TIME STEPPING

$$\mathcal{M}\partial_t \vec{X} + \mathcal{L}\vec{X} = \mathcal{N}(\vec{X})$$

$\mathcal{M}$, $\mathcal{L}\vec{X}$, $\mathcal{N}(\vec{X})$ $\Rightarrow$ VIA SPECTRAL METHOD, DISCRETIZE ∇, $\int$

LEFT W/ ODEs FOR COEFFICIENTS $\hat{\vec{X}}$

$$\hat{\vec{X}} = \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{v}_0 \\ \hat{v}_1 \\ \vdots \end{bmatrix}$$

$$\mathcal{M}\partial_t \hat{\vec{X}} + \underbrace{\hat{L}\hat{\vec{X}}}_{\text{MATRIX SOLVE}} = \underbrace{\mathcal{N}(\vec{X})}_{\text{EVALUATION IN GRID SPACE}}$$

CAN USE ANY STANDARD ODE SOLVER!

SIMPLEST

$$f(t+\Delta t) - f(t) = \Delta t \, \mathrm{RHS}$$

$$f(t+\Delta t) = f(t) + \Delta t \, \mathrm{RHS} \qquad \text{"FORWARD EULER"}$$

MANY ODE SOLVERS, BUT 2 CLASSES WE CARE ABOUT:

EXPLICIT

NEXT STEP DEPENDS ONLY ON PAST DATA

OBVIOUS EXAMPLE $y'(t) = f(y)$

$$y_{n+1} = y_n + \Delta t\, f(y_n)$$ "FORWARD EULER"

$$y_{n+1} = y_n + \Delta t \lambda y_n = (1 + \Delta t \lambda) y_n$$

$$1 + \Delta t \lambda \leq 0$$

$$\Delta t \leq \frac{-1}{\lambda}$$

IMPLICIT

NEXT STEP DEPENDS ON PAST + ITSELF

OBVIOUS EXAMPLE

$$y_{n+1} = y_n + \Delta t\, f(y_{n+1})$$ "BACKWARD EULER"

IF $f(y_{n+1})$ IS LINEAR, THEN NEED TO SOLVE LINEAR SYSTEM

$$\Delta t\, f(y_{n+1}) - y_{n+1} - y_n = 0$$

$$\partial_t \hat{\vec{X}} = \breve{L} \cdot \hat{\vec{X}} + \underbrace{\widehat{N(\vec{x})}}_{=0}$$

$$\Downarrow$$

$$y' = -\lambda y$$

$\lambda \in \mathrm{EIG}(\breve{L})$

WHY?

$$L \vec{x} = \lambda \vec{x}$$

$$L \vec{y} = \vec{b}$$

FOR A GIVEN VECTOR NORM, INDUCED MATRIX NORM

$$\|A\|_2 = \max_{\vec{x}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

FOR L_2, $\|\vec{x}\| = \langle \vec{x} | \vec{x} \rangle^{1/2}$

$$\|A\|_2 = \sqrt{\rho(A^* A)}$$

$\rho(A) \equiv$ "SPECTRAL RADIUS"

