Rapidly rotating Bose-Einstein condensates
in harmonic traps

Alexander Fetter, Stanford University
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*for general references, see [1, 2, 3, 4]
1 Physics of one vortex line in harmonic trap

Assume general three-dimensional trap potential

$$V_{\text{tr}}(\mathbf{r}) = \frac{1}{2} M \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right)$$

Basic idea (Bogoliubov): for weak interparticle potentials, nearly all particles remain in condensate for $T \ll T_c$

- dilute: $s$-wave scattering length $a_s \ll$ interparticle spacing $n^{-1/3}$
- equivalently, require $n a_s^3 \ll 1$
- assume self-consistent condensate wave function $\Psi(\mathbf{r})$
- gives nonuniform condensate density $n(\mathbf{r}) = |\Psi(\mathbf{r})|^2$
- for $T \ll T_c$, normalization requires $N = \int dV |\Psi(\mathbf{r})|^2$
- assume an energy functional

$$E[\Psi] = \int dV \left[ \Psi^* (\mathcal{T} + V_{\text{tr}}) \Psi + \frac{1}{2} g |\Psi|^4 \right] ,$$

where $\mathcal{T} = -\hbar^2 \nabla^2 / 2M$ is kinetic energy operator and $g = 4\pi a_s \hbar^2 / M$ is interaction coupling parameter
• balance of kinetic energy $\langle T \rangle$ and trap energy $\langle V_{tr} \rangle$ gives mean oscillator length $d_0 = \sqrt{\hbar/M \omega_0}$ where $\omega_0 = (\omega_x \omega_y \omega_z)^{1/3}$ is geometric mean.

• balance of kinetic energy $\langle T \rangle$ and interaction energy $\langle gn \rangle$ gives healing length

$$\xi = \frac{\hbar}{\sqrt{2Mgn}} = \frac{1}{\sqrt{8\pi a_s n}}$$

• treat energy $E[\Psi]$ as a functional of $\Psi$ and seek stationary solution

• with fixed normalization and $\mu$ the chemical potential, this gives Gross-Pitaevskii (GP) equation

$$\left( T + V_{tr} + g|\Psi|^2 \right)_{\text{Hartree}} \Psi = \mu \Psi$$

• can interpret nonlinear term as a Hartree potential $V_H(r) = gn(r)$, giving interaction with nonuniform condensate density

• generalize to time-dependent GP equation

$$i\hbar \frac{\partial \Psi}{\partial t} = (T + V_{tr} + V_H) \Psi$$

• this result implies that stationary solutions have time dependence $\exp(-i\mu t/\hbar)$
Introduce hydrodynamic variables

- write $\Psi(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)| \exp[iS(\mathbf{r}, t)]$ with phase $S$
- condensate density is $n(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2$
- current is
  \[ j = \frac{\hbar}{2Mi} [\Psi^* \nabla \Psi - \Psi \nabla \Psi^*] = |\Psi|^2 \frac{\hbar \nabla S}{M} = n \mathbf{v} \]
- identify last factor as velocity $\mathbf{v} = \hbar \nabla S / M$
- note that $\mathbf{v}$ is irrotational so $\nabla \wedge \mathbf{v} = 0$
- general property: circulation around contour $\mathcal{C}$ is
  \[ \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{v} = \frac{\hbar}{M} \oint_{\mathcal{C}} d\mathbf{l} \cdot \nabla S = \frac{\hbar}{M} \Delta S|_{\mathcal{C}} \]
- change of phase $\Delta S|_{\mathcal{C}}$ must be integer times $2\pi$ since $\Psi$ is single-valued
- hence circulation in BEC is \textit{quantized} in units of $\kappa \equiv 2\pi \hbar / M = \hbar / M$
- rewrite time-dependent GP equation in terms of $|\Psi|$ and $S$
  - imaginary part: $\partial n / \partial t + \nabla \cdot (n \mathbf{v}) = 0$
  - real part: generalized Bernoulli equation
Introduction of harmonic trap yields much richer system than a uniform interacting Bose gas

- trap gives new \emph{energy} scale $\hbar \omega_0$ and new \emph{length} scale $d_0 = \sqrt{\hbar/M \omega_0}$
- assume repulsive interactions with $a_s > 0$
- repulsive interactions expand the condensate to larger mean radius $R_0 > d_0$
- as order of magnitude, ground-state energy $E_g$ has the form \cite{5}

$$
\frac{E_g}{N} \sim \hbar \omega_0 \left( \frac{1}{\mathcal{R}^2_{\text{kinetic}}} + \frac{\mathcal{R}^2_{\text{potential}}}{d_0} + \frac{N a_s}{d_0} \frac{1}{\mathcal{R}^3_{\text{interaction}}} \right),
$$

with $\mathcal{R} = R_0/d_0$ the dimensionless expansion ratio of radius

- new dimensionless parameter $N a_s/d_0$ arises from trap
- minimize $E_g$ with respect to $\mathcal{R}$
- if $N a_s/d_0 \lesssim 1$, minimum $E_g$ gives $\mathcal{R} \sim 1$ (ideal gas)
Properties of Thomas-Fermi (TF) limit

• if $Na_s/d_0 \gg 1$, kinetic energy is small and minimum $E_g$ gives

$$\mathcal{R} = \frac{R_0}{d_0} \sim \left( \frac{Na_s}{d_0} \right)^{1/5} \gg 1 \quad (\text{"Thomas-Fermi" limit})$$

