For Boulder Summer school 2013.

Quantum many-body thermalization + localization.

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thanks especially to Vadim Oganesyan. + many others to be mentioned when we get to their work.

Outline:

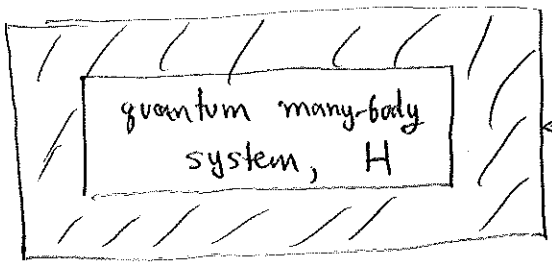
- Review of quantum stat. mech. + many-body dynamics.
- Quantum thermalization
  Eigenstate Thermalization Hypothesis (ETH)
- Many-body localization (MBL) (failure of thermalization)
  eigenstate phase transitions.

Topic: Dynamics of many-body quantum systems.

Strongly-interacting s-corelated s-entangled internally, but isolated from their environment (or we are including the environment as part of our system).

At energies that corespond to $T \neq 0$ (T = temperature)
(even $T \to \infty$ or $T < 0$)

Not "conventional" ~~en~~ quantum condensed matter focus on ground state + a few excitations.



← isolated from any environment.

H is time-independent, strongly-interacting.

- Thermo., Stat. Mech. idealization often discussed.
- E.g., spins in an insulator, ignore coupling to phonons…, keep only spin Hamiltonian (Anderson 1958)

- Well approximated in trapped atomic systems (ions, neutral atomic gases)
- An idealization to consider for quantum information "processing".

⋮

Examples: N interacting 2-state systems $\vec{\sigma}_i$ $i = 1, \dots N$

(spin-1/2's, qubits, orbitals for fermions, …) ← Pauli operators. 3 components.

or "hard-core" bosons

$$H = \sum_{i=1}^{N} \vec{h}_i \cdot \vec{\sigma}_i + \sum_{\langle ij \rangle} \vec{\sigma}_i \overleftrightarrow{J}_{ij} \vec{\sigma}_j$$

(hopping is $\sigma_i^+ \sigma_j^- + h.c.$)

single-spin energy

interaction between nearby spins

in general, 3×3 matrix of interactions.

short-range in real space interactions / hoppings

spins at fixed positions $\vec{r}_i$, can be on a lattice in 1D, 2D, 3D, …

E.g., TFIM $H = \sum_i h_i \sigma_{xi} + \sum_{\langle ij \rangle} J_{ij} \sigma_{zi} \sigma_{zj}$

Quantum stat. mech. + dynamics:

Closed system, time-indpt Hamiltonian H.

Pure states $|\psi\rangle$ are a nice idealization and tool, but only an approximation to "real" mixed states ρ.

ρ: probability operator (a.k.a. "density matrix") describes general ("mixed") quantum states of our system. $\rho^\dagger = \rho$ $\mathrm{Tr}\,\rho = 1$

spectrum of ρ is in $[0,1]$

We will work in "Schrodinger picture":

Time evolution is "simply" given by $e^{-iHt/\hbar}$ unitary operator:

$$\rho(t) = e^{-iHt/\hbar}\,\rho(0)\,e^{iHt/\hbar} \qquad \text{(} \rho(0) \text{ at } t=0\text{)}$$

in D.E. form: $i\hbar\dot{\rho} = [H, \rho]$

[$|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(t=0)\rangle$ for pure states.]

$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$ (S's eqn.)

"Pictures"

Heisenberg: Operators have time evolution, state does not. Convenient if states are simple. But here we are interested in complicated, highly-correlated + entangled states.

Schrodinger: States have time evolution, not operators. "Preferred" here because at least simple operators remain simple.

Say our "full" quantum system consists of N 2-state subsystems ("spins").

There are 2^N linearly-independent pure states. 2^N-dimensional Hilbert space.

Common orthonormal basis to think about + use:

Nonentangled pure product states that are simultaneously eigenstates of all $\{\sigma_{zi}\}$:

E.g.: $|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \otimes |\downarrow\rangle_3 \otimes \cdots \otimes |\uparrow\rangle_i \otimes \cdots \otimes |\downarrow\rangle_N$, etc.