RECALL

$A = A^*$ HERM

$A^* = A^{-1}$ UNITARY

$A^* A = A A^*$ NORMAL

$\rho(A) = \max_{1 \le i \le n} \lambda_i \qquad A\vec{x} = \lambda \vec{x}$

$$\underset{\sim}{A} \in \mathbb{R}^{n \times n}$$

So, λ gives some idea of the "size" of $\underset{\sim}{A}$

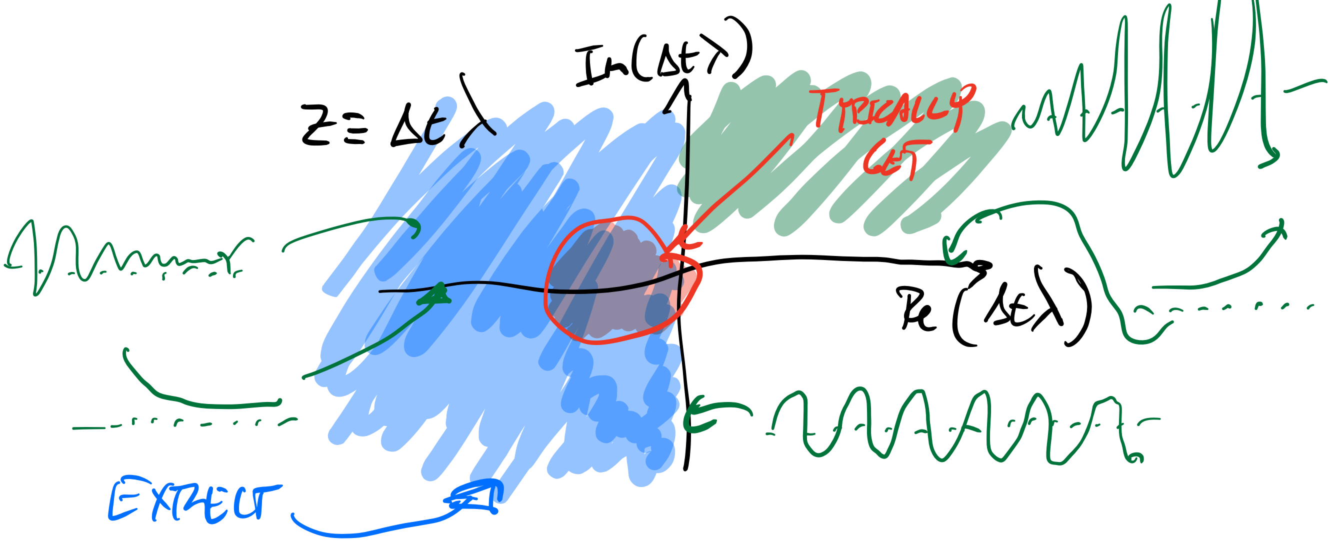
$$y' = \lambda y \rightarrow y(t) = y_0 e^{\lambda t}$$

$$|y(t+\Delta t)| \le |y(t)|$$

if $\operatorname{Re} \lambda < 0$

For an ODE method, "absolute stability region"

$$|y_{n+1}| \le |y_n| \qquad \operatorname{Re} z < 0$$



BACK TO HEAT EQN,

$$\partial_t T = \varkappa \nabla^2 T \;\xrightarrow{FT}\; -\varkappa k^2 \hat{T}$$

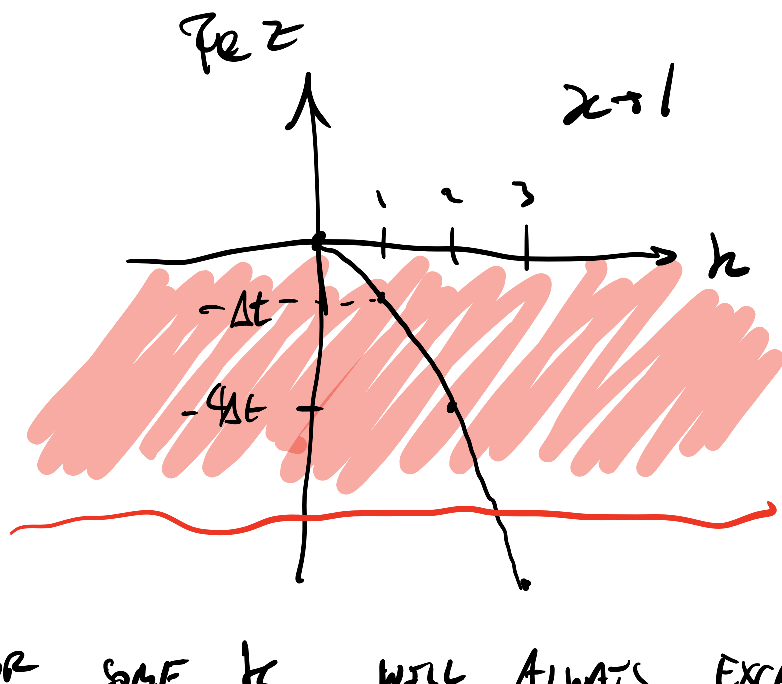
$$\partial_t \hat{T} = -\varkappa k^2 \hat{T}$$

