Transport theory of mesoscopic systems - random matrix approach to chaotic and disordered conductors

- Modeled by non-interacting Fermi gas of scattering states
- Conductance and noise properties determined by S-matrix elements and their dependence on energy, magnetic field, gate voltage...
- 2 probe case: 
  
  \[ G = (e^2/h) \text{Tr} \{tt^\dagger\} = (e^2/h) \sum_a T_a \]
  
  \[ P = (2e^2/h)(\Delta v eV) \sum_a T_a (1 - T_a) \]

We will focus on 2-probe case
Disordered metal

Weak localization UCF, anomalous shot noise, anomalous thermopower…

Ballistic point contact

But not a result of adiabaticity! See Szafer and Stone PRL 1989
Two approaches to understanding S-matrix:

- **Semiclassical/statistical: dynamical/flexible** (JBS 1990, 93, Marcus 1992)


- **Disordered Quasi-1D**: Imry 1986, Muttalib et al. 1987, Dorokhov (1983) & MPK (1988), Beenakker (97) - relation to localization goes back to 1960’s

- **Belief**: SC approach gave dynamical info, but not quantitative (see 1994 LH lectures); 2002 - Richter and Sieber solved problem!
In both disordered and chaotic case it will be necessary to define ensembles and calculate averages over them - compare to exp’t? Ergodic hypothesis: (Lee and Stone, 1985)

Actual data from ballistic junction -  
M. Keller and D. Prober 1995
RMT of Ballistic Microstructures

Need $P(\{T_n\})$, then can calculate $<G>$, $\text{Var}(G)$, $<P_{\text{shot}}>$ …

$\{T_n\}$ derived from $S$-matrix, need to define ensemble of $S$-matrices:

$\Rightarrow$ Most random distribution allowed by symmetry

$SS^\dagger = 1$ (no TR), $S = S^T$ (with TR symmetry)

$S = \begin{bmatrix} r & t \\ t' & r' \end{bmatrix}$

$S \begin{bmatrix} I \\ I' \end{bmatrix} = \begin{bmatrix} 0 \\ O' \end{bmatrix}$

$S$ relates flux in to flux out, e-vectors and e-values not simply related to $\{T_n\}$

$M \begin{bmatrix} I \\ O \end{bmatrix} = \begin{bmatrix} O' \\ I' \end{bmatrix}$

$M = \begin{bmatrix} (t'^\dagger)^{-1} & r'(t')^{-1} \\ (t')^{-1} & (t')^{-1} \end{bmatrix}$

$e$-values of $MM^\dagger$ related to $\{T_n\}$, also $M$ multiplicative - crucial for disordered case (later) - defined parameterization of $S$ we need.
What does “most random” mean for an ensemble of matrices?

e.g 2D space: $P(x,y)dxdy$

Most random: $P(x,y)dxdy = dxdy/(Area)$

$$\mu(\mathbf{dr}) = \frac{dxdy}{A}$$

Change variables: $r, \theta \rightarrow \mu(\mathbf{dr}) = r dr d\theta / A$

$$\mathbf{dr} \cdot \mathbf{dr} = dx^2 + dy^2 = dr^2 + (r d\theta)^2 = \sum_{ij} g_{ij} q_i q_j \Rightarrow \mu(\mathbf{dr}) = (\det[g])^{1/2}$$

Need $P(S), P(M), P(H) \ldots$ must define space and metric for matrices

Dim. = # of ind. Parameters = $4N^2, N(2N+1)$; $N$ channels, $2N \times 2N$ matrices - what is metric? $dS^2 = \text{Tr}\{dS dS^\dagger\} \rightarrow g \rightarrow \mu(dS) = (\det[g])^{1/2}$

Example: $2 \times 2$ real symmetric matrix (TR inv. TLS hamiltonian)
$H = \begin{bmatrix} h_1 & h_3 \\ h_3 & h_2 \end{bmatrix} \quad H + dH = \begin{bmatrix} h_1 + dh_1 & h_3 + dh_3 \\ h_3 + dh_3 & h_2 + dh_2 \end{bmatrix}$

$Tr\{dHdH^T\} = dh_1^2 + dh_2^2 + 2dh_3^2$

$\mu(dH) = dh_1 dh_2 2dh_3$

More useful coordinate system: E-values + e-vectors

$E_{1,2} = \frac{1}{\sqrt{2}}[\sigma \pm \sqrt{\Delta^2 + 2h_3^2}] \quad \sigma = \frac{1}{\sqrt{2}}[h_1 + h_2] \quad \Delta = \frac{1}{\sqrt{2}}[h_1 - h_2]$

$dh_1 dh_2 2dh_3 = d\sigma d\Delta 2dh_3 = J(E_1, E_2, h_3) dE_1 dE_2 2dh_3$

$J(E_1, E_2, h_3) = J(E_1, E_2) = \begin{vmatrix} \partial\sigma/\partial E_1 & \partial\sigma/\partial E_2 \\ \partial\Delta/\partial E_1 & \partial\Delta/\partial E_2 \end{vmatrix}$

$\partial\Delta/\partial E_1 = \sqrt{(\Delta^2 + 2h_3^2)/2} = (E_1 - E_2)/2$

$\mu(dH) \propto (E_1 - E_2) dE_1 dE_2 dh_3$ 

Eigenvalue repulsion, non-trivial metric

$\mu(dH) \propto (E_1 - E_2)^\beta dE_1 dE_2$ 

$\beta = 1,2,4$ for 3 symm. classes
Parameterize $M$, then $S$: “polar decomposition”

\[
M = \begin{bmatrix}
  u_1 & 0 \\
  0 & u_3
\end{bmatrix}
\begin{bmatrix}
  \sqrt{(1 + \lambda)} & \sqrt{\lambda} \\
  \sqrt{\lambda} & \sqrt{(1 + \lambda)}
\end{bmatrix}
\begin{bmatrix}
  u_2 & 0 \\
  0 & u_4
\end{bmatrix}
\]

\[
\lambda = \text{Diag}(\lambda_1, \lambda_2, \ldots \lambda_N), \quad u_i \text{ are } N \times N \text{ unitary matrices}
\]

with TR: $u_3 = u_1^*$, $u_4 = u_2^*$.