Mixed states: in a given basis for the pure states and observables ρ is represented as an Hermitian $2^N \times 2^N$ matrix.

(also any observable is an Hermitian $2^N \times 2^N$ matrix).

There are 4^N linearly-independent mixed states / observables (with $\mathrm{Tr}\,\rho = 1$ constraint, really $4^N - 1$ independent mixed states)

There are 4 Hermitian operators for each "spin" i

$$\mathbb{1}_i \equiv \sigma_{0i}, \sigma_{xi}, \sigma_{yi}, \sigma_{zi}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ in basis of σ_{zi}'s eigenstates.

The 4^N mixed states/operators on the full system are all outer products of these basic operators (4 choices for each spin)

E.g.: $\sigma_{x1} \otimes \sigma_{z2} \otimes \mathbb{1}_3 \otimes \cdots \otimes \sigma_{yi} \otimes \cdots \otimes \mathbb{1}_N$, etc.

Usually all the $\mathbb{1}$'s (and $\otimes$'s) are left implicit, such as in writing $\left(J_{ij}\sigma_{zi}\sigma_{zj}\right)$

Local vs. "Global" operators:

"m-local operator": no more than m of the σ_i's in the outer product are not identity operators.

Any linear combo of m-local operators is also an m-local operator.

→ These are the operators that are *in practice* observables, ← usually with m=1 or m=2.

For large N almost all operators are "global" or "nonlocal" with of order N of the non-identity spin operators in the product. These global operators are (presently) inaccessible to observation for large N...

[there are $3^m \binom{N}{m}$ linearly-indpt operators that are m-local but not (m-1)-local]

Meaning of quantum thermalization:

→ System at long time goes to thermal equilibrium. ← Usual "ergodic" assumption to justify standard eq'm stat. mech.

It does *not* mean $\rho(t) \to \rho_{eq'm}$ for the *full system*

Under unitary time evolution $\rho(t) = e^{-iHt/\hbar} \rho(t=0) e^{iHt/\hbar}$

"remembers" *all* information about the initial state.

No information is *lost*, it is only *moved* and "hidden", in "inaccessible" observables, due to being "encoded" by entanglement. To detect it at long time usually requires measuring "nonlocal" observables" and thus not really feasible (today).

For a system that does go to thermal equilibrium, what thermalizes are <u>subsystems</u>. The full system successfully serves as a reservoir and thermalizes any subsystem that is $\ll$ full system (in entropy).

Does our system thermalize? $\Longleftrightarrow$ Is it a functioning reservoir that can thermalize itself?

$\rightarrow$ Some systems are many-body localized and fail to thermalize. $\leftarrow$

Sharp distinction is only present in $N \to \infty$ (like all phase transitions)

Need a "family" of initial states $\rho_N(t=0)$ for a sequence of N, so we can take limit $N \to \infty$. (and H_N)

Full system has conserved total energy (H_N) and possibly other conserved <u>densities</u> (<u>extensive</u> conserved quantities that the system can be a reservoir of), such as particle number, total spin, ... (an infinite number of such conserved densities for integrable systems)

Conserved <u>densities</u> are each assumed to go to a limit for $N \to \infty$

Consider any subsystem S:

degrees of freedom:

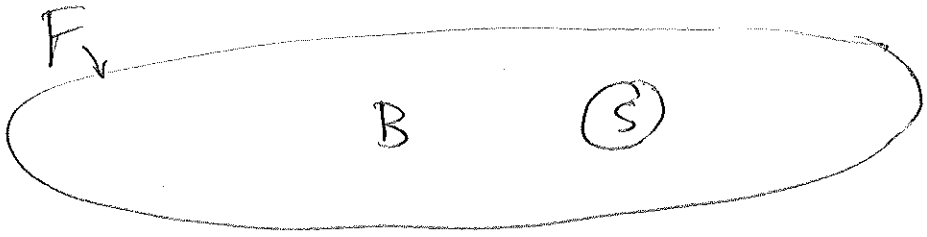
$$F = S \oplus B$$

F: full system; S: subsystem; B: "bath" = remainder of system.

S/B split need not be in real space.

