- COARSENING: qualitative understanding, but no quantitative description
- GLASSES: also conceptual difficulty: no qualitative understanding!

Are there thermodynamics properties that are enjoyed also in the out-of-equilibrium regime?

- arXiv: 1010.0149 growing length scales

P-spin model: (Mean-field) different p or Ising vs Spherical spins (soft spins)

$\rightarrow$ fragile glasses ($p \geq 3$)

$\rightarrow$ domain growth

$\rightarrow$ spin-glasses ($p = 2$)

Convexity of the free energy - F. ZAMPONI

Ferromagnetic Ising model:

$$H = \underbrace{-\sum_{\langle ij \rangle} J s_i s_j}_{H_0(s)} - h \sum_i s_i \qquad s_i = \pm 1$$

$$F(h) = -\frac{1}{N} \log \sum_s e^{\beta J \sum_{\langle ij \rangle} s_i s_j + \beta h \sum_i s_i}$$ which is convex.

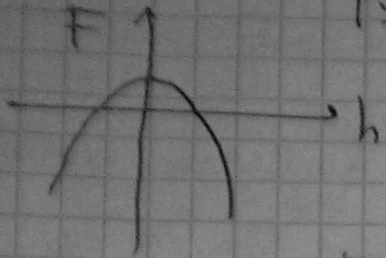
$$\frac{dF}{dh} = -\frac{1}{N} \left\langle \sum_i s_i \right\rangle = -m$$

$$\left\langle \sum_i s_i \right\rangle = \frac{1}{Z} \sum_s \left(\sum_i s_i \right) e^{-\beta H}$$

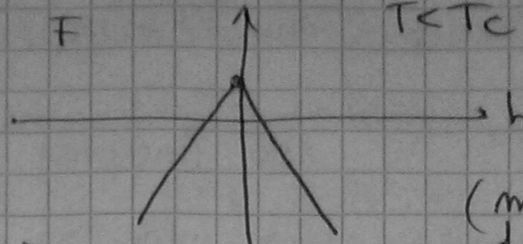
$$\frac{d^2F}{dh^2} = -\frac{\beta}{N} \left\langle \left(\sum_i s_i \right)^2 \right\rangle_c ,$$ where connected means that we substract the product of $\langle \sum_i s_i \rangle$

$$= -\frac{\beta}{N} \left\langle \left(\sum_i s_i - \left\langle \sum_i s_i \right\rangle \right)^2 \right\rangle \quad \boxed{< 0}$$ always negative

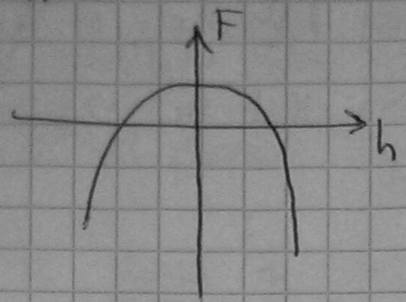
$T>T_c$



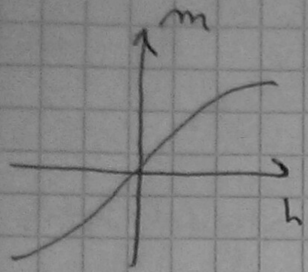
$T<T_c$



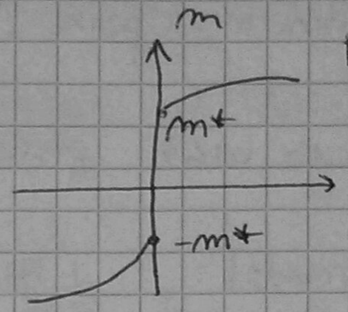
(m is discontinuous)

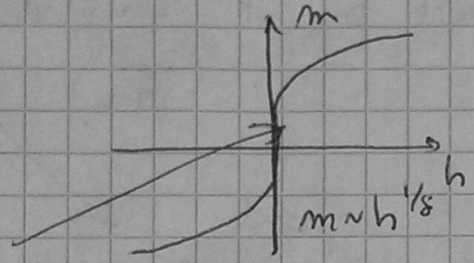


$T=T_c$



$T<T_c$





$m \sim h^{1/\delta}$

∞ derivative

$\delta = 3$ MF

$\delta = 15$ 2d

($m \sim \varepsilon$, $h \sim Q$ in Lectovic's lectures)

What about $F(m)$? We can take a Legendre transform

$$F(m) = F(h(m)) + \underbrace{m h(m)}_{\text{corresponds to a certain value of } m}$$