FOR EACH MODE

so $\text{Re}\,\lambda \leq 0$ FOR ALL k

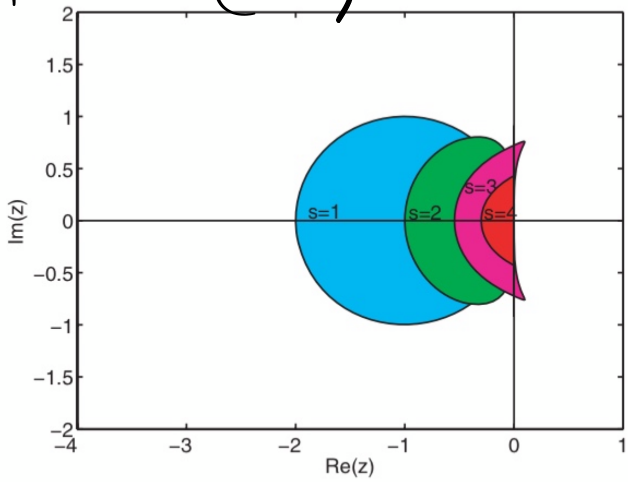
WANT Δt AS LARGE AS POSSIBLE

$$z = \Delta t \lambda = -\varkappa k^2 \Delta t$$

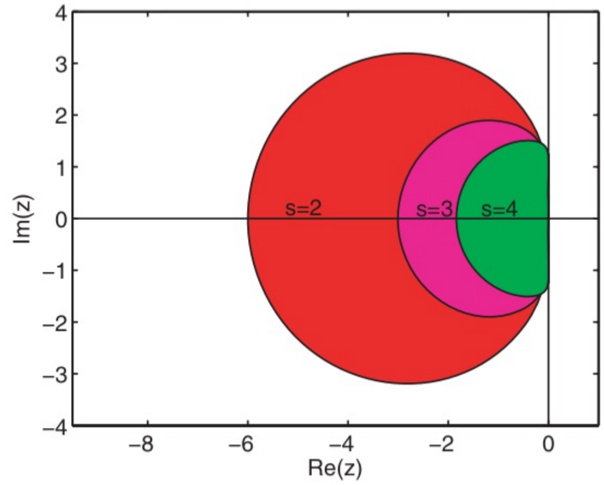


FOR SOME k, WILL ALWAYS EXCEED STABILITY

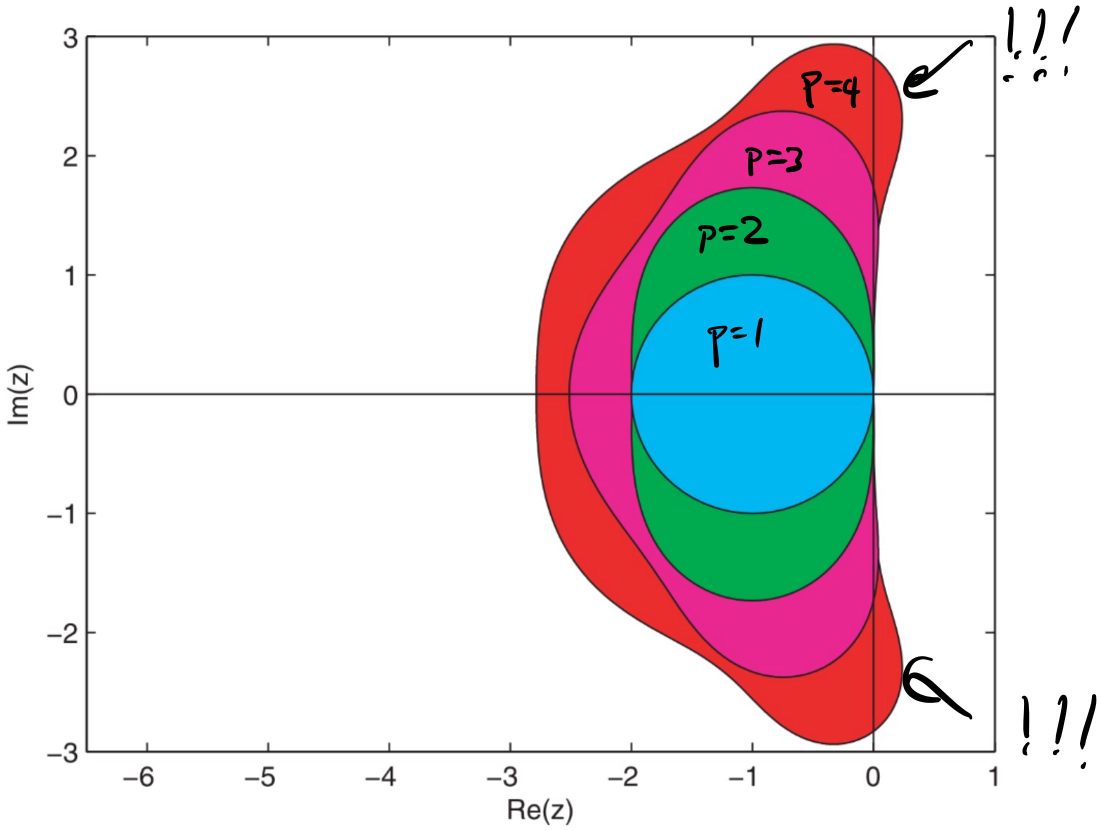
ASCHER (2008)