Will take $F, B \to \infty$ at finite S. All Conserved quantities are "shared" and move between S and B.

this picture need not be real space.



F = full system
S = sub system
B = "bath"
F = S ⊕ B

$N, F, B \to \infty$ at finite S

probability operator on S (a.k.a. "reduced" density matrix):

$$\rho_S(t) = \underset{B}{\mathrm{Tr}}\, \rho_N(t)$$

Tr_B: trace over "bath"; $\rho_N(t)$: Full system

$$\rho_N(t) = e^{-iH_N t/\hbar}\, \rho_N(t=0)\, e^{iH_N t/\hbar}$$

exact dynamics of full system

Thermalization:

eq. = equilibrium

$$\lim_{t\to\infty} \lim_{N\to\infty} \rho_S(t) = \rho_S^{eq.}(T, \{\mu_j\}, \cdots) = \lim_{N\to\infty} \underset{B}{\mathrm{Tr}}\, \rho_N^{eq}(T, \mu, \cdots)$$

one parameter for each conserved density

$$\rho^{eq} = \frac{1}{Z} e^{-\beta(H - \mu N)\cdots}$$

SubSystems go to equilibrium:

Expect that for systems that do thermalize, this occurs for <u>all</u> subsystems (excluding ones that capture all of some conserved quantity) and <u>all</u> initial states of the considered $E, \{N_j\}$, etc. that correspond to this $T, \{\mu_j\}, \ldots$

→ Note: this thermalization is "stronger" for quantum systems than for classical systems.

Eigenstate Thermalization Hypothesis (ETH): Deutsch 1991, Srednicki 1994, Rigol, Dunjko, Olshanii 2008

One motivation:

If system thermalizes from all initial conditions, then exact many-body eigenstates must all be thermal (since they are time-independent)

$$H|n\rangle = E_n|n\rangle$$

← eigenstates of the full H.

take limit $N \to \infty$ at fixed T, μ, etc... (series of E_N's, $|n\rangle$'s)

probability of α in eigenstate $|n\rangle$:

$$\rho_S^{(n)} = \underset{B}{\mathrm{Tr}}\, |n\rangle\langle n| \xrightarrow[\text{for all } |n\rangle]{N\to\infty} \rho_S^{eq.}(T, \{\mu_i\}, \dots)$$

if ETH is true

In each exact many-body eigenstate of full system, all subsystems are thermalized. Have thermal entropy of entanglement. "Volume-law" entanglement entropy.

The H in ETH is Hypothesis. (not theorem). ETH appears to be true for many systems, but false for systems that are Anderson localized.

Writing ρ of full system in basis of exact many-body eigenstates:

$$\rho = \sum_{n,m} |n\rangle \rho_{nm} \langle m|$$

double sum over all eigenstates of full H.

If ETH is true, diagonal terms give only equilibrium properties, same for all ρ_{nn} with a given energy... densities (thus $T, \{\mu_i\}, \dots$)

Being out of equilibrium is special coherence off-the-diagonal in ρ (in this basis)

When ETH is true:

For the full system, thermalization is "just" dephasing:

"scrambling" of the phases of the off-diagonal parts of ρ

$$\rho_{nm}(t) = \underbrace{e^{i(E_m - E_n)t/\hbar}}_{\text{phase factor.}} \rho_{nm}(t=0) \qquad \left(\text{exact dynamics of } \rho\right)$$

Full thermalization requires that when these phases are made essentially random (by $t \to \infty$), these off-diagonal terms no longer contribute to ρ_S. This also appears to be true for systems obeying ETH (and not for Anderson-localized systems).

This motivates the "diagonal ensemble"

- $\rho_D = \sum_n |n\rangle \rho_{nn} \langle n|$ that neglects all off-diagonal terms.

  $\rho_{nn}(t) = \rho_{nn}(t=0)$ set by initial condition

Other ensembles:

- ρ^{eq} = Boltzmann-Gibbs ensemble $= e^{-\beta(H - \sum_i \mu_i N_i) + \cdots}$

  ∞ = Generalized Gibbs ensemble if we have an ∞ series of conserved densities

- $\rho^{(n)} = |n\rangle\langle n| \to$ micro canonical ensemble in the limit of one many-body eigenstate.