\[
[2 + MM^\dagger + (MM^\dagger)^{-1}]^{-1} = \frac{1}{4}
\begin{pmatrix}
  tt^\dagger & 0 \\
  0 & t'^\dagger t'
\end{pmatrix}
\]

\[
T_a = \frac{1}{1 + \lambda_a}, \quad R_a = \frac{\lambda_a}{1 + \lambda_a}
\]

Find $\mu(dM)$ in terms of $\{\lambda_a\}$, then $P(\{\lambda_a\}) \Rightarrow P(\{T_a\})$, avg.s of $g$

Can work directly with $S$

\[
S = \begin{pmatrix}
  U & 0 \\
  0 & V
\end{pmatrix}
\begin{pmatrix}
  -\sqrt{1-T} & \sqrt{T} \\
  \sqrt{T} & \sqrt{1-T}
\end{pmatrix}
\begin{pmatrix}
  U' & 0 \\
  0 & V'
\end{pmatrix}
\]

\[
T = \text{Diag}(T_1, T_2 \ldots T_N)
\]

\[
d\mu(S) = J \prod_{\alpha} d\mu(U_\alpha) \prod_i dT_i
\]
\[
J(\{T_n\}) = \prod_{i<j} |T_i - T_j|^{-\beta} \prod_k T_k^{-1+\beta/2}
\]

\[
P(\{T_n\}) = c \prod_{i<i} |T_i - T_j|^{-\beta} \prod_k T_k^{-1+\beta/2} \exp[-\beta f(T_k)]
\]

\[
P(\{T_a\}) = C_\beta \exp[-\beta \sum_{a<b} \ln |T_a - T_b| + \sum_c V_\beta(T_c)]
\]

Have the j.p.d, what do we need to do with it?

\[
< g > = \sum_a^{N} T_a, \quad < g^2 > = \sum_{a,b}^{N} T_a T_b, \quad < P > = \sum_a^{N} T_a (1 - T_a)
\]

For Var(g) need

\[
K(T, T') = \int \prod_{c=3}^{N} dT_c P(\{T_c\}) - \rho(T) \rho(T')
\]

Need 1-pt and 2-pt correlation fcns of the jpdf of \{T_n\}
Many methods to find these fcns and the two-pt corr. fcn is “universal” upon rescaling if only logarithmic correlations

Nice approach for $\beta=2$ is method of orthogonal polynomials

$$\prod_{a<b}^{N} |T_a - T_b|^2 = \begin{vmatrix} p_1(T_1) & p_1(T_2) & \cdots & p_1(T_N) \\ p_2(T_1) & p_2(T_2) & \cdots & p_N(T_N) \\ \cdots \\ p_N(T_1) & p_N(T_2) & \cdots & p_N(T_N) \end{vmatrix}^2$$

$$\equiv <\Psi_N|\Psi_N>$$

$p_n = \text{orthog poly, choose Legendre, [0,1]}

$$\rho(T) = <\Psi_N|\sum_{a} \delta(T - T_a)|\Psi_N> = \sum_{n=1}^{N} p_n^2(T)$$

Use recursion relations, asymptotic form of $p_n$:

$$\rho(T) = \frac{N}{\pi \sqrt{T(1-T)}}$$

(normalized to $N$ - so that $G = (e^2/h)T$)

Same method gives $K(T,T')$ in terms of $p_N p_{N-1}$
What do we expect for this system?

Classical symmetry between reflection and transmission => \( <R> = <T> = N/2 \)

\[
< g > = \frac{N}{\pi} \int_0^1 dT \sqrt{\frac{T}{1 - T}} = \frac{N}{2}
\]

Need to go to next order in \( N^{-1} \) to get WL effect

\[
< P_{shot} > \propto \frac{N}{\pi} \int_0^1 dT \sqrt{T(1 - T)}
\]

\[
= \frac{N}{8} \Rightarrow \frac{1}{4} P_{tunnel}
\]
Due to symmetry between reflection and transmission, one can get order \(1/N\) effects easily for the circular ensemble - don’t need to distinguish \(T_{ab}, R_{ab} = S_{ab}\) just do averages over unitary group \(U(2N)\):

\[
g = \sum_{a,b}^{N} |t_{ab}|^2 = \sum_{a,b}^{N} S_{ab}S_{ab}^*, \quad S \in U(2N)
\]

\[
\langle S_{ab}S_{cd}^* \rangle_{CUE} = \int d\mu(S) S_{ab}S_{cd}^* = \frac{\delta_{ac}\delta_{bd}}{2N}
\]

\[
\langle S_{ab}S_{ab}^* \rangle_{COE} = \int d\mu(U) [UU^\dagger]_{ab} = \frac{1 + \delta_{ab}}{2N + 1}
\]

\[
\langle g \rangle_{CUE} = N^2 \cdot \frac{1}{2N} = \frac{N}{2}
\]

\[
\langle g \rangle_{COE} = N^2 \cdot \frac{1}{2N + 1} + N \cdot \frac{1}{2N + 1} \approx \frac{N}{2} + \frac{1}{4}
\]

\[
\delta G_{WL} = (e^2/h)(1/4)
\]

\(The \ Mystery\)

Similarly \(\text{Var}(g) = (1/8\beta)\)
All of these results are subtly different for a disordered/diffusive wire - will analyze next lecture