Evolutionary Game Theory

non-equilibrium and non-linear dynamics of interacting particle systems

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Game Theory

John Nash:
"An equilibrium is reached as soon as no party can increase its profit by unilaterally deciding differently."

John Maynard-Smith and George R. Price:
"A strategy is called evolutionary stable if a population of individuals homogenously playing this strategy is able to outperform and eliminate a small amount of any mutant strategy introduced into the population."

Strategic Games

Mathematical description of strategic situations, in which an individual’s success in making choices depends on the choices of others.

Prisoner’s Dilemma:

<table>
<thead>
<tr>
<th></th>
<th>Cooperator (C)</th>
<th>Defector (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1 year</td>
<td>10 years</td>
</tr>
<tr>
<td>D</td>
<td>0 years</td>
<td>0 years</td>
</tr>
</tbody>
</table>

(D,D) is a Nash equilibrium where unilateral deviation does not pay off.
Two suspects of a crime are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal. If one testifies (defects from the other) for the prosecution against the other and the other remains silent (cooperates with the other), the betrayer goes free and the silent accomplice receives the full 10-year sentence. If both remain silent, both prisoners are sentenced to only 1 year in jail for a minor charge. If each betrays the other, each receives a five-year sentence. Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation. How should the prisoners act?

The fundamental problem of cooperation:

<table>
<thead>
<tr>
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<th>Defector (D)</th>
</tr>
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<tbody>
<tr>
<td>C</td>
<td>( b - c )</td>
<td>( -c )</td>
</tr>
<tr>
<td>D</td>
<td>( b )</td>
<td>0</td>
</tr>
</tbody>
</table>

General two-player games

<table>
<thead>
<tr>
<th></th>
<th>Reward</th>
<th>Sucker's payoff</th>
<th>Punishment</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
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The snowdrift game:

<table>
<thead>
<tr>
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<th>Defector (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>( b - c/2 )</td>
<td>( b - c )</td>
</tr>
<tr>
<td>D</td>
<td>( b )</td>
<td>0</td>
</tr>
</tbody>
</table>
Evolutionary Game Theory

Consider a population of size $N$

$N_i$ individuals play strategy $A_i$: $a_i = N_i/N$ (frequency)

Composition of the population is updated by some (evolutionary) rules: $N_i(t) \to N_i(t+dt)$

Moran process:
- pick two at random
- the fitter wins

Rate Equations

“Chemical” reactions:

$A + B \xrightarrow{k_A} A + A$
$B + C \xrightarrow{k_B} B + B$
$C + A \xrightarrow{k_C} C + C$

Rate equations:

$\frac{da}{dt} = a(k_{AB} - k_{C}}$
$\frac{db}{dt} = b(k_{BC} - k_{AA})$
$\frac{dc}{dt} = c(k_{CA} - k_{BB})$

Fitness and replicator equations

Payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
</tr>
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<tr>
<td>$A$</td>
<td>$p_{11} := R$</td>
<td>$p_{12} := S$</td>
</tr>
<tr>
<td>$B$</td>
<td>$p_{21} := T$</td>
<td>$p_{22} := P$</td>
</tr>
</tbody>
</table>

Frequencies: $a = N_A/N$, $b = N_B/N = (1 - a)$

Fitness = expected payoff:

$f_A(a) = Ra + S(1 - a)$, $f_B(a) = Ta + P(1 - a)$

$\dot{f}(a) = a f_A(a) + (1 - a) f_B(a)$

Replicator dynamics:

$\frac{da}{dt} = (f_A(a) - f(a)) a$
$\frac{db}{dt} = (f_B(a) - f(a)) b$
Note that in populations the strategy of an individual is fixed. What varies is the number of individuals with a particular strategy.
Microbial Laboratory Communities:
model systems for competition, cooperation, ...

Colicinogenic Bacteria

Toxin producing (colicinogenic) E. coli (C) carry a 'col' plasmid; genes for colicin, colicin specific immunity proteins, lysis protein

Colicin-sensitive bacteria (S)

Colicin-resistant bacteria (R) are mutations of S with altered cell membrane proteins that bind and translocate colicin

R outgrows C; no col for 'col'

C kills S

S outgrows R; better nutrient uptake

Nonlinear Dynamics of 2-Player Games

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<td>Temptation</td>
<td>Punishment</td>
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Replicator dynamics:
\[
\frac{\partial a}{\partial t} = a(1-a)(f_A - f_B) = a(1-a)[\mu_A(1-a) - \mu_B a] =: F(a)
\]

\[
\mu_A := S - P, \quad \mu_B := T - R.
\]

\[
\begin{array}{c|cc}
    & A & B \\
\hline
    A & 1 & 1 + \mu_A \\
    B & 1 + \mu_B & 1
\end{array}
\]

Nonlinear Dynamics

\[
\frac{\partial a}{\partial t} = a(1-a)[\mu_A(1-a) - \mu_B a] =: F(a)
\]

\[
\alpha_{\text{int}}^* = \frac{\mu_A}{\mu_A + \mu_B}
\]

Zeros of \(F(a)\) are fixed points \(a^*\).
Slope of \(F(a^*)\) determines stability.
Recommended Reading:

Examples for game theory problems in biology:
J. Gore et al., Nature 07921 (2009)

Background in nonlinear dynamics:
S.H. Strogatz, Nonlinear Dynamics and Chaos, Westview, chapters 2&3