When ETH is true, usual eq'm stat. mech. works and these ensembles all give the same ρ_S for all subsystems, (eigenstates, initial conditions.)
(eigenstate)
(provided initial condition has the specified densities of all conserved quantities).

But they can all be observably different on small subsystems (thus local observables) for Anderson-localized systems, where ETH is false,

For generic systems, arguments in support of ETH are heuristic (plausible, but not at all rigorous).
numerical

Testing ETH is hard: requires examining exact many-body eigenstates. See Rigol, Dunjko, Olshanii (2008)
(N up to ~20) in practice
Pal and Huse (2010), Kim + Huse, unpublished and many others.

One Convenient type of test: compare $\rho_S^{(n)}$ to $\rho_S^{(n+1)}$ for many-body eigenstates that are adjacent in energy

$$E_{n+1} - E_n \sim 2^{-N}, \text{ so}$$

$|\rho_S^{(n)} - \rho_S^{(n+1)}|$ should decrease exponentially with N

we see this, and even the "worst-case" decreases, indicating that ETH is true for all eigenstates (Kim + Huse, unpublished)
for a nonintegrable Ising spin chain

Anderson localization: H has static randomness, e.g. particles in random potential. spins in random local fields.

3 "versions":

- Many-body localization: spins or particles with interactions at $T>0$ (our topic here)
- Noninteracting particles (or harmonic waves/oscillators) in a random potential with randomness

} Anderson 1958; most theory work over the years since then

- Interacting particles at $T=0$ and possibly coupled to an environment (phonons, etc.): metal-insulator or superconductor-insulator ground-state quantum phase transitions. (not our present topic).

} most exp'l work is about this

Let's follow Basko, Aleiner, Altshuler (BAA) 2006 and start with a tight-binding model of noninteracting fermions. (for $T>0$, bosons can behave similarly, but fermions are simpler + more familiar to some of us)

N sites.

$$H = \sum_{i=1}^{N} W_i c_i^\dagger c_i + \sum_{\langle ij \rangle} \left(t_{ij} c_i^\dagger c_j + \text{h.c.} \right)$$

W_i: on-site random potential with $\overline{W_i} = 0$ (avg. over prob. dist. of W)

t_{ij}: short-range hopping; can be random; let's keep them real (for simplicity)

This is a single-particle Hamiltonian:

single-particle <u>eigenstates</u> can be: (with nonzero randomness)

localized: $\psi_i^{(\alpha)} \sim e^{-|\vec{r}_i - \vec{r}_0^{(\alpha)}|/\xi}$ near $\vec{r}_0^{(\alpha)}$

each eigenstate α has its $\vec{r}_0^{(\alpha)}$, energy ϵ.
ξ = localization length.

extended: $|\psi_i| \sim N^{-1/2}$ "covers" all sites, diffusive dynamics. This can happen only for $d \geq 3$, weak enough randomness.

critical: special states <u>at</u> transition between localization + diffusion in $d \geq 3$.

tune randomness or energy: localization transition = "mobility edge".
An Eigenstate Phase Transition.

Now assume randomness in H is strong enough so <u>all</u> N single-particle eigenstates are localized with finite ξ.
(e.g., $d=1$). Eigenstates: $\psi_i^{(\alpha)}$ particle added to this state by $c_\alpha^\dagger$

noninteracting particles

$$H_0 = \sum_{\alpha=1}^{N} \epsilon_\alpha c_\alpha^\dagger c_\alpha$$

or: say $\sigma_{z\alpha} = (2 c_\alpha^\dagger c_\alpha - 1)$

~~(fermions)~~ ("hard-core" bosons instead of fermions)

noninteracting spins (q-bits)

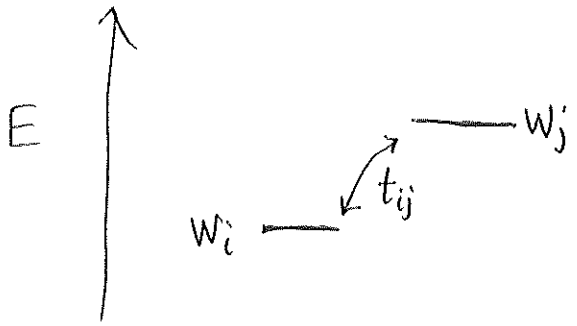
$$H_0 = \sum_{\alpha=1}^{N} \frac{\epsilon_\alpha}{2}(\sigma_{z\alpha} + 1)$$

"equivalent" to N spin-½'s in a field (except different commutation relations).

