

$T_{NL} / T_A = f(k)$

Ratio of time-scales
Lack of universality in MHD turbulence, $B_0 = 0$, $F=0$

Do these spectral indices persist in time for a given flow?

Would it be observed as well in the statistically steady state?

Symmetric MHD Taylor-Green $2048^3$ equivalent res. $R_{\text{lam}} \sim 1300$, Lee et al. 2010
Lack of universality in MHD turbulence \( B_0 = 0, F=0 \)

- Do these spectral indices persist in time for a given flow?
- Would it be observed as well in the statistically steady state?

- Does the difference in indices persist at higher Reynolds number?
- Is there, in the IK case, a follow-up steeper spectrum?
Extreme events in the Solar Wind

Fast vs. slow solar wind magnetic field, and velocity data (Marino et al.)
Extreme events in solar active regions

*(Abramenko, review, 2007)*

![Graph showing scaling exponents ζ(q) of structure functions of order q for eight active regions. The NOAA number and the strongest flare (X-ray class/optical class) of each active region is shown. Increase of the flaring activity of active regions (from the top down to the bottom) is accompanied by general increase in concavity of ζ(q) functions.]

Figure 16: Scaling exponents ζ(q) of structure functions of order q calculated for eight active regions by Abramenko et al. (2002). The straight dotted line has a slope of 1/3 and refers to the state of Kolmogorov turbulence. The NOAA number and the strongest flare (X-ray class/optical class) of each active region is shown. Increase of the flaring activity of active regions (from the top down to the bottom) is accompanied by general increase in concavity of ζ(q) functions.
Extreme events in numerical 3D MHD

• Scaling exponents, $512^3$
  DNS with varying $B_0$:

• As $B_0$ increases, so does the intermittency, i.e. the departure from a straight line

Solar observation (Abramenkova et al.)
MDI line of sight high res. magnetograms

• .6 x .6 arc sec., B > 17G, with 400 X 270 pixels and for long time series (up to 500 magnetograms)

• Is this a manifestation of weak MHD turbulence in the presence of a strong B?

The inertial slopes $\alpha$ measured from 3 to 10 Mm (larger scale: sunspots) vary from $\alpha$=-2.3 (X-flare with index 130), To $\alpha$ below 2.0 (flare index of 18), To $\alpha$=-1.6 (flare index of 0)
Solar observations (Abramenko et al.)
MDI - line of sight high res. magnetograms

Temporal variation of inertial index $\alpha$ in dark black line, and flux in thin grey line

Kolmogorov 1941

Very active region with 37 flares (13 M, 3 X class)
Solar observations (Abramenko, 2005)

- Temporal variation of **inertial index** $\alpha$ in dark black line, and flux in grey line.

- Or is it a manifestation of something else (like non-universal RMHD behavior)?

Do quiet regions follow Kolmogorov law? But where is the intermittency?
Solar observations (Abramenko, 2005)

Variation of inertial index $\alpha$ with flare type

Stationarity (quiet) vs. bursty (chaotic, catastrophic) behavior?
Dmitruk, Gomez and Matthaeus, PoF 2003

Reduced MHD
Numerical data

Rapazzo et al., 2008

Fig. 8.—Total energy spectra as a function of the wavenumber $n$ for simulations F, G, H and I. To higher values of $c_A = v_A / v_{ph}$, the ratio between the Alfvén and photospheric velocities, correspond steeper spectra, with spectral index respectively 1.8, 2, 2.3 and 2.7.
Dmitruk, Gomez and Matthaeus, PoF

Reduced MHD
Numerical data

Rapazzo et al., 2008

Total energy spectra as a function of the wave number $k$ for simulations F, G, H and I. To higher Alfvénic velocity, the ratio between the Alfvén and sound velocities, correspond steeper spectra, with exponents respectively 1.8, 2, 2.3 and 2.7.
Free decay, \( k_0 = 4 \), \( EM = EV \), \( PM = 1 \)
\( R = 2700 \), \( HC \sim HM \sim 0 \)
\( 1024^3 \), 9 T*

Also: Maron et al, 2008: \( k_{\text{perp}}^{-3/2} \)
Going **beyond**, using models of turbulence

- Are spectral indices universal or do they change
  - with Rossby number, *at fixed Reynolds number*?
  - with Reynolds number, *at fixed Rossby number*?

Large Eddy Simulation (LES) with spectral modeling of turbulent flows *(Chollet & Lesieur, 1981)* but implementing:
  - A dynamical fit to the computed energy spectrum instead of imposing Kolmogorov law
  - Inclusion of helicity in both the eddy viscosity and the eddy noise
  - (somewhat phase-preserving) eddy noise reconstruction
Numerical modeling

Direct Numerical Simulations (DNS)

versus

Large Eddy Simulations (LES)

* Resolve all scales

vs.

* Model (many) small scales

Slide from Comte, Cargese Summer school on turbulence, July 2007
Ratio of time scales: not constant in inertial range

Lagrangian model, $6000^3$ equiv. res.