ADAMS BASHFORTH



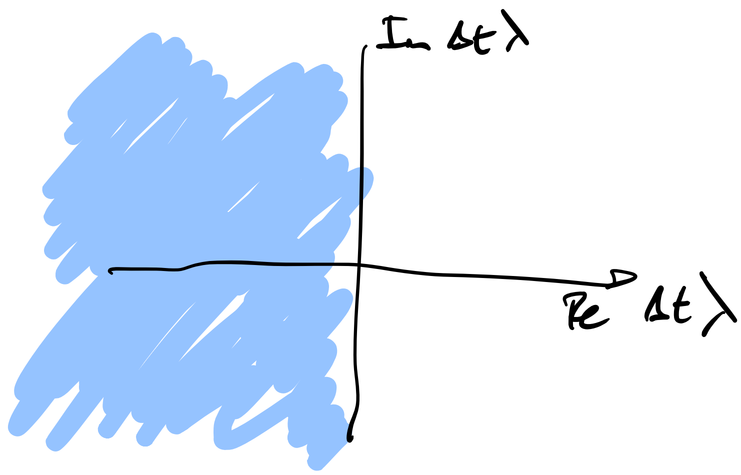
ADAMS - MOULTON



EXPLICIT RUNGE-KUTTA

SO, IF WE HAVE DIFFUSIVE OR OTHER HIGH ORDER DERIVATIVES, TIMESTEPPING STABILITY GETS HARDER & HARDER FOR SMALL SCALES...

ENTER IMPLICIT TIMESTEPPERS



← GET WHAT YOU EXPECT!

BUT

STABILITY ≠ ACCURACY!

AND IF f → NON-LINEAR,

$$q' = f(q)$$

$$q_{n+1} = q_n + f(q_{n+1})$$

$$\underbrace{|q_{n+1}\rangle = |q_n\rangle + f(|q_{n+1}\rangle)}$$

SYSTEM OF NL ALGEBRAIC EQNS

- NEWTON'S METHOD, E.G. REQUIRES GOOD GUESS!
- EXTREMELY EXPENSIVE

DEDALUS SIDESTEPS THE 2ND PROBLEM BY USING IMEX TIMESTEPPING

↳ BEST OF BOTH WORLDS:

- LINEAR TERMS IMPLICIT ⇒ ABSOLUTE STABILITY
- NL TERMS EXPLICIT ⇒ CHEAP TO TIME SIZE

DETAILS BEYOND THESE LECTURES, BUT SEE

ASCHER (2006)

SBDF REF

RK443 REF

DEDALUS HAS MANY TIMESTEPPERS

ADDING NEW ONE IS STRAIGHTFORWARD

2 MAIN CLASSES

- MULTISTEP / ADVANTAGE

  DISADVANTAGE : BOOTSTRAPPING

  USES S PRIOR STEPS

- MULTISTAGE AKA RUNGE-KUTTA

  WHAT ABOUT STEPS $n < S$?

LOTS OF CHOICES, MANY TRADEOFFS

→ SIZE OF Δt

→ COST PER STEP

BUT NOT MUCH TO WORRY ABOUT: CHOOSE RK222

RK443

UNLESS YOU'RE REALLY INTO TIMESTEPPING

OPEN QUESTION

FOR A REAL, NON-LINEAR TURBULENT FLUID SYSTEM, WHAT CHOICE OF TIMESTEPPER PRODUCES OPTIMAL PERFORMANCE FOR A GIVEN ACCURACY?

IMPORTANT: NEED TO CHOOSE TIMESTEP

TYPICALLY,

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u$$

$$\partial_t u + \nabla p - \nu \nabla^2 u = \underbrace{- u \cdot \nabla u}$$

NL TERM D ADVECTION

SIMPLE HEURISTIC

CFL CONDITION

SAFETY FACTOR

$$\Delta t < C \frac{\min(\Delta x_i)}{\max(u[i,j])}$$

IMPORTANT: CHEBYSHEV GRID IS NOT UNIFORMLY SPACED