↑ = occupied state
↓ = empty state

(aside) Qualitatively, what is (strong) localization?

Single-particle states are strongly localized ($\xi \lesssim$ lattice spacing) when $|t_{ij}| \ll |W_i - W_j|$ for almost all pairs of sites:



← 2 typical adjacent sites on the lattice.

Eigenstates have <u>some</u> hybridization between i and j, but only with amplitude $\sim \frac{t_{ij}}{|W_i - W_j|} \ll 1$ when this ratio is small (treating t_{ij} in perturbation theory)

In $d \geq 3$ ~~mobility edge~~ regime of extended states is when this local hybridization becomes strong for <u>many</u> pairs, allowing a single single-particle eigenstate to ("percolate") have weight on many sites + to thus occupy sites in all regions of space.

H_0 (noninteracting)

These are also many-body Hamiltonians (many particles or many spins)

They have 2^N localized many-body eigenstates. (in this case "trivially" localized)

These systems do not obey E. T. H.

do not transport energy or particles (except at length scales $\lesssim \xi$)

do not act as a reservoir to thermally equilibrate subsystems.

Eigenstates are simple product states like

spin ↔ particle

↑ ↔ 1 occupied

↓ ↔ 0 empty "orbital"

$$|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \otimes |\downarrow\rangle_3 \otimes \cdots \otimes |\uparrow\rangle_N$$

or $|1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \cdots \otimes |1\rangle_N$

with no entanglement between these degrees of freedom.

(when not in an eigenstate)

Dynamics of spin model: each spin precesses about its z axis at frequency $|\epsilon_\alpha|/\hbar$

particle model: each $\langle C_\alpha \rangle(t)$ oscillates as $\sim e^{-i\epsilon_\alpha t/\hbar}$

(particles also do move around on length scales $\lesssim \xi$)

Now add short-range

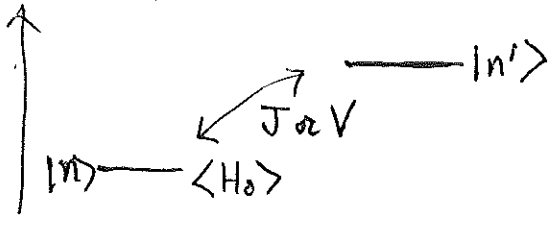
- particle-particle or
- spin-spin interactions.

When these interactions are weak, treat them as perturbations to the noninteracting localized H_0

particles: $$H = \underbrace{\sum_{\alpha=1}^{N} \epsilon_\alpha c_\alpha^\dagger c_\alpha}_{H_0} + \underbrace{\sum_{\langle\alpha\beta\gamma\delta\rangle} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta}_{\text{interaction. localized states } \alpha,\beta,\gamma,\delta \text{ all near each other.}}$$

spins: $$H = \underbrace{\sum_{i=1}^{N} \vec{h}_i \cdot \vec{\sigma}_i}_{\substack{\uparrow \\ \text{local static} \\ \text{random fields} \\ H_0}} + \underbrace{\sum_{\langle ij\rangle} \vec{\sigma}_i \overleftrightarrow{J}_{ij} \vec{\sigma}_j}_{\substack{\text{general short-range} \\ \text{interactions.}}} + \underbrace{3\text{-spin} + \text{higher-order term, if present.}}_{\substack{\text{in both cases:} \\ \text{no coupling to} \\ \text{any external} \\ \text{bath or environment.}}}$$

View many-body $|n\rangle$ eigenstates of H_0 as "sites" (2^N corners of a hypercube) with random energies $\langle H_0\rangle$. Interactions allow "hops" = particle-hole processes or spin flips



BAA 2006 showed many-body localization (MBL) survives weak non-zero interaction at $T>0$ for particles.

Oganesyan + Huse 2007 generalized to ~~spin models~~ $T \to \infty$. (all eigenstates)
arguments apply also to spins.