Ratio $E_M(k)/E_V(k)$: constant in inertial range
Ratio of time scales: not constant in inertial range

Lagrangian model, $6000^3$ equiv. res.

Ratio $E_M(k)/E_V(k)$, constant in inertial range: new paradigm?
Lagrangian-averaged (or alpha) Model for Navier-Stokes and MHD (LAMHD):

the velocity & induction are smoothed on lengths $\alpha_v$ & $\alpha_M$, but not their sources (vorticity & current)

$$v = u_s + \delta v, \quad B = B_s + \delta B,$$

$$G_\alpha(r, t) = \exp[-r/\alpha]/4\pi\alpha^2 r.$$

$$u_s = G_{\alpha_v} \otimes v, \quad B_s = G_{\alpha_M} \otimes B,$$

$$v = (1 - \alpha_v^2 \nabla^2) u_s \text{ and } B = (1 - \alpha_M^2 \nabla^2) B_s$$

Equations preserve invariants (in modified - filtered $L_2 \rightarrow H_1$ form)

McIntyre (mid ‘70s), Holm (2002), Marsden, Titi, …, Montgomery & AP (2002)
Lagrangian-averaged NS & MHD Model Equations for the \textit{ideal} case
Invariants of the MHD-alpha equations in two space dimensions

\[ E = \frac{1}{2} \int d^2x \left( \mathbf{u}_s \cdot \mathbf{v} + B_s \cdot \mathbf{B} \right) , \]

\[ H_C = \frac{1}{2} \int d^2x \mathbf{v} \cdot \mathbf{B}_s , \]

\[ \mathcal{A} = \frac{1}{2} \int d^2x A^2_{s_z} . \]

- The invariants, to which the usual Kolmogorov -like phenomenology will apply, involve BOTH the smoothed fields and the raw ones (H_1 norm instead of L_2)
- In 3D, replace A by magnetic helicity
Lagrangian-averaged NS & MHD dissipative Model Equations

• Advection by smooth velocity field
• The velocity equation involves both the smooth and rough fields
• The induction equation involves only smooth fields, except for dissipation which, in terms of $B_s$, is hyperdiffusive

\[ v_k \sim (1 - \alpha^2_v k^2) u_{s,k}, \quad B_k \sim (1 - \alpha^2_m k^2) B_{s,k} \]
Scientific framework

• In the MHD, understanding the processes by which energy is distributed and dissipated down to kinetic scales, and the role of nonlinear interactions and turbulence in the Sun and for space weather

• Modeling of turbulent flows with magnetic fields in three dimensions, taking into account long-range interactions between eddies and waves, and the geometrical shape of small-scale eddies

• Understanding Cluster observations in preparation for a new remote sensing NASA mission (MMS: Magnetospheric Multi-Scale)
Cancellation Exponent

Rapid change of sign of fields of zero mean (& derivatives)

Ott et al., 1992: sign-singular measure $\mu_i(l)$ as the difference of two probabilities, for disjoint subsets $Q_i(l)$ of size $l$ covering $Q(L)$

$$\mu_i(l) = \frac{1}{\int_{Q(L)} d\mathbf{r} \left| f(\mathbf{r}) \right|} \int_{Q_i(l)} d\mathbf{r} f(\mathbf{r}) \quad (1)$$

As $l$ grows, cancellations between structures of opposite signs occur

$$\chi(l) = \sum_{Q_i(l)} |\mu_i(l)| \sim l^{-\kappa} \quad (2)$$
FIG. 3. (Color) The coarse-grained signed measure of the current $J$ at time $t=7.3$ for four different box sizes, namely $l/L=0.001$, $l/L=0.016$, $l/L=0.059$, $l/L=0.12$, from top to bottom. Colors range from cyan for negative $J$ values to yellow for positive ones, going through blue and brown. Cancellations at large scales are responsible for the decrease in magnitude of the measure.
Cancellation exponent $\kappa$ (upper curve) and magnetic dissipation (lower curve): comparison of DNS and LAMHD

Graham et al., PRE 72, 2005

$\kappa = \frac{d - d_F}{2}$ (see Sorriso-Valvo et al. PoP, 2002)
Variation of correlation with scale

- Solar data, active region
2D-MHD

- Three invariants:
  - Total energy \( E_T = \langle v^2/2 + b^2/2 \rangle \)
  - Cross-helicity \( H_c = \langle v \cdot b \rangle \)
  - Magnetic potential \( E_A = \langle a^2 \rangle \), \( b = \text{curl} \ a \)

- These invariants have different physical dimensions: \( E_A \) is more "large-scale" than \( E_T \) and \( H_c \)

- Statistical equilibria: possibility of an inverse cascade of \( E_A \) together with direct cascades of \( E_T \) and of \( H_c \) (all observed numerically), like in 3D.

- Is \( H_c \) "more" or "less" invariant than \( E_T \)?
  What happens to the ratio \( H_c / E_T \)? (selective decay)
Cascades in forced 2D-MHD

- **Energies** (top: magnetic; bottom: kinetic) as a function of time are correctly represented in the model.