• typically, $a_s \sim$ a few nm and $d_0 \sim$ a few $\mu$m

• thus $Na_s/d_0 \sim 10^3$ for $N \sim 10^6$

• ignore kinetic energy (radial gradient of density) and GP equation reduces to simple equation for density

$$gn(r) = g|\Psi(r)|^2 = \mu - V_{tr}(r)$$

where right side is positive and zero elsewhere

• central density is $n(0) = \mu/g$

• TF density is $n(r) = n(0) \left(1 - r^2/R_0^2\right)$ for spherical condensate in isotropic harmonic trap

• condensate radius given by $R_0^2 = 2\mu/M\omega_0^2$

• easily generalized to anisotropic trap: take $R_j^2 = 2\mu/M\omega_j^2$ for $j = x, y, z$
• normalization integral $\int dV \ n(r) = N$ for TF density gives $N(\mu_{TF})$
• easy to obtain $\mu_{TF}/\hbar \omega_0 = \frac{1}{2} (15Na_s/d_0)^{2/5} \gg 1$
• expansion ratio is $R_0/d_0 = (15Na_s/d_0)^{1/5} \gg 1$
• define healing length in terms of the central density
  $$\xi^2 = \frac{1}{8\pi n(0)a_s}$$
• easily obtain the result $\xi R_0 = d_0^2$
• TF limit gives hierarchy of length scales $\xi \ll d_0 \ll R_0$
• $\xi$ will be seen to characterize the vortex-core radius, so TF limit corresponds to vortices with small cores in a large condensate
(a) One vortex line in trapped BEC

First assume bulk condensate with uniform density \( n \) and a single straight vortex line along \( z \) axis

- Gross and Pitaevskii [6, 7]: take condensate wave function
  \[
  \Psi(r) = \sqrt{n} e^{i\phi} f \left( \frac{r_{\perp}}{\xi} \right)
  \]
  where \( r_{\perp} \) and \( \phi \) are two-dimensional polar coordinates
- chemical potential is \( \mu = gn \)
- speed of sound is \( s = \sqrt{\mu/M} \)
- assume \( f(0) = 0 \) and \( f(x) \rightarrow 1 \) for \( x \gg 1 \)
- velocity has circular streamlines with \( \mathbf{v} = (\hbar/M r_{\perp}) \hat{\phi} \)
- this is a quantized vortex line with \( \oint \mathbf{l} \cdot \mathbf{v} = \hbar/M \)
- \( v \sim s \) when \( r_{\perp} \sim \xi \), so vortex core forms by cavitation
- equivalently, centrifugal barrier gives vortex core of radius \( \xi \)
- energy per unit length of vortex is
  \[
  E_v \approx \frac{\pi \hbar^2 n}{M} \ln \left( \frac{1.46 R}{\xi} \right)
  \]
Static behavior of straight vortex line in a trap

Assume axisymmetric trap with

\[ V_{tr}(r_\perp, z) = \frac{1}{2} M \left( \omega_\perp^2 r_\perp^2 + \omega_z^2 z^2 \right) \]

- If \( \omega_z/\omega_\perp \gg 1 \), strong axial confinement gives disk-shaped condensate
- If \( \omega_z/\omega_\perp \ll 1 \), strong radial confinement gives cigar-shaped condensate
- Axisymmetric shape means angular momentum \( L_z \) is conserved for a single vortex on symmetry axis
- Condensate wave function has the form
  \[ \Psi \left( r_\perp, z \right) = e^{i\phi} |\Psi \left( r_\perp, z \right)| \]
- Velocity is \( \mathbf{v} = (\hbar/M r_\perp) \hat{\phi} \), like uniform condensate
- Centrifugal energy again forces wave function to vanish for \( r_\perp \lesssim \xi \)
- Hence density is now toroidal, with a hole along the symmetry axis
In TF limit, the separation of length scales $\xi \ll d_0 \ll R_0$ means that TF density is essentially unchanged

- to calculate energy, use the density of vortex-free TF condensate and cut off logarithmic divergences at core radius $\xi$
- if condensate is in rotational equilibrium at angular velocity $\Omega$, the appropriate energy functional is \cite{8} $E'[\Psi] = E[\Psi] - \Omega \cdot L[\Psi]$ where $L$ is the angular momentum
- let $E'_0$ be energy of rotating vortex-free condensate
- let $E'_1(r_0, \Omega)$ be energy of a rotating condensate with straight vortex that is displaced laterally by distance $r_0$ from symmetry axis
- approximation of straight vortex works best for disk-shaped condensate ($\omega_z \gtrsim \omega_{\perp}$)
- Difference of these two energies is energy associated with formation of vortex $\Delta E'(r_0, \Omega) = E'_1(r_0, \Omega) - E'_0$
- $\Delta E'(r_0, \Omega)$ depends on position $r_0$ of vortex and on $\Omega$
Plot $\Delta E'(r_0, \Omega)$ as function of $\zeta_0 = r_0/R_\perp$ for various fixed $\Omega$ [9]

curve (a) is $\Delta E'(r_0, \Omega)$ for $\Omega = 0$

- $\Delta E'(r_0, 0)$ decreases monotonically with increasing $\zeta_0$
- curvature is negative at $\zeta_0 = 0$
- for no dissipation, fixed energy means constant $\zeta_0$, so that only allowed motion is uniform precession at a fixed distance from origin

- angular velocity is given by variational Lagrangian method [10, 11, 3]

$$\dot{\phi}_0 = \frac{\partial E(r_0)/\partial r_0}{\partial L_z(r_0)/\partial r_0} = \frac{\Omega_m}{1 - r_0^2/R_\perp^2},$$

where $\Omega_m = \frac{3}{2} \left( \hbar/M R_\perp^2 \right) \ln \left( R_\perp/\xi \right)$ is critical angular velocity for onset of metastability for central vortex (discussed below)