MBL is localization of energy/particles in real space.
($d=1,2,3$ typically are the ~~cases~~ dimensions of interest.)
+ also localization in the many-body Hilbert space
(corners of high-dimensional hypercube).

Eigenstates of H are essentially eigenstates of H_0
"dressed" with finite, local quantum "fluctuations"
caused by the interactions. (~~we~~ we'll return to this).

Increase the interactions and/or decrease the randomness:
there is a MBL $\leftrightarrow$ ETH eigenstate (and dynamical) phase transition.

this transition is poorly understood (no mean field theory
no scaling theory)

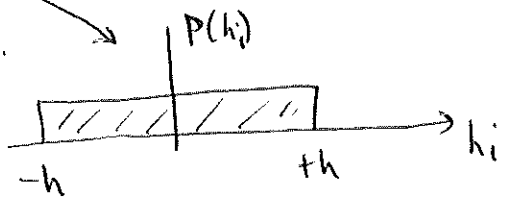
Model of Pal+Huse (2010) spin-1/2 chain

$$H = \sum_i \left(\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + h_i \sigma_{zi} \right) + \underline{\text{nothing else}}$$

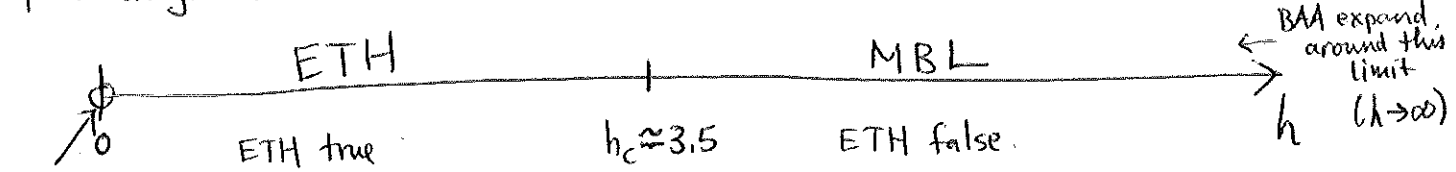
$L+1 = 1$ (p.b.c.)

did exact diagonalization $L \leq 16$

look at all eigenstates ("$T = \infty$")



phase diagram:



ETH true

system is →

Nonzero DC thermal + spin conductivity

"volume-law" →

locally decoherent: any q. info. present spreads over entire system.

Dynamics:
Energy, ~~particles~~ ~~and~~ σ_z diffuses

Entanglement spreads ballistically

(see Kim + Huse 2013 for a different nonrandom model)

ETH false.

is not a thermal reservoir for itself

strictly zero DC conductivity

"area-law" entanglement in eigenstates.

can store a q-bit locally with no decoherence.

no diffusion of energy or σ_z

entanglement spreads as $\sim \log t$

(Bardarson, et al., PRL 2012).

↑ an eigenstate phase transition:

shows up in dynamics, but not in thermodynamics"

For MBL phase to exist we need no thermal reservoir

Situations that appear to generically make a reservoir and thus forbid MBL:

Spontaneously-broken continuous symmetries → gapless long-wavelength Goldstone modes (e.g. acoustic phonons, spin waves....)

Particles moving in continuous space in $d \geq 3$: highest energy single-particle states are delocalized + make a reservoir. ($d=2$?)

Decoherence + spreading of entanglement in the MBL phase.

Simple 2-spin example:

$$H = h_1 \sigma_{z1} + h_2 \sigma_{z2} + J \sigma_{z1} \sigma_{z2}$$

All terms commute with each other and with H:

No "transport" of σ_z or energy. "Trivially" localized.

Eigenstates of H are all nonentangled product states.