← need to remove the energy term introduced to fix the magnetisation

$$= \max_h \left[F(h) + hm \right]$$

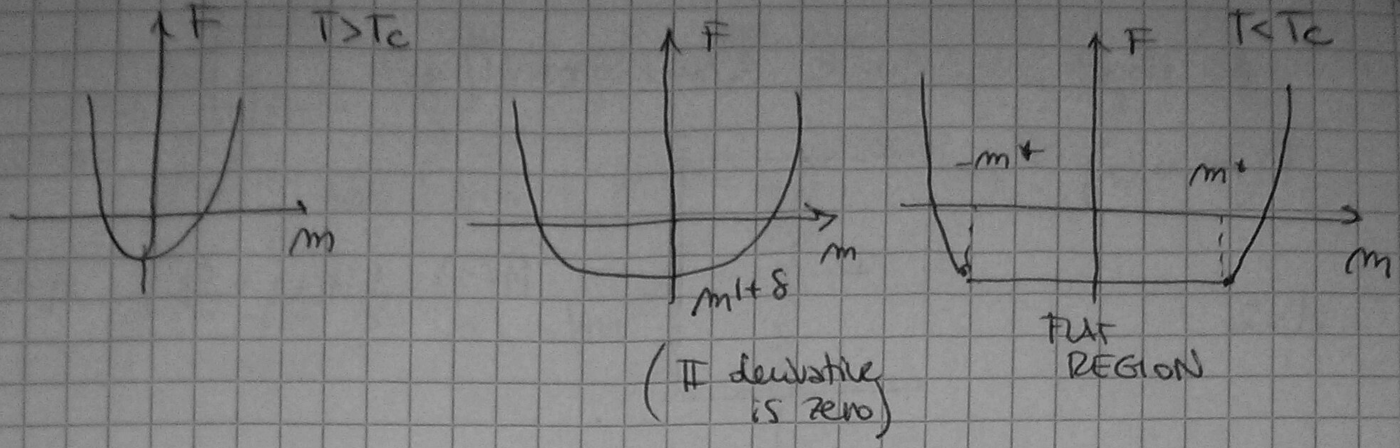
the maximum is unique, F_h is convex

$$\frac{dF}{dh} = -m$$

Legendre transforms are always convex:

$$\frac{dF}{dm} = \cancel{\frac{dF}{dh}\frac{dh}{dm}} + h(m) + \cancel{m\frac{dh}{dm}} = h(m)$$

$$\frac{d^2F}{dm^2} = \frac{dh}{dm} = 1 \Big/ \frac{dm}{dh} \geq 0 \quad \text{is always concave}$$



We can try another definition : ~ microcanonical ensemble

$$Z(m) = \sum_S e^{-\beta H_0(S)} \underbrace{\delta\left(m - \frac{1}{N}\sum_i s_i\right)}_{N \text{ finite: binning in the mag. space}} \propto P(m)$$

$$F(m) = -\frac{T}{N} \log Z(m)$$

- fully-connected case : $H_0 = -\frac{J}{2N} \sum_{ij} s_i s_j$ (Curie-Weiss Model)

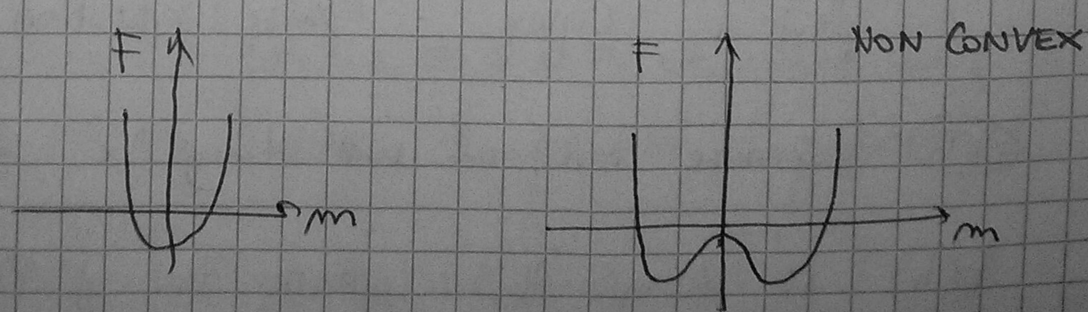
$$= -\frac{N J}{2} \left(\frac{1}{N} \sum_i s_i\right)^2$$

$$= -\frac{N J}{2} m^2$$

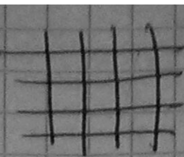
$$Z(m) = e^{\frac{\beta N J m^2}{2}} \binom{N}{N\left(\frac{1+m}{2}\right)}$$

$$\underset{(N \to \infty)}{F(m)} = -\frac{J m^2}{2} - T\left[-\left(\frac{1+m}{2}\right)\log\left(\frac{1+m}{2}\right) - \left(\frac{1-m}{2}\right)\log\left(\frac{1-m}{2}\right)\right]$$

$T > T_c = J$

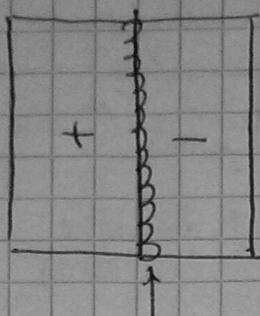


• what about in finite dimensions?