- $E(r_0)$ and $L_z(r_0)$ are energy and angular momentum of off-center vortex in nonrotating TF condensate
• precession arises from nonuniform trap potential \((\textit{not image vortex})\) and nonuniform condensate density
• for vortex near the center, \(\dot{\phi}_0 \approx \Omega_m\)
• for larger \(r_0\), precession increases because of reduced TF density near the edge \((\textit{not from image vortex})\)
• compare with experimental studies at JILA [12]
  – theory predicts \(\dot{\phi}/2\pi \approx 1.58 \pm 0.16\) Hz, and
  – experiment finds \(\dot{\phi}/2\pi \approx 1.8 \pm 0.1\) Hz
• in presence of weak dissipation, vortex slowly moves outward along curve (a), following spiral orbit in \(xy\) plane
As $\Omega$ increases, curvature near $r_0 = 0$ decreases

- curve (b) is when curvature near $r_0 = 0$ vanishes
- it corresponds to angular velocity

$$\Omega_m = \frac{3}{2} \frac{\hbar}{M R_{\perp}^2} \ln \left( \frac{R_{\perp}}{\xi} \right) = \frac{3}{5} \Omega_c$$

- for $\Omega \gtrsim \Omega_m$, energy $\Delta E'(r_0, \Omega)$ has local minimum near $r_0 = 0$
- dissipation would now drive vortex back toward symmetry axis
- $\Omega_m$ is angular velocity for onset of metastability
- vortex at center is locally stable for $\Omega > \Omega_m$, but not globally stable, since $\Delta E'(0, \Omega_m)$ is positive
As $\Omega$ increases beyond $\Omega_m$, local minimum of $\Delta E'(r_0, \Omega)$ near center decreases

- curve (c) is for $\Omega_c$ when $\Delta E'(0, \Omega_c)$ vanishes
- central vortex is degenerate in energy with vortex-free state at $\Omega_c$

$$\Omega_c = \frac{5}{2} \frac{\hbar}{2MR_\perp^2} \ln \left( \frac{R_\perp}{\xi} \right) = \frac{5}{3} \Omega_m$$

- for $\Omega > \Omega_c$, central vortex is both locally and globally stable
- as $\Omega$ increases beyond $\Omega_c$, energy barrier near outer edge becomes thinner
- curve (d) illustrates behavior for $\Omega = \frac{3}{2} \Omega_c$
(b) Feynman’s relation for vortex density in rotating superfluids

- solid-body rotation has $v_{sb} = \Omega \wedge r$
- $v_{sb}$ has constant vorticity $\nabla \wedge v_{sb} = 2\Omega$
- each quantized vortex at $r_j$ has localized vorticity
  \[ \nabla \wedge v = \frac{2\pi \hbar}{M} \delta^{(2)}(r_\perp - r_j) \hat{z} \]
- assume $N_v$ vortices uniformly distributed in area $A$ bounded by contour $C$
- circulation around $C$ is $N_v \times 2\pi \hbar / M$
- but circulation in $A$ is also $2\Omega A$
- hence vortex density is $n_v = N_v / A = M\Omega / \pi \hbar$
- area per vortex $1/n_v$ is $\pi \hbar / M\Omega \equiv \pi l^2$ which defines radius $l = \sqrt{\hbar / M\Omega}$ of circular cell
- intervortex spacing $\sim 2l$ decreases like $1/\sqrt{\Omega}$
- analogous to quantized flux lines (charged vortices) in type-II superconductors
(c) Experimental creation and detection of vortices in a dilute trapped BEC

- first vortex made at JILA (1999) [13]
- used nearly spherical $^{87}\text{Rb}$ condensate containing two different hyperfine components
- spin up condensate by coupling the two components with a stirring perturbation
- turn off coupling, leaving one component with trapped quantized vortex surrounding nonrotating core of other component
- use selective tuning to make nondestructive image of either component
- study precession of this vortex with filled core around trap center (also with empty core [12])
- find good fit to theory
- see no outward radial motion for $\sim 1$ s, so dissipation is small on this time scale
École Normale Supérieure (ENS) group in Paris studied vortex creation in elongated rotating cigar-shaped condensate with one component [14, 15]

- used off-center toggled rotating laser beam to deform the transverse trap potential and stir the condensate at an applied frequency $\Omega/2\pi \lesssim 200 \text{ Hz}$
- find vortex appears at a critical frequency $\Omega_c \approx 0.7\omega_\perp$ (detected by expanding the condensate, which now has a disk shape, with vortex core as expanded hole)
- this value of $\Omega_c$ is significantly ($\sim 70\%$) higher than that predicted by TF thermodynamic critical angular velocity
- vortex nucleation is dynamical process associated with surface instability (quadrupole oscillation)
• ENS group observed small vortex arrays of up to 11 vortices (arranged in two concentric rings)
• like patterns predicted and seen in superfluid $^4$He
• MIT group has prepared considerably larger rotating condensates in less elongated trap
• they have observed triangular vortex lattices with up to 130 vortices [16]
• like Abrikosov lattice of quantized flux lines (which are charged vortices) in type-II superconductors
• JILA group has now made large rotating condensates with several hundred vortices and angular velocity $\Omega/\omega_\perp \approx 0.995$ [17]
• these rapidly rotating systems open many exciting new possibilities (discussed below)
2  Vortex arrays in mean-field (GP) regime (these are coherent states)

Qualitative features

As $\Omega$ increases, the vortex density $n_v = M\Omega/\pi\hbar$ increases linearly following the Feynman relation