- However, the growth of squared magnetic potential due to the inverse cascade is always smaller than for DNS (solid line).

- Negative resistivity instability which involves the small scales where the filtering occurs.
Inverse Cascade of Magnetic Potential

- Turbulent magnetic resistivity $\eta_{turb} \sim E_V E_M$ in the small scales is $<0$ when $E_M > E_v$, and is responsible for the inverse cascade (AP, JFM 1978; turbulence closure result)

**Solid line: DNS; other lines: $\alpha$ runs**
*arrows indicate values of alpha*

- Change of sign of $\eta_{turb}$ in small scales for $\alpha$ runs (except for the one with $\alpha_m = 0$)
In conclusion:

- The LAMHD model works up to the cut-off length $\alpha$
- The errors are smaller than for under-resolved runs
- The growth rate of large-scale instabilities is OK in 3D

The model allows for computations in regimes of turbulence never explored before at a known given Reynolds number

*Phys. Fluids* 17, 035112; and 18, 045106 and 20, 035107; *Phys Rev E* 76, 056310
Validation of LES: temporal evolution of total energy

Savings in CPU: $0.5^4 \times \frac{1536}{96}^4 \approx 30,000$ (also for memory)
Validation of LES, energy spectrum

![Graph showing energy spectrum comparison between DNS and LES-PH methods.](image)
Validation of LES, Helicity spectrum
Phenomenologies for MHD turbulence

- **MHD could be like fluids** → Kolmogorov spectrum $E_{K41}(k) \sim k^{-5/3}$

Or

- **Slowing-down** of energy transfer to small scales because of Alfvén waves propagation along a (quasi)-uniform field $B_0$: $E_{IK}(k) \sim (\epsilon^T B_0)^{1/2} k^{-3/2}$

(Iroshnikov - Kraichnan (IK), mid ’60s)

$$T_{\text{transfer}} \sim T_{\text{NL}} \ast [T_{\text{NL}}/T_A], \text{ or 3-wave interactions but still with isotropy.}$$

Eddy turn-over time $T_{\text{NL}} \sim l/u_l$ and wave (Alfvén) time $T_A \sim l/B_0$

And

- **Weak turbulence** theory for MHD (Galtier et al PoP 2000): anisotropy develops and the exact spectrum is:

$$E_{\text{WT}}(k) = C_w k_{\text{perp}}^{-2} f(k_{//})$$

Note: WT is IK-compatible when isotropy ($k_{//} \sim k_{\text{perp}}$) is assumed: $T_{\text{NL}} \sim l_{\text{perp}}/u_l$ and $T_A \sim l_{//}/B_0$

Or $k_{\text{perp}}^{-5/3}$ (Goldreich Sridhar, APJ ‘95) ? Or $k_{\text{perp}}^{-3/2}$ (Nakayama ‘99; Boldyrev ‘06, Yoshida ‘07) ?
**Moore’s law**: Doubling of speed of processors every 18 months implies doubling of resolution for DNS in 3D every 6 years …

- Develop models of turbulent flows (*Large Eddy Simulations*, closures, Lagrangian-averaged, …)
- Improve numerical techniques

* Be patient

- Is Adaptive Mesh Refinement (AMR) a solution?

- *If so, how do we adapt? How much accuracy do we need?*
The need for Adaptive Mesh Refinement

Figure 1: From Bill Skamarock, showing the lack of convergence with model resolution.
AMR on 2D Navier-Stokes

Rosenberg et al., JCP 2006;
Aimé Fournier et al., 2008

- Decay for long times (incompressible)
- Formation of dipolar vortex structures
- Lesser number of degrees of freedom (~ 1/4) with AMR, compared to an equivalent pseudo-spectral code (periodic boundary conditions)

(but ....)
AMR in incompressible 2D - MHD turbulence at $R \sim 1000$

- AMR using spectral elements of **different orders**, $P$;
- DNS is in black

- No noticeable differences when using the $L_2$ norms (energy and its dissipation)

- **But accuracy matters** when looking at Max norms, here the current

---

*Rosenberg et al., New J. Phys., 2007*
AMR in 2D - MHD turbulence

- Magnetic X-point configuration in 2D
- Temporal variation of:
  - Dissipation
  - $J_{\text{max}}$
  - Degrees of freedom normalized by the number of modes in a pseudo-spectral code at the same $R_v$, ~33%

Refinement and coarsening criteria …

Rosenberg et al., New J. Phys. 2007
2D -MHD Orszag-Tang vortex with AMR

• **Error** in temporal derivative of total energy (compared to dissipation) is ~ $10^{-3}$
  
  *(computed every 10 time steps)*

• Error in $\mathbf{\psi} \cdot \mathbf{v}$ is ~ $10^{-5}$ *(controlled by a code parameter)*

*Rosenberg et al., New J. Phys. 2007*
Examples of AMR

Hairpin vortex, Euler case
Grauer et al. PRL 80 (1998)

Parallel flux tubes in 3D, ideal run with effective resolution up to $4096^3$
Grauer Marliani PRL 84 (2000)
Thank you for your attention!