Hydrodynamics

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Thermodynamics / statistical mechanics :

N>>1; large number of interacting particles !

Hydrodynamics :

L >> l; T >> t; long wavelength, slow time - average over (some) microscopic length and time scales ... continuum field theories !

microscopic length : l  ?  (particle size, mean-free path, pore size, .... )

microscopic time : t  ?  (particle relaxation times, hopping times, ... )

Continuum theory ?

1. Microscopic picture + systematic averaging ... "rigorous" !

2. Equations of state, constitutive equations ... "empirical" !

3. Symmetry, invariance ... "intuitive" !

Balance laws: mass, momentum (linear, angular), energy, entropy ...
Variables - independent? \( (r, t) = (x, y, z, t) \) space-time

- dependent? \( \rho(r, t), v(r, t), \sigma(r, t), \ldots \)

  density (scalar), velocity (vector), stress (tensor), ...

Frames:

1. Lagrangian - material frame: follow parcels of material ... finite bodies.

2. Eulerian - spatial (inertial) frame: lab-based ... infinite bodies.

Laws + boundary conditions:

Qualitative behavior of solutions:


**Laws: differential form** (integral form - useful in the presence of discontinuities - e.g. shocks)

**Conservation of mass**

Conservation of mass
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

Eulerian

\[
\frac{D\rho(r, t)}{Dt} = (\partial_t + \mathbf{u} \cdot \nabla)\rho = -\rho \nabla \cdot \mathbf{u}
\]

where
\[
\frac{dr}{dt} = \mathbf{u}
\]

Material derivative

**Incompressible fluid**

\[
\frac{D\rho}{Dt} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{u} = 0
\]

**Conservation of linear momentum**

\[
\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} + \nabla \cdot \mathbf{\sigma}
\]

Newton II for deformable continua (per unit volume)

accln force stress

\( \sim \) force/area (interactions !)

**Cauchy stress theorem**

\[
\mathbf{T}^{(n)} dA = \sum_i T^{(e_i)} dA_i
\]

\[
dA_i = (\mathbf{n} \cdot \mathbf{e}_i) dA
\]

\[
\mathbf{T}^{(n)} = \sigma^T \mathbf{n}
\]

Stress tensor completely defines force/area locally !
Conservation of angular momentum?

\[ \sigma = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + [-p + (\zeta - \frac{2}{3}\mu)(\nabla \cdot \mathbf{u})] \mathbf{I} \]

2 constant fluid!

(anisotropic) shear \hspace{1cm} (isotropic) "pressure" = thermodynamic + dynamic

\[ \rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} + (\zeta + \mu/3) \nabla(\nabla \cdot \mathbf{u}) + \mathbf{f} \]

acceleration \hspace{1cm} shear viscosity \hspace{1cm} body force

Pressure \hspace{1cm} dilatation viscosity

Conservation of entropy?

\[ \rho T(\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s) = \kappa \nabla^2 T + \frac{1}{2} \mu [\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \nabla \mathbf{u}]^2 + \zeta(\nabla \cdot \mathbf{u})^2 + \mathbf{f} \cdot \mathbf{u} \]

entropy production \hspace{1cm} viscous dissipation \hspace{1cm} external work

Stress tensor (Newtonian fluid) - isotropic, homogeneous

\[ \sigma = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \]

heat conduction

Initial conditions:

\[ \mathbf{u}(\mathbf{r}, 0) \quad T(\mathbf{r}, 0) \]

unknowns \[ \mathbf{u}, p, T \]

Boundary conditions

\[ \mathbf{u}|_{\partial \Omega} = \mathbf{U}_b \quad \text{or} \quad p|_{\partial \Omega} = p_b \]

open boundaries

\[ s|_{\partial \Omega} = s_b \quad \text{or} \quad -\rho T \nabla s|_{\partial \Omega} = \mathbf{q}_b \]
Solid (rigid) boundaries
\[ u|_{\partial \Omega} = 0 \]
no slip

A very slippery condition!

Fails at
- low density
- porous interface
- hydrophobic surface
- contact line (triple phase boundary)

Free boundaries
\[ \sigma n = -\gamma \kappa n \]
transmission is continuous

Surface tension

Generalizing Laplace's law
\[ \delta p = \gamma \kappa \]
\[ \kappa = \frac{1}{R_1} + \frac{1}{R_2} \]
2 \times mean curvature

Physical parameters:
- \( \mu \)
- \( \rho \) fluid (intrinsic)
- \( \gamma \)
- \( L \) extrinsic
- \( U \) extrinsic

Dynamic viscosity?