- in addition, centrifugal forces expand the condensate radially, so that the area also increases
- hence the number of vortices $N_v = n_v\pi R^2_\perp = M\Omega R^2_\perp/\hbar$ increases faster than linearly with $\Omega$
- conservation of particles implies that the condensate also shrinks axially
- TF approximation assumes that interaction energy $\langle g|\Psi|^4\rangle$ and trap energy $\langle V_{tr}|\Psi|^2\rangle$ are large relative to kinetic energy for density variations $(\hbar^2/M)\langle(\nabla|\Psi|)^2\rangle$
- expansion of condensate means that central density eventually becomes small and TF picture fails
(a) Mean-field Thomas-Fermi regime

Quantitative description of rotating TF condensate

Kinetic energy of condensate involves

$$\frac{\hbar^2}{2M} \int dV \left| \nabla \Psi \right|^2 = \int dV \left( \frac{1}{2} M v^2 |\Psi|^2 + \frac{\hbar^2}{2M} \int dV \left( \nabla |\Psi| \right)^2 \right)$$

where $\Psi = \exp(iS)|\Psi|$ and $v = \hbar \nabla S/M$ is flow velocity

- generalized TF approximation: retain the energy of superflow but ignore the energy from density variation
- this approximation will fail eventually when vortex lattice becomes dense and cores start to overlap
- assume axisymmetric trap $V_{tr} = \frac{1}{2} M \left( \omega^2_\perp r^2_\perp + \omega^2_z z^2 \right)$
- in rotating frame, generalized TF energy functional is

$$E'[\Psi] = \int dV \left[ \left( \frac{1}{2} M v^2 + V_{tr} - M \Omega \cdot r \wedge v \right) |\Psi|^2 + \frac{1}{2} g |\Psi|^4 \right]$$
For $\Omega$ along $z$, can rewrite $E'[\Psi]$ as

$$E'[\Psi] = \int dV \left[ \frac{1}{2} M \left( \mathbf{v} - \mathbf{v}_{sb} \right)^2 |\Psi|^2 + \frac{1}{2} M \omega_z^2 z^2 |\Psi|^2 ight.$$ 

$$+ \frac{1}{2} \left( \omega_\perp^2 - \Omega^2 \right) r_\perp^2 |\Psi|^2 + \frac{1}{2} g |\Psi|^4 \right]$$

- here, $\mathbf{v}$ is flow velocity generated by all the vortices and $\mathbf{v}_{sb} \equiv \mathbf{v} \wedge \mathbf{r}$ is solid-body rotation
- in the rotating frame, the dominant effect of the dense vortex array is that spatially averaged flow velocity $\langle \mathbf{v} \rangle$ is close to $\mathbf{v}_{sb}$
- hence can ignore first term in $E'[\Psi]$, giving

$$E'[\Psi] \approx \int dV \left[ \frac{1}{2} M \omega_z^2 z^2 |\Psi|^2 + \frac{1}{2} \left( \omega_\perp^2 - \Omega^2 \right) r_\perp^2 |\Psi|^2 ight.$$ 

$$+ \frac{1}{2} g |\Psi|^4 \right]$$

- $E'$ now looks exactly like TF energy for nonrotating condensate but with a \textit{reduced} radial trap frequency $\omega_\perp^2 \rightarrow \omega_\perp^2 - \Omega^2$
Now have TF wave function that depends explicitly on $\Omega$ through the altered radial trap frequency $\omega^2 \rightarrow \omega^2 - \Omega^2$

$$|\Psi(r_\perp, z)|^2 = n(0) \left( 1 - \frac{r_\perp^2}{R^2} - \frac{z^2}{R_z^2} \right)$$

where $R^2_\perp = 2\mu/[M(\omega^2_\perp - \Omega^2)]$ and $R^2_z = 2\mu/M\omega^2_z$

- must have $\Omega < \omega_\perp$ to retain radial confinement
- normalization $\int dV |\Psi|^2 = N$ shows that
  $$\frac{\mu(\Omega)}{\mu(0)} = \left( 1 - \frac{\Omega^2}{\omega^2_\perp} \right)^{2/5}$$

  in three dimensions
- central density given by $n(0) = \mu(\Omega)/g$
- $n(0)$ decreases with increasing $\Omega$ because of reduced radial confinement
• TF formulas for condensate radii show that
\[
\frac{R_z(\Omega)}{R_z(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{1/5}, \quad \frac{R_\perp(\Omega)}{R_\perp(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{-3/10}
\]
confirming axial shrinkage and radial expansion
• aspect ratio changes
\[
\frac{R_z(\Omega)}{R_\perp(\Omega)} = \frac{R_z(0)}{R_\perp(0)} \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{1/2}
\]
• this last effect provides an important diagnostic tool to determine actual angular velocity \(\Omega\) [18]
• these JILA experiments [18] obtain rapidly rotating condensates by rotating thermal cloud above \(T_c\) and then cooling to \(T \ll T_c\)
• method works by deforming the normal cloud from disk-shaped to cigar-shaped, then removing atoms near the long ends (they have small angular momentum)
• measured aspect ratio indicated that \(\Omega/\omega_\perp\) became as large as 0.94
How uniform is the vortex array?

