Elastic and “Plastic” Depinning in Driven Systems.

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Outline

- **Motivation**
  - Good understanding of the depinning transition of elastic media as a dynamical critical phenomenon (see talks by D.S. Fisher).
  - Many experimental systems are plagued with history dependence of the response and exhibit a crossover from elastic to “plastic” depinning as the parameters are varied (see talks by S. Bhattacharya and E. Andrei).
  - Need for a simple theoretical model to describe such systems.

- **The Model**
  - First ingredient: the generic coarse-grained model for an elastic medium driven over quenched disorder described by Daniel Fisher.
  - Second ingredient: the phenomenology of viscoelasticity described in one of Dave Weitz’s lectures→ frequency-dependent shear modulus and shear viscosity.
  - A model for a driven viscoelastic medium where elastic couplings to displacement are replaced by nonlocal (in time) couplings to local velocities.

- **Mean-Field Solution**
  - Transition from continuous to first order hysteretic depinning as a MF critical point.
  - Stick-slip-type instabilities for pinning potentials with cusps.

- **Relation to other systems and models**
  - The Random Field Ising Model of hysteresis in magnets discussed by Karin Dahmen.
  - Models of crack propagation in heterogeneous solids.

- **Work in progress**
  - Study of the model in finite dimensions by RG methods and numerical simulations.
Rain down dirty windshields


Vortices in superconductors

A. Tonomura, Micron 30, 479 (1999) (Nb films)
From the lectures of S. Bhattacharya we learned that, while theoretical models of the depinning of driven elastic media yield a critical scaling of the velocity $v \sim (F - F_c)^\beta$, with $\beta < 1$ in both 2d and 3d, experiments in CDWs sometime show critical scaling of the IV characteristic with either $\beta < 1$ or $\beta > 1$, sometime show history dependence and hysteresis with “switching” of the type shown below:

![Graph showing IV characteristic with critical scaling](image)

**A. Maeda et al., 1990 ($K_{0.3}MoO_3$)**

- constant Voltage ($\sim$ driving force)
- constant CDW current ($\sim$ CDW velocity)

**History dependence of response generally associated with tearing of the CDW due to phase slips**

⇒ breakdown of the elastic model
Evolution of shape of IV with applied field $H$:

Crossover from elastic to plastic depinning with decreasing interaction strength:

Bhattacharya & Higgins, 1993 (NbSE$_2$)

2d vortex simulations:
(Faleski, AAM & MCM)

Just above $F_c$ vortex motion is confined to a network of channels.
General Continuum Model
of driven system, as introduced by Daniel Fisher

\[ \gamma \partial u(r, t) = F_{\text{ext}} + \int_{r', t'} J(r - r', t - t') \left[ u(r', t') - u(r, t) \right] + F_p(r, u(r, t)) \quad (1) \]

Two classes:

- **Monotonic Models** \( \Rightarrow J(r, t) \geq 0 \)
  
  “no passing rule“ (AAM 1992, scalar field)

  - uniqueness of sliding state
  - no hysteresis

  \( J(r, t) = \mu \delta(t) \nabla^2 \delta(r) \Rightarrow \) conventional short-range elasticity.

- **Nonmonotonic Models** \( \Rightarrow J(r, t) \) can be \( < 0 \)

  possibility of coexistence of pinned and sliding states and **first order** phase transitions.
Generic Model of Driven Elastic Medium

Generic coarse-grained model of an overdamped elastic medium driven through disorder:
local deformations (CDW phase) → \( u(\vec{r}_i) = u_i \)

\[ d = 1 \]

The pinning potential \( V_i(u_i) \) at site \( i \) is periodic in \( u_i \), random in \( i \), e.g.:
\[ V_i(u_i) = h_i \cos(u_i - \beta_i) \]
with \( h_i \) and \( \beta_i \) random variables.

**Mean motion:** \( v(F) = \langle \dot{u}_i \rangle \)

Model for: CDWs, vortex lattice, driven interfaces, ...
Properties of elastic depinning
(see talks by D.S. Fisher)

- **Continuous transition** at a threshold $F_C$ from pinned ($v = 0$) to **sliding** ($v \neq 0$) state.

  ⇒ **Universal critical behavior:**
  
  \[ v \sim (F_{\text{ext}} - F_C)^\beta, \]
  \[ \beta = 1 - (4 - d)/6 \]
  (Fisher (1985), Middleton, Narayan & Fisher, Natterman)

- **No hysteresis:** for a given $F_{\text{ext}}(t)$, **unique** long-time solution (AAM, 1992).