Surface tension?

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Dynamic Viscosity</th>
<th>Surface Tension</th>
</tr>
</thead>
<tbody>
<tr>
<td>air</td>
<td>0.00001 Pa.s</td>
<td></td>
</tr>
<tr>
<td>water</td>
<td>0.001 Pa.s</td>
<td>0.07 Pa.m</td>
</tr>
<tr>
<td>honey</td>
<td>1.0 Pa.s</td>
<td>0.02 Pa.m</td>
</tr>
</tbody>
</table>
Incompressible, inviscid fluid (and $f = 0$)

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = D \nabla^2 T + \frac{\mu}{2 \rho C_p} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + \frac{\mathbf{f} \cdot \mathbf{u}}{\rho C_p} \]

Incompressible, viscous fluid

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = D \nabla^2 T + \frac{\mu}{\rho C_p} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + \frac{\mathbf{f} \cdot \mathbf{u}}{\rho C_p} \]

Steady incompressible, inviscid flow

\[ \partial \mathbf{u} / \partial t = 0; \quad \rho \mathbf{u}^2 / 2 + p = \text{const} \]

Bernoulli eqn. (energy cons.)

\[ \frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = \frac{Ds}{Dt} = 0 \]

Incompressible, inertia-less flow

\[ \nabla p = \mu \nabla^2 \mathbf{u} + \mathbf{f} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

Stokes eqns.

2d flow ? axisymmetric flow ?

Euler eqns.

Navier-Stokes eqns.

Steady incompressible, inviscid flow

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\[ \nabla \cdot \mathbf{u} = 0 \]

Stokes eqns.

Linear !
Laws: symmetry + invariance? a hydrodynamic theory of self-propelled objects?

birds, bees, fishes, pedestrians, ...

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + \lambda_1 \mathbf{u} \cdot \nabla \mathbf{u} + \lambda_2 (\nabla \cdot \mathbf{u}) \mathbf{u} + \lambda_3 \nabla (|\mathbf{u}|^2) = \\
\alpha \mathbf{u} - \beta |\mathbf{u}|^2 \mathbf{u} - \nabla p + D_1 \nabla^2 \mathbf{u} + D_2 \nabla (\nabla \cdot \mathbf{u}) + D_3 (\mathbf{u} \cdot \nabla)^2 \mathbf{u} + \mathbf{f}
\]

not Galilean invariant

1. Truncation? 2nd order in space + cubic in velocity ...

2. Coefficients? empirical / experimental ...

- Unusual phases/ instabilities
- Long range fluctuation effects

Compare with balance laws: symmetry/invariance used in constitutive equation.

What next?

- Nonlinear PDE ... all terms are not equally relevant in all situations.

- Approximations: analysis (perturbation/ asymptotics), computation, SCALING
Dimensional analysis and the Pi theorem

In any equation, all terms must have the same dimensions!

\[ f(q_1, q_2, \ldots, q_n) = 0 \quad F(\pi_1, \pi_2, \ldots, \pi_p) = 0 \]

\[ \pi_i = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n} \]

- \( n \) number of physical variables (dimensional)
- \( k \) number of independent physical units
- \( p = n - k \) dimensionless numbers

E. Buckingham (1914)

examples:

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} \]

force/volume

\[ \mu, \rho \] fluid parameters

\[ \frac{L}{U} \] extrinsic

\( n = 4; \ k = 3; \ p = 1 \)

dimensionless variables

\[ \hat{t} = tU/L; \ \hat{u} = u/U; \ \hat{p} = pL/\mu U \]

dimensionless equation:

\[ \frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \hat{u} = \frac{\mu}{\rho U L} (-\hat{\nabla} \hat{p} + \hat{\nabla}^2 \hat{u}) \]

\( \nu = \mu/\rho \) kinematic viscosity (momentum diffusivity)

\[ Re = \frac{UL}{\nu} = \frac{\rho U^2}{\mu U/L} = \frac{\text{inertial pressure}}{\text{viscous stress}} \]

Reynolds #

\( Re = 0 \); \( Re = \infty \)

Stokes \quad Euler
Fun with scaling!