The analysis of the TF density profile $|\Psi_{TF}|^2 = n_{TF}$ in the rotating condensate assumed that the flow velocity $\mathbf{v}$ was precisely the solid-body value $\mathbf{v}_{sb} = \Omega \wedge \mathbf{r}$

- this led to the cancellation of the contribution
  \[ \int dV (\mathbf{v} - \Omega \wedge \mathbf{r})^2 n_{TF} \]
  in the TF energy functional
- a more careful study [19] shows that there is a small nonuniformity in the vortex lattice
- specifically, each regular vortex lattice position vector $\mathbf{r}_j$ experiences a small displacement field $\mathbf{u}(\mathbf{r})$, so that $\mathbf{r}_j \rightarrow \mathbf{r}_j + \mathbf{u}(\mathbf{r}_j)$
- as a result, the two-dimensional vortex density changes to
  \[ n_v(\mathbf{r}) \approx \overline{n}_v (1 - \nabla \cdot \mathbf{u}) \]
  where $\overline{n}_v = M\Omega/\pi \hbar$ is the uniform Feynman value
• near the $j$th vortex core, the flow velocity is a singular part

$$\mathbf{v}_{\text{sing}} = \frac{\hbar}{M} \mathbf{\hat{z}} \wedge (\mathbf{r} - \mathbf{r}_j) \left| \mathbf{r} - \mathbf{r}_j \right|^2$$

plus a smooth background $\mathbf{v}(\mathbf{r})$

• the smooth background velocity can be evaluated as an integral over the slightly nonuniform vortex density

$$\mathbf{v}(\mathbf{r}) \approx \frac{\hbar}{M} \int d^2\mathbf{r'} n_v [1 - \nabla' \cdot \mathbf{u}(\mathbf{r'})] \frac{\mathbf{\hat{z}} \wedge (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^2} \approx \Omega \wedge [\mathbf{r} - 2\mathbf{u}(\mathbf{r})]$$

where the second term follows with an integration by parts using $\nabla^2 \ln |\mathbf{r} - \mathbf{r'}| = 2\pi \delta^{(2)}(\mathbf{r} - \mathbf{r'})$

• the first term is the usual solid-body rotation $\Omega \wedge \mathbf{r}$, and the second term shows how the distortion in the vortex lattice affects the mean induced velocity
the new term in the energy is nonzero contribution from the local integral inside the \( j \)th unit cell

\[
\sum_j \int dV_j \frac{M}{2} (v_{\text{sing}} + \bar{v} - \Omega \wedge r_j)^2 n_{TF}(r_j)
\]

and then summed over the vortex lattice

since the particle density and the vortex density vary slowly over each unit cell, replace \( \sum_j \) with an integral weighted with the nonuniform vortex density \( n_v(r) \)

the dominant solid-body contribution \( \Omega \wedge r \) cancels, and the remaining parts are the energy of the \( j \)th vortex inside the local circular cell (from \( v_{\text{sing}} \)) and the contribution from the distortion of the lattice

the radius of the local unit cell is \( l(r) = 1/\sqrt{\pi n_v(r)} \), which includes the lattice distortion

the additional kinetic energy becomes approximately

\[
\int dV \ n_{TF} \left[ \frac{\pi \hbar^2}{2M} n_v (1 - \nabla \cdot u) \ln \left( \frac{1}{\pi \bar{n}_v \xi^2} \right) + 2M \Omega^2 u^2 \right]
\]

with no other dependence on \( u \) to leading logarithmic order
• vary this energy with respect to \( u \) and obtain the Euler-Lagrange equation, which can be solved to give

\[
\mathbf{u}(r) \approx -\frac{1}{8\pi n_v} \ln \left( \frac{\bar{l}^2}{\xi^2} \right) \nabla \ln n_{TF}(r)
\]

\[
\approx \frac{\bar{l}^2}{4R_\perp} \ln \left( \frac{\bar{l}^2}{\xi^2} \right) \frac{r}{1 - r^2/R_\perp^2}
\]

where \( \bar{l}^2 = 1/\pi n_v \) can be taken as the mean circular cell radius inside the slowly varying logarithm.

• the deformation of the regular vortex lattice is purely radial (as expected from symmetry).

• \( R_\perp^2/\bar{l}^2 \) is the number of vortices \( \mathcal{N}_v \) in the rotating condensate, so that the nonuniform distortion is small, of order \( 1/\mathcal{N}_v \) (at most a few %), even though the TF number density \( n_{TF} \) changes dramatically near edge.

• recent JILA experiments [20] confirm these predicted small distortions for relatively dense vortex lattices.

• correspondingly, the vortex density becomes

\[
n_v(r) \approx \bar{n}_v - \frac{1}{2\pi R_\perp^2} \ln \left( \frac{\bar{l}^2}{\xi^2} \right) \frac{1}{(1 - r^2/R_\perp^2)^2}
\]

(the correction is again of order \( 1/\mathcal{N}_v \))
Tkachenko oscillations of the vortex lattice

In 1966, Tkachenko [21] studied the equilibrium arrangement of a rotating vortex array as model for superfluid $^4$He.

- assumed two-dimensional incompressible fluid with straight vortices
- showed that a triangular lattice has lowest energy in rotating frame
- studied small perturbations about the equilibrium positions and found unusual collective motion in which the vortices undergo a nearly transverse wave of lattice distortions (analogous to two-dimensional transverse “phonons” in the vortex lattice, but with no change in fluid density)
- for long wavelengths (small $k$), Tkachenko found a linear dispersion relation $\omega_k \approx c_T k$
- speed of Tkachenko wave is $c_T = \sqrt{\frac{1}{4} \hbar \Omega / M} = \frac{1}{2} \bar{l} \Omega$, where $\bar{l} = \sqrt{\hbar / M \Omega}$ is radius of circular vortex cell
More generally, the vortices can also undergo bending motions, leading to a collective version of Kelvin helical wave on a single vortex (not discussed here)