- **Avalanches** when $F_{\text{ext}}$ is slowly increased below $F_C$ with power-law size distribution. Infinite avalanche at $F_C$. (related to Bak-Tang-Wiesenfeld sandpile models, self-organized criticality, Gutenberg-Richter distribution of earthquakes)
Phenomenology of Viscoelasticity
(see lectures by Dave Weitz)

When a frequency-dependent shear stress (of frequency $\omega$) is applied to a viscous liquid, the liquid will respond elastically if $\omega >> 1/\tau$ and will flow if $\omega << 1/\tau$, with $\tau$ the characteristic structural relaxation time discussed by Jorge Kurchan.

- **Elastic response:**
  \[
  \text{elastic stress} = \sigma_{xy}^{\text{el}} = \mu \gamma
  \]
  \[
  \gamma = \text{strain}
  \mu = \text{shear modulus}
  \gamma = \frac{\partial u_x}{\partial y}
  \]

- **Viscous Flow:**
  \[
  \text{viscous stress} = \sigma_{xy}^{\text{vis}} = \eta \dot{\gamma}
  \]
  \[
  \dot{\gamma} = \text{strain rate}
  \eta = \text{shear viscosity}
  \dot{\gamma} = \frac{\partial v_x}{\partial y}
  \]

Here $u_x$ denotes the elastic displacement field and $v_x$ the flow velocity.

To describe viscoelasticity we introduce frequency-dependent moduli and viscosities
(which are indeed probed in experiments)¹:

\[
\sigma_{xy}^{\text{vis}-\text{el}}(\omega) = \mu(\omega)\gamma(\omega) = \bar{\eta}(\omega)\dot{\gamma}(\omega)
\]

or in the time domain:

\[
\sigma_{xy}^{\text{vis}-\text{el}}(t) = \int_0^t dt' C(t-t') \frac{\partial v_x(t')}{\partial y}
\]

It can be shown by a microscopic derivation of the above equation that $C(t)$ is the stress tensor autocorrelation function that describes the relaxation of local stresses in the system. It is related to the shear viscosity via a Green-Kubo formula (see, e.g., Hansen & McDonald, *Theory of Simple Liquids*, Academic Press; or Boon & Yip, *Molecular Hydrodynamics*, McGraw-Hill, 1980), which in this case simply reads

\[
\bar{\eta}(\omega) = \int_0^\infty dt e^{i\omega t} C(t)
\]

Note that more in general the stress tensor autocorrelation function introduces both temporal and spatial nonlocaties in the viscous response. It has the general properties:

\[
\int_0^\infty dt C(t) = \eta
\]

\[
C(t = 0) = G_\infty
\]

Finally the viscoelastic force is given by

\[
F_x^{\text{vis}-\text{el}}(t) = \partial_y \sigma_{xy}^{\text{vis}-\text{el}} = \int_0^t dt' C(t-t') \partial_y^2 v_x(t').
\]
Maxwell Model:
The Maxwell model assumes that \( C(t) \) is simply an exponential, i.e., \( C(t) = G_\infty e^{-t/\tau} \), corresponding to:

\[
G(\omega) = G_\infty \frac{i\omega\tau}{1 + i\omega\tau}
\]

\[
\tilde{\eta}(\omega) = \frac{\eta}{1 + i\omega\tau} \quad G_\infty \tau = \eta_0
\]

We will use this form below. This approximation simplifies things considerably, but is not essential. The viscous force has the following limiting forms:

\[
F_x^{vis} = \left. \eta_0 \frac{\partial^2 v_x}{\partial y^2} \right|_{\tau = 0} \quad \text{for} \quad \tau = 0
\]

\[
F_x^{vis} = \left. \eta_0 \frac{\partial^2 u_x}{\partial y^2} \right|_{\tau \to \infty} \quad \text{for} \quad \tau \to \infty
\]
A Model for Plastic Depinning

Replace elastic couplings by viscoelastic couplings:

\[
\mu (u_j - u_i) \rightarrow \int_{-\infty}^{t} C(t - s) \left[ \dot{u}_j(s) - \dot{u}_i(s) \right] ds
\]

with \( C(t) \) time-correlation function of local stress, e.g.

\[
\begin{align*}
C(t) &= \mu e^{-t/\tau} \\
C(t = 0) &= \mu \quad \text{(elastic modulus)} \\
\int_{0}^{\infty} C(t)dt &= \eta \quad \text{(viscosity)}
\end{align*}
\]

\[
\eta = \mu \tau
\]

- Nonlocal couplings to velocity generated generically upon coarse-graining (cf. derivation of hydrodynamics.)
- The model exhibits both elastic restoring forces and viscous forces on different length and time scales.