	E
$\uparrow\uparrow$	$h_1 + h_2 + J$
$\uparrow\downarrow$	$h_1 - h_2 - J$
$\downarrow\uparrow$	$-h_1 + h_2 - J$
$\downarrow\downarrow$	$-h_1 - h_2 + J$

Start at $t=0$ with general nonentangled product state ($\hbar=1$)

$$(\alpha\uparrow_1 + \beta\downarrow_1)\otimes(\gamma\uparrow_2 + \delta\downarrow_2)$$

$$|t=0\rangle = \alpha\uparrow_1(\gamma\uparrow_2+\delta\downarrow_2) + \beta\downarrow_1(\gamma\uparrow_2+\delta\downarrow_2)$$

spin 2 has same state for $\uparrow_1$ and $\downarrow_1$: no entanglement

time-evolve this state

$$|t\rangle = e^{-ih_1 t}\alpha\uparrow_1\left(e^{-i(h_2+J)t}\gamma\uparrow_2 + e^{i(h_2+J)t}\delta\downarrow_2\right)$$
$$+ e^{ih_1 t}\beta\downarrow_1\left(e^{-i(h_2-J)}\gamma\uparrow_2 + e^{i(h_2-J)}\delta\downarrow_2\right)$$

spin 2 precesses at a rate given by σ_{z1}

no longer a product state (except at certain special times)

→ entanglement without any transport of energy or spin

→ decoherence: quantum info initially on spin 1 is shared between spins 1 and 2.

Note: this decoherence does not happen except when neither spin is started in an eigenstate.

Any two localized degrees of freedom that are <u>not</u> in a local eigenstate will interact and become entangled at long times.

"Description" of ~~many~~ MBL phase

(Serbyn, et al., Huse + Oganesyan, Bauer + Nayak, Swingle 2013)

Elementary excitations are particles or spins "dressed" with multiparticle, multispin operators.

In spin language: pseudospins $\vec{\tau}_i$ ("ℓ-bits") ℓ = localized

In MBL phase, when all eigenstates of H are localized, operators $\vec{\tau}_i$ are local in real space when expressed in terms of $\vec{\sigma}_j$'s (and vice versa)

$$\vec{\tau}_i = \sum_j \overleftrightarrow{K}_{ij} \vec{\sigma}_j + \sum_{jk} \overleftrightarrow{L}_{ijk} \vec{\sigma}_j \vec{\sigma}_k + \cdots$$

~~all terms fall off~~ typical value of $K, L, \ldots$ / probability of being large fall off exponentially with distance from "site i" (with localization length ξ)

Map from $\vec{\sigma}$'s to $\vec{\tau}$'s is a "local unitary" (Bauer + Nayak)

H in terms of $\vec{\tau}$'s:

$$H = \sum_{i=1}^{N} h_i \tau_{zi} + \sum_{ij} J_{ij} \tau_{zi} \tau_{zj} + \cdots$$

(just like our 2-spin example)

J_{ij} falls off with distance as $J_{ij} \sim e^{-r_{ij}/\xi}$

τ_{zi}'s all commute with each other and with H.

the 2^N many body eigenstates of H are simply the 2^N simultaneous eigenstates of all τ_{zi}'s

Eigenstate of H in MBL phase is

$|n\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots |\downarrow\rangle_N$ in terms of eigenstates of τ_{zi}

Can flip one ℓ-bit to make another eigenstate of H:

$$|\bar{n}\rangle = \tau_{xi}|n\rangle$$

$|\bar{n}\rangle$ and $|n\rangle$ have all τ_{zj}'s the same except $j=i$ is flipped.

Write a q-bit on ℓ-bit i:

$$|\psi\rangle = \alpha|n\rangle + \beta|\bar{n}\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

$= \text{eigenstate} \otimes (\alpha|\uparrow\rangle_i + \beta|\downarrow\rangle_i) \otimes \text{eigenstate}$ arbitrary state of this q-bit.

With this H, this q-bit will live forever, no decoherence.

(only precession of $\vec{\tau}_i$ about its z-axis): An ideal quantum memory.

But: Preparation of exact eigenstates is unrealistic.

If q-bits are "written" (perhaps by "accident") on other $\vec{\tau}$'s, then precession rate of $\vec{\tau}_i$ depends on all the τ_{zj}'s $\Rightarrow$ entanglement + decoherence.

$J_{ij} \sim e^{-r_{ij}/\xi} \rightarrow$ entanglement time $\sim e^{+r_{ij}/\xi} \rightarrow$ entanglement volume $\sim (\log t)^d$.

But: "bath" of all τ_{zj}'s is static, so decoherence can in principle be reversed by spin-echo techniques.