- analysis of small perturbations in a vortex lattice yields the long-wavelength dispersion relation [22, 23]

\[
\omega^2 \approx (2\Omega)^2 \frac{k_z^2 + \frac{1}{16} k^4_{\perp} l^2}{k_z^2 + k^2_{\perp}}
\]

where \(k_z\) and \(k_{\perp}\) are the components of \(k\) parallel and perpendicular to the rotation axis

- for \(k_z \to 0\), this expression reproduces the Tkachenko result \(\omega \approx c_T k_{\perp}\)

- for \(k_{\perp} \to 0\), reproduces classical inertial waves with \(\omega = \pm 2\Omega\)

- these modes have not been observed in superfluid \(^4\)He because visualizing vortices is very difficult
In a rotating gas, the compressibility becomes important, as shown by Sonin [24, 25] and Baym [26]

- let the speed of sound in the compressible gas be $c_s$
- for a wave propagating in the $xy$ plane, the coupling between the vortices and the compressible fluid leads to generalized dispersion relation

$$\omega^2 = c_T^2 \frac{c_s^2 k^4}{4\Omega^2 + c_s^2 k^2}$$

- if $k \gg \Omega/c_s$, recover Tkachenko’s result $\omega = c_T k$ (incompressible limit)
- but if $k \ll \Omega/c_s$, mode becomes soft with $\omega \propto k^2$
- Sonin [25] obtains dynamical equations for waves in a nonuniform condensate, along with appropriate boundary conditions at the outer surface
- Baym [26] uses theory for uniform condensate plus approximate boundary conditions from Anglin and Crescimanno [27]
- rough agreement with JILA experiments [28] on low-lying Tkachenko modes in rapidly rotating BEC (up to $\Omega/\omega_\perp \approx 0.975$)
What happens to vortex core radius as $\Omega \to \omega_\perp$?

- recall simple estimate
  \[ \xi^2 = \frac{1}{8\pi n(0)a_s} \]

- this expression implies that vortex core size $\xi$ diverges for $\Omega \to \omega_\perp$ because $n(0) \propto \mu(\Omega) \propto (1 - \Omega^2/\omega_\perp^2)^{2/5}$

- improved description generalizes TF model to include the circulating flow velocity around each core with a mean Wigner-Seitz circular cell of radius $l = \sqrt{\hbar/M\Omega}$

- includes spatial variation of density near core and treats $\xi$ as a variational parameter [29, 30]

- as $\Omega$ increases and $l$ decreases, predict that $\xi$ increases until $\xi^2/l^2 \sim 0.5$ and this ratio then remains fixed as $\Omega$ continues to increase

- in this limit, the vortex cores occupy a constant finite fraction ($\sim 0.5$) of unit cell

- Recent JILA experiments [17] have reached rotation rates $\Omega/\omega_\perp \gtrsim 0.99$, and more detailed studies confirm this growth and saturation of the core size [20]
(b) Mean-field quantum Hall regime

Lowest-Landau-Level (quantum Hall) behavior

When the vortex cores overlap, kinetic energy associated with density variation around each vortex core becomes important

- hence the TF approximation breaks down (it ignores this kinetic energy from density variations)

- it is preferable to return to full GP energy functional $E'[\Psi]$ in the rotating frame.

- in this limit of rapid rotations ($\Omega \lesssim \omega_\perp$), Ho [31] suggested an important rewriting of the same quantity that incorporates kinetic energy exactly

- in this limit of rapid rotation, the condensate expands and becomes effectively two dimensional

- for simplicity, treat a two-dimensional condensate that is uniform in the $z$ direction over a length $Z$

- condensate wave function $\Psi(r_\perp, z)$ can be written as $\sqrt{N/Z} \psi(r_\perp)$, where $\psi(r_\perp)$ is a two-dimensional wave function with unit normalization $\int d^2r |\psi|^2 = 1$
General two-dimensional energy functional in rotating frame becomes

\[ E'[\psi] = \int d^2r \psi^* \left( \frac{p^2}{2M} + \frac{1}{2} M \omega_\perp^2 r_\perp^2 - \Omega L_z + \frac{1}{2} g_{2D} |\psi|^2 \right) \psi, \]

where \( p = -i\hbar \nabla \), \( L_z = \hat{z} \cdot r \times p \), and \( g_{2D} = N g/Z \).

This energy functional can be rewritten exactly as

\[ E'[\psi] = \int d^2r \psi^* \left[ \frac{(p - M \omega_\perp \times r_\perp)^2}{2M} \right. \]

\[ + (\omega_\perp - \Omega) L_z + \frac{1}{2} g_{2D} |\psi|^2 \left. \right] \psi, \]

where \( \omega_\perp \equiv \hat{z} \omega_\perp \)

- assume that \( \Omega/\omega_\perp \rightarrow 1 \) and that interaction energy \( \int d^2r \frac{1}{2} g_{2D} |\psi|^4 \) is small
- hence focus on the first line
• in this limit, energy becomes

\[ E'_L[\psi] = \int d^2r \psi^* \frac{(\mathbf{p} - M\omega_\perp \times \mathbf{r}_\perp)^2}{2M} \psi \]

• define equivalent uniform magnetic field \( \mathbf{B} = -2M\omega_\perp/|e| \)

• define equivalent vector potential \( \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} \)

• here, we use symmetric gauge to describe magnetic field, with \( \mathbf{B} = \nabla \times \mathbf{A} \)