1. Frequency of a bobbing buoy?
   \[ \omega = f(\Delta \rho, \rho_w, R, g) \]

2. Oscillations of a drop?
   \[ \omega = f(\gamma, \rho, R) \]

3. Diffusion of a polymer?
   \[ D = f(k_B T, \mu, R) \]

4. Tsunami warning time?
   \[ T = f(L, g, H, \lambda) \]

5. Yield of an atomic explosion?
   \[ T + 0.006s \]
   \[ E = f(R, t, \rho) \]

6. Drag on a body?
   \[ F = f(R, \rho, U, \mu) \]
Small does not mean negligible!

\[ F = \rho U^2 g \left( \frac{UL}{\nu} \right) \quad C_d = \frac{F}{\rho U^2} = g(Re) \]

\[ Re \to \infty ? \]

Example

\[ \mu \ddot{x} + kx = f; \quad x(0) = 0 \]

exact solution ... exponentials!

i.e.

\[ \tau x + x = f/k; \quad x(0) = 0 \]
natural time scale

\[ \tau = \frac{\mu}{k} \]

2 regimes

\[ t \ll \tau; \quad x(t) = ft/k\tau = ft/\mu \]

\[ t \gg \tau; \quad x(t) = f/k \]

initial layer ... memory of init. condn.

- Singular ODE
- Divide and conquer!

Home work?

\[ \epsilon \ddot{x} + \dot{x} = \zeta(t); \quad x(0) = 0, \quad \dot{x}(0) = 0 \]

\[ < \zeta(t) >= 0, \quad < \zeta(t)\zeta(t') >= G\delta(t - t') \]

In space - boundary layers .... Prandtl (1905), but also Laplace, Stokes, Rayleigh, Lamb,

Stokes' problems: fluid driven by transient wall motion

Boundary data:

I \[ u(x, y = 0, t) = UH(t); \]

II \[ u(x, y = 0, t) = U \sin \omega t \]

\[ u = (u(x, y, t), v(x, y, t)) \]

- Scaling approach
- Analytical approach
**I** Suddenly moving boundary ... no intrinsic/extrinsic length scale!

zone of influence: \( y \sim (\nu t)^{1/2} \)  

momentum diffusion

**II** Oscillating boundary .. extrinsic frequency scale \( \omega \)

zone of influence: \( l_S \sim (\nu/\omega)^{1/2} \)  

Stokes length

\[ \nabla \cdot \mathbf{u} = 0 \ \checkmark \]

**N.S.** \( \partial_t u = \nu \partial_{yy} u \)

**I** similarity solution (heat equation)

**II** separation of variables

Prandtl boundary layer

Steady flow past

a semi-infinite flat plate

- effect of wall is limited to a (small) neighborhood of the wall ... at low viscosity (high Re !)

- zone of influence cannot depend on the length of the plate (infinite !) ... must be self-similar

Home work!
\[ \nabla \cdot \mathbf{u} = 0 \rightarrow \frac{u}{x} \sim \frac{v}{\delta} \]

\[ \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} \]

boundary layer assumption \[ \delta \ll x \rightarrow v \ll u \]
\[ \rho u^2/x \sim \mu u/\delta^2 (\sim p/x) \rightarrow \delta \sim (\frac{\nu x}{U})^{1/2} \]

\[ \frac{\delta}{x} \sim Re_x^{-1/2} \]

where \[ Re_x = \frac{Ux}{\nu} \]
so that \[ Re_x \gg 1 \rightarrow \delta \ll x \]

Wall shear stress \[ \sigma \sim \mu \partial_y u \sim \mu U/\delta \sim (\frac{\rho \mu U^3}{x})^{1/2} \]
decreases with distance along plate!

Total drag force / width \[ \int_0^l \sigma dx \sim (\rho \mu U^3 l)^{1/2} \]
skin drag

Comparison \[ F_p = C_d \rho U^2 l \]
pressure drag / width \[ F_s = C \mu U \]
Stokes drag / length

Careful analysis: Stream function approach (see Wikipedia or Batchelor, Landau/Lifshitz)

Q. Are these steady state solutions stable ?

Q. Instability and transition (to turbulence) ?

NEXT TIME: Hydrodynamic Instability, Turbulence (briefly) + Free surface flows