Equations of motion:

\[
\dot{u}_i = \sum_{\text{nbrs}} j \int_{-\infty}^{t} C(t - s) \left[ \dot{u}_j(s) - \dot{u}_i(s) \right] ds + F_{\text{ext}} - \frac{\partial V_i(u_i)}{\partial u_i}
\]
Viscoelastic Model

With the Maxwell form of $C(t) = \mu e^{-t/\tau}$, the integro-differential equation of motion can be transformed into a second order differential equation:

$$
\tau \ddot{u}_i + \left[ 1 + \eta + \tau \frac{\partial^2 V_i(u_i)}{\partial u_i^2} \right] \dot{u}_i = \eta \sum_{\text{nbrs } j} (\dot{u}_j - \dot{u}_i) + F_{\text{ext}} - \frac{\partial V_i(u_i)}{\partial u_i}
$$

Pinning potential:

periodic, piecewise quadratic (scallops):

$V(\phi) = \frac{h}{2}(\phi^2 - \phi + \frac{1}{4})$  distribution $\rho(h)$ of $h$

$\rightarrow$ linear pinning force

\[\text{\includegraphics{diagram}}\]

Parameters:

- $\eta$ interaction strength
- $\tau$ duration of velocity coupling
- $\rho(h)$ disorder distribution of width $h_0$

Below $\rho(h) = \frac{1}{h_0} e^{-h/h_0}$
Mean-Field Solution

MFT \((d = \infty)\): \[
\frac{1}{N} \sum_{\text{nbrs } j} C(t) \dot{u}_j \rightarrow C(t)\bar{v}
\]

\[
\bar{v} = \frac{1}{N} \sum_i \dot{u}_i = \langle \dot{u}_i \rangle
\]

Mean Field Equation:

\[
\tau \ddot{u}_i + \gamma(\eta, \tau, h; u_i) \dot{u}_i = F + \eta \bar{v} - \frac{\partial V_i(u_i)}{\partial u_i}
\]

with

\[
\gamma(\eta, \tau, h; u_i) = 1 + \eta + \tau \frac{\partial^2 V_i(u_i)}{\partial u_i^2}
\]

For scalloped potential MFT can be solved exactly under the assumption of steady \(\bar{v}\).

- Solve 2nd order linear DE.
- Find \(T_i(\eta \bar{v} + F_{\text{ext}}, h_i, \eta, \tau)\).
- Impose self-consistency \(\langle T_i^{-1} \rangle = \bar{v}\).
Which Pinning Potential?

Mean-field Theory: Solved for

\[ V_i = h_i \sin[2\pi (u_i - u_{i,0})] \]

(Fisher, 1985)

For piecewise quadratic potential (scallops):

\[ \beta = 1 \]

Which is more generic?

\( d \neq \infty: \)

The effective potential for \( u_i \) changes rapidly when a neighbor “hops”.

\[ \Rightarrow \text{ proper MFT given by scalloped potential.} \]

(supported by functional renormalization arguments)
\( \tau = 0 \)

\[
(1 + \eta) u_i = F_{\text{ext}} + \eta \ddot{V} + \frac{h_i}{2} - h_i u_i
\]

cf. equation for an overdamped single particle:

• **effective friction**: \( 1 + \eta \)

• **effective driving force**: \( F_{\text{ext}} + \eta \ddot{V} \) – depends on particle dynamics.

Viscoelastic particle:

\[
\overline{\dot{u}} = \int_0^\infty dh \frac{h}{1+\eta} \frac{\rho(h)}{\ln \left( \frac{F_{\text{ext}} + \eta \overline{\dot{V}} + h/2}{F_{\text{ext}} + \eta \overline{\dot{V}} - h/2} \right)}
\]

Single particle:

\[
\overline{\dot{u}} = \int_0^\infty dh \frac{h \rho(h)}{\ln \left( \frac{F_{\text{ext}} + h/2}{F_{\text{ext}} - h/2} \right)}
\]
Solution for fixed $h$

$$\rho(h) = \delta(h - 1)$$

$$\mu = 1, \quad \tau = 0.01 \to 100$$

For fixed $h$, nonuniform initial conditions must be chosen to reflect the disorder.