• approximate \( E'_L[\psi] \) is precisely the Hamiltonian of a single particle with charge \(-|e|\) moving in the \( xy \) plane in this magnetic field \( \mathbf{B} \)

\[ \mathcal{H}_L = \int d^2r \psi^* \frac{(\mathbf{p} - |e|\mathbf{A})^2}{2M} \psi \]

• this one-body Hamiltonian was solved by Landau in 1930, but in different gauge (now known as “Landau gauge”)

• for solution in symmetric gauge, see Ref. [32]
Here, the exact eigenfunctions can be written \( \psi_{nm}(r_\perp) \), where \( n \geq 0 \) and \( m \geq 0 \) are non-negative integers and \( n \) specifies the “Landau level”

- for these Landau eigenfunctions, the eigenvalues of \( \mathcal{H}_L \) are \( \epsilon_{nm} = \hbar \omega_\perp (2n + 1) \)
- evidently, the eigenvalues are independent of \( m \), so that the states in a given Landau level are massively degenerate
- these eigenfunctions are also eigenstates of \( L_z \) with eigenvalues \( \hbar (m - n) \)
- apart from the interaction energy, the Landau-level eigenfunction \( \psi_{nm} \) is an eigenstate of full one-particle Hamiltonian

\[
\frac{(p - |e|A)^2}{2M} + (\omega_\perp - \Omega) L_z
\]

with eigenvalue

\[
\hbar [(\omega_\perp + \Omega) n + (\omega_\perp - \Omega) m + \omega_\perp]
\]

- small positive value of \( \omega_\perp - \Omega \ll \omega_\perp + \Omega \) lifts the degeneracy associated with the index \( m \)
Interaction effects tend to mix the various single-particle eigenfunctions $\psi_{nm}$

- if $\omega_\perp - \Omega$ is sufficiently small and if interaction energy is small, then there is energy gap $2\hbar \omega_\perp$ between the lowest Landau level and the excited Landau levels, and energy is independent of $m$

- this requires $g_{2D} n \lesssim \hbar \omega_\perp$, where $n$ is the mean two-dimensional particle density (note that $g_{2D} n \sim \mu$, where $\mu$ is the chemical potential)

- the assumption of small interaction energy may be valid because centrifugal forces dramatically expand the condensate as $\Omega \to \omega_\perp$

- hence assume that the system is solely in the lowest Landau level ("LLL") and construct the approximate solution of the GP equation from this restricted set of eigenfunctions $\psi_{0m}$
LLL eigenfunctions have a very simple form

$$\psi_{0m}(r_\perp) \propto r_\perp^m e^{im\phi} e^{-r_\perp^2/2d_\perp^2}$$

- here, $d_\perp = \sqrt{\hbar/M\omega_\perp}$ is analogous to the “magnetic length” in the Landau problem

- in terms of a complex variable $\zeta \equiv x + iy$, these LLL eigenfunctions have an extremely simple form

$$\psi_{0m} \propto \zeta^m e^{-r_\perp^2/2d_\perp^2}$$

with $m \geq 0$ (note that $\zeta = r_\perp e^{i\phi}$ when expressed in two-dimensional polar coordinates)

- assume that the GP wave function is a finite linear combination of these LLL eigenfunctions

$$\psi_{GP}(r_\perp) = \sum_m c_m \psi_{0m}(r_\perp) = f(\zeta) e^{-r_\perp^2/2d_\perp^2}$$

where $f(\zeta)$ is an analytic function of the complex variable $\zeta$

- specifically, $f(\zeta)$ is a finite polynomial and thus can be factorized as $f(\zeta) = \prod_j (\zeta - \zeta_j)$ apart from overall constant
In this way, we are led to the very simple approximate GP solution

$$\psi_{LLL}(r_\perp) = C \prod_j (\zeta - \zeta_j) e^{-r^2_\perp/2d^2_\perp}$$

where $C$ is a normalization constant

- the product $\prod_j (\zeta - \zeta_j)$ is a complex polynomial that vanishes at each of the points $\{\zeta_j\}$, so that these are the positions of the nodes of $\psi$
- in addition, phase of wave function increases by $2\pi$ whenever $\zeta$ moves around any of these zeros $\{\zeta_j\}$
- we conclude that the LLL trial solution has singly quantized vortices located at positions of zeros $\{\zeta_j\}$
- spatial variation of number density $n(r_\perp) = |\psi_{LLL}(r_\perp)|^2$ is determined by spacing of the vortices, so that core size is comparable with the intervortex spacing $\bar{l} = \sqrt{\hbar/M\Omega}$ which is simply $d_\perp$ in the limit $\Omega \approx \omega_\perp$
- this approximate solution thus generalizes previous TF wave function in the limit $\Omega \rightarrow \omega_\perp$
Take this LLL trial function seriously and study its properties

• since LLL wave functions play a crucial role in the quantum Hall effect (two-dimensional electrons in a strong magnetic field), this LLL regime has been called “mean-field quantum Hall” limit [33]

• note that we are still in the regime governed by GP equation, so there is still a BEC

• corresponding many-body ground state is simply a Hartree product with each particle in same one-body solution $\psi_{LLL}(r_\perp)$, namely

$$\Psi_{GP}(r_1, r_2, \cdots, r_N) \propto \prod_{j=1}^{N} \psi_{LLL}(r_j)$$

• this is coherent (superfluid) state, since a single GP state $\psi_{LLL}$ has macroscopic occupation
• study logarithm of the particle density for this LLL state \( \ln n_{LLL}(\mathbf{r}_\perp) = \ln |\psi_{LLL}(\mathbf{r}_\perp)|^2 \)