$P(m \sim 0)$?

$d > 2$



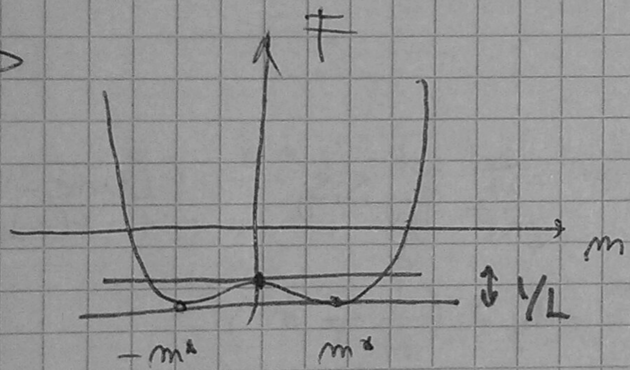
$H(s) = \frac{1}{2} H(+) + \frac{1}{2} H(-) + A L^{d-1}$

$Z(m \sim 0) \sim Z(+) \, e^{-\beta A L^{d-1}}$ $\Rightarrow F(m) \sim F(m^*) + \frac{A L^{d-1}}{N = L^d}$

surface tension per unit length

$\frac{A}{L}$ correction

$\Rightarrow$



when $L \to \infty$ we recover the flat part of $F(m)$.

* in MF the barriers are of order N: to go from all spins + to all spins − you have to overcome a barrier of order N.

$\Rightarrow$ in MF not convex : no phase separation

finite d convex : phase separation

BUT Legendre transform will always be concave.

$F(h)$ is always the Legendre tr. of $F(m)$, if you do it back you get the convex envelope! Not invertible.

$$F(h) = -\frac{T}{N}\log\int dm \sum_{\{s\}} e^{-\beta H_0 + \beta h N m}\,\delta\left(m - \frac{1}{N}\sum_i s_i\right)$$

$$= -\frac{T}{N}\log\int dm\; e^{\beta h N m}\, e^{-\beta N f(m)}$$

$$= \min_m \left[F(m) - hm\right]$$

BUT being not convex is not always bad ⇒ useful to identify <u>metastable states</u>

$F[m_i]$

↓

$F[m(x)]$

$m_i = \langle s_i \rangle$ → you cannot impose $\delta(m_i - s_i)$, you need some average

- put a local magnetic field and do a Legendre transform: $H = H_0(s) - \sum_i h_i s_i$

  $F(m)$ always convex

  BUT impossible to compute, $F(h_i)$ + Leg.

  moreover it gives no info on the states: there's just one minimum.

$$\frac{\partial F}{\partial h_i \partial h_j} = \beta\langle (s_i - m_i)(s_j - m_j)\rangle$$

$$= \beta\langle v_i v_j\rangle \Rightarrow \text{eigenvalues always positive}$$

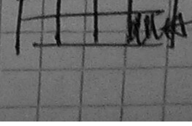
- High temperature expansion of the Legendre transform

  ↳ TAP equations → minima ≡ STATES we want to study

  loose the convexity because the expansion is performed

- KAC MODELS

$\delta\left(m(x) - \frac{1}{N}\sum_{\text{volume}} s_i\right)$ well defined constraint



V

$V \to \infty$

$\lambda \gg a$

$$\Rightarrow F[m(x)] = \int dx \left[\underbrace{(\nabla m(x))^2}_{\text{m wants to be uniform}} + \underbrace{V(m(x))}_{\text{non-convex: little volume where spins interact very strongly (long range interactions } \lambda \gg a)} \right]$$

$\Rightarrow$ fully-connected model

- if you force m to be uniform $\rightarrow$ NON-CONVEX
- if m is $\underbrace{\text{free to behave}}_{\text{phase separate}}$, then an interface will form.
  $\hookrightarrow$ FLAT free energy.

(Convex envelope of the free energy is what describes phase separations!)

F. KRZKALA

Free energies and algorithms

- Replica free energy : $-H = \sum_{i \le j} \left(-\frac{\lambda}{2N} x_i^2 x_j^2 + \frac{\lambda}{N} x_i x_j x_i^* x_j^* + \sqrt{\frac{\lambda}{N}} x_i x_j w_{ij} \right)$

hidden vector x^*

you observe $y = \sqrt{\frac{\lambda}{N}} \vec{x} \vec{x}^{*t} + w$

$$f = \lim_{N \to \infty} \frac{\mathbb{E}_{w,x^*}(-\log Z)}{N} = \min_m \Big[\underbrace{\varphi_{DEN}(\lambda m, \lambda m) + \frac{\lambda m^2}{4}}_{f_{RS}(m)} \Big]$$

$$m^{eq} = \operatorname{argmin} f_{RS}(m)$$

* In the cavity method you know the chain reasoning, in replicas the procedure is not very clear.