- **Elastic model:** $\bar{v} = F_{\text{ext}} - F_{c}$, $F_{c} = 0.25$

- **Single particle:** $\bar{v} = \left[ \ln \left( \frac{F_{\text{ext}} + 1/2}{F_{\text{ext}} - 1/2} \right) \right]^{-1}$
Analytical MF Solution

There is a critical line $\eta_c(\tau)$ separating single-valued solution (continuous depinning) from multi-valued solutions (hysteretic depinning).

$$\rho(h) = \frac{1}{h_0} \exp\left(-\frac{h}{h_0}\right)$$

$\mu = 3.0$: 
Simulation of MF Model

- Check stability of constant $\bar{v}$ solution.
- Constant $F_{ext}$ or constant $\bar{v}$ drive.
- RK integration
- “Events” are “crossings” where $u_i$ crosses from one scallop to another.

Record $\overline{v}(t)$, event times for $F(t)$ histories.
Numerical Simulation of MF Model

$\eta = 32.0, \ \tau = 0.8, \ N = 16384$:

Comparison of MFT simulation with analytic results:

- Noise in upper branch due to finite system size (decreases as $N^{-1/2}$).

- In the lower branch of the hysteresis cycle the mean field velocity occasionally spikes due to macroscopic events. These fluctuations saturate at large $N$ and are due to stick-slip type instability.

- Premature switching from lower to upper branch.
The Stick-Slip Instability

$\eta = 32.0$, $\tau = 0.0$, $N = 16384$:

Dots are time of crossing events where a $u_i$ with pinning $h_i$ crosses from one scallop to another.

Maximum pinning force is $h$; maximum drive from external and other $u_i$’s is $\eta \overline{v} + F_{ext}$.

(This instability is absent for a smooth pinning potential.)
Scaling behavior near critical point: \((\tau = 0)\) \(\eta_c, F_c\)

\[\Delta v \sim (F - F_c)^{1/\delta}\]
\[\delta_{MF} = 3\]

\[\Delta v \sim (\eta - \eta_c)^{\delta}\]
\[\beta_{MF} = 1/2\]

\[\Delta v \sim (\eta - \eta_c)^{\beta}\]

cf. RFIM (Dahmen, Sethna, ...)

jump from system-spanning avalanche
RFIM of Hysteresis in Magnets

Dahmen & Sethna, 1996

\[
\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} S_i S_j - \sum_i (H + f_i) S_i
\]

random fields \( f_i \) with distribution \( \rho(f) \).

Dynamics:

\[
\frac{1}{\tau_0} \partial_t S_i = \sum_{\langle ij \rangle} J_{ij} S_j - H - f_i
\]

driving force \( F \leftrightarrow \) applied field \( H \)

mean velocity \( v \leftrightarrow \) Magnetization \( M \)

local velocities \( v_i \leftrightarrow \) spins \( S_i \)

Work in progress:

RG calculation to derive a \( \delta - \epsilon \) expansion of the critical exponents about their MF values.

Expect same exponents as RFIM.
Crack Propagation in Brittle Materials

- **Elastic** (long-range) coupling between points on the crack front.

- **Stress overshoot**: inertial stress transfer in which the motion of one segment creates a transient stress (additional to the elastic stress) on other segments.

**J. Schwarz & D.S. Fisher** (cond-mat/0012246):

\[ \sigma_i(t) = \sum_{ij} [h_j(t) - h_i(t)] + M \sum_j [h_j(t) - h_j(t - 1)] \]

From continuous \((M < M_c)\) to discontinuous hysteretic depinning \((M > M_c)\).
Summary and Outlook

- Model of a driven disordered system that exhibits a transition from continuous to hysteretic depinning with disorder and driving force as tuning parameters.

- In MF the transition is a critical point in the universality class of the noneq. RFIM.

- Work in progress: preliminary RG calculations indicate that the critical point survives in finite dimensions.

- Connection with model of crack propagation and earthquakes.
  (J. Schwarz & D.S. Fisher, cond-mat/0012246).