• use \( \psi_{LLL} \) to find

\[
\ln n_{LLL}(\mathbf{r}_\perp) = -\frac{r_\perp^2}{d_\perp^2} + 2 \sum_j \ln |\mathbf{r}_\perp - \mathbf{r}_j|
\]

• apply two-dimensional Laplacian

• use \( \nabla^2 \ln |\mathbf{r} - \mathbf{r}_j| = 2\pi \delta^{(2)} (\mathbf{r} - \mathbf{r}_j) \)

• find

\[
\nabla^2 \ln n_{LLL}(\mathbf{r}_\perp) = -\frac{4}{d_\perp^2} + 4\pi \sum_j \delta^{(2)} (\mathbf{r}_\perp - \mathbf{r}_j)
\]

• here, sum over delta functions is precisely the vortex density \( n_v(\mathbf{r}_\perp) \)

• this result relates \textit{particle} density \( n_{LLL}(\mathbf{r}_\perp) \) in LLL approximation to \textit{vortex} density \( n_v(\mathbf{r}_\perp) \) [31, 33]

\[
\frac{1}{4} \nabla^2 \ln n_{LLL}(\mathbf{r}_\perp) = -\frac{1}{d_\perp^2} + \pi n_v(\mathbf{r}_\perp)
\]
• if vortex lattice is exactly uniform (so $n_v$ is constant),
  then density profile is strictly Gaussian, with $n_{LLL}(r_\perp) \propto \exp(-r_\perp^2/\sigma^2)$ and $\sigma^{-2} = d_\perp^{-2} - \pi n_v$

• in this case, $\sigma^2 \gg d_\perp^2$

• more precisely, $\sigma^{-2} \propto \omega_\perp - \Omega$

• Watanabe et al. [33] argue that the density profile should independently have an inverted parabolic (TF) shape $n_{LLL}(r_\perp) \propto 1 - r_\perp^2/R_\perp^2$

• then find nonuniform vortex density with

$$n_v(r_\perp) \approx \frac{1}{\pi d_\perp^2} - \frac{1}{\pi R_\perp^2} \frac{1}{(1 - r_\perp^2/R_\perp^2)^2}$$

similar to result at lower $\Omega$ [19]

• independently, numerical work by Cooper et al. [34] shows that allowing the vortices in the LLL to deviate from the triangular array near the outer edge lowers the energy
3 Behavior for $\Omega \gtrsim \omega_\perp$

What happens beyond the “mean-field quantum Hall” regime is still subject to vigorous debate

(a) Beyond GP regime (*correlated states*)

- define the ratio $\nu \equiv N/N_v$ of the number of atoms per vortex
- because of similarities to a two-dimensional electron gas in a strong magnetic field, $\nu$ is called the “filling fraction” [35, 36]
- current experiments [17] have $N \sim 10^5$ and $N_v \sim$ several hundred, so $\nu \sim$ a few hundred
- numerical studies [36] for small number of vortices ($N_v \lesssim 8$) and variable $N$ indicate that the coherent GP state is favored for $\nu \gtrsim 6$
• for smaller $\nu$ there is a sequence of highly correlated states similar to some known from the quantum Hall effect, in particular a bosonic version of the Laughlin state [36] (here $z_j = x_j + iy_j$ refers to $j$th particle)

$$\Psi_{\text{Lau}}(r_1, r_2, \cdots, r_N) \propto \prod_{j<k} (z_j - z_k)^2 \exp \left( -\sum_{j=1}^{N} \frac{|z_j|^2}{2d_\perp} \right)$$

• these correlated many-body states are qualitatively different from coherent GP form

- $\Psi_{GP}(r_1, r_2, \cdots, r_N) \propto \prod_j \psi(r_j)$ is the Hartree product of $N$ factors of same one-body function $\psi(r)$
- the product $\prod_{jk}(z_j - z_k)^2$ in $\Psi_{\text{Lau}}(r_1, r_2, \cdots, r_N)$ involves $N(N-1)/2$ factors for all possible pairs of particles and vanishes whenever two particles are close together
- this is the source of the correlations
- for large $N$, correlated form $\Psi_{\text{Lau}}$ is much more difficult to use
(b) Addition of quartic potential

One way to avoid singularity when $\Omega \to \omega_\perp$ is to add a quartic confining potential \cite{37, 38, 39}

- now have a total potential with quadratic and quartic terms
  \[ V_{tr} = \frac{1}{2} M \omega_\perp^2 \left( r^2 + \lambda \frac{r^4}{d_\perp^2} \right) \]
  where the dimensionless constant $\lambda$ fixes the quartic admixture
- allows access to regime $\Omega/\omega_\perp \geq 1$
- studied experimentally at ENS, Paris \cite{40}, where a blue-detuned axial laser provided the weak quartic confinement ($\lambda \sim 10^{-3}$ and $\omega_\perp/2\pi \approx 64.8$ Hz)
- find regular vortex lattice for $\Omega \lesssim \omega_\perp$
- find disordered vortex lattice for $\omega_\perp \lesssim \Omega$
- near $\Omega \approx 1.05 \omega_\perp$, the system seems to break up
- TF theory predicts a reduced density at center, which is observed
What is happening?

- ENS condensate is nearly spherical for $\Omega \sim \omega_\perp$, so three-dimensional effects are important.

- They suggest repeating the experiment with strong axial confinement to see if three-dimensional effects dominate and cause instability.

- GP analysis in two dimensions finds nothing like the observed break up [38, 39, 41].

- Is there some sort of transition from a GP state to a highly correlated state in the regime $\Omega \gtrsim \omega_\perp$?

- This issue remains very uncertain.
References


[34] N. R. Cooper, S. Komineas, and N. Read, cond-mat/0404112.