Rigidity Transition: From Random Networks to Jamming

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Introduction

Studying the mechanical stability of various systems can help us understand the elastic properties of different materials ranging from crystals to granular media and glasses. In the recent years there has been considerable interest in the elastic properties of random disordered materials and jammed disk packings near their stability transition point. These systems are usually modeled as networks consisting of points in space (sites) and a number of constraints (bonds) where a constraint is a fixed connection between two points. A finite two dimensional network is considered rigid if there are enough bonds present so that the system has only three degrees of freedom for translations and rotations of the whole network. If we start removing bonds from a random network, at some point it will lose its rigidity. This random bond dilution process is called rigidity percolation (RP). Packings of soft spheres or disks show the same behavior when we lower their confining pressure towards zero in a process known as unjamming. Despite their close correspondence, random and jammed networks show significant differences when they are close to their marginal states. In this study we will have a closer look at the qualitative differences in the rigidity transition of these networks.

Pebble Game Analysis

The first methodical study of mechanical stability in networks was done by Maxwell in the 1860s. He used ideas previously introduced by Lagrange to develop the equation today we refer to as *modified* Maxwell's count:

$$F = dN_S - N_B + R \qquad \text{(I)}$$

Where F represents the number of degrees of freedom, d the dimensionality of network, N_S the number of sites, N_B the number of bonds and R the number of redundant bonds. A redundant bond is a bond that introduces extra constraints to the system and can be removed without changing the rigid regions in the network.

A finite 2D network is said to be isostatic if $R=0$ and $F=3$. If $R=0$ and $F>3$ it means there are some internal degrees of freedom that we call floppy modes and if $R\neq 0$ and $F<3$ there are over constrained (stressed) regions in the network. Here we determine these regions using the **pebble game** which is an integer algorithm for book keeping between degrees of freedom and constraints that allows a rigid region decomposition of generic spring networks by analyzing their topology. Figure 1 illustrates such an analysis for a small network.

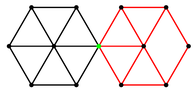

Figure 1. Red means isostatic, Black means stressed and Green means hinge.

Results

Figure 2 shows noticeable differences in the nature of the marginal states of three types of networks.
At the top row a jammed network is shown at the marginal state (center), one bond above (right) and one bond below it (left). At the bottom row the soft transition of a RP network through its marginal state is shown. There is a third family of networks introduced in the middle row that becomes isostatic everywhere at their marginal point (as in jamming). This stress-relieved (SR) network is obtained by cutting bonds randomly, but only if they are stressed in a triangular network.

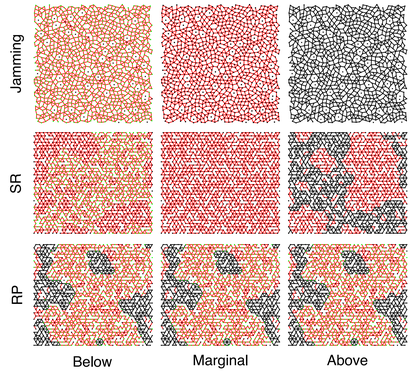

Figure 2.

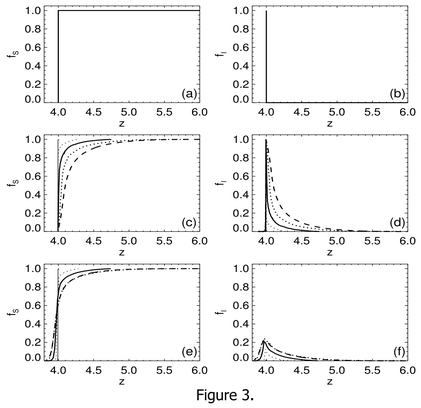

Figure 3.

Figure 3 shows a comparison between the fractions of stressed (left) and isostatic (right) bonds for jamming (top row), SR (middle row), and RP (bottom row) averaged over 50 networks. The latter two are plotted for a triangular net plus configurations corresponding to jammed networks at four different values of initial mean coordination z. The initial z for different line styles are: Solid-Thin: z_J=4.01, Dotted Thin: z_J=4.3, Solid Thick: z_J=4.7, Dotted Thick: z_J=5.98, Dashed Thick: z_J=6 (Triangular)

Figure 4 shows shear (G) and bulk (K) moduli as a function of z both for random dilution (a) and for stressed dilution (b) of networks shown in Figure 3. As can be seen from the picture:
(1) For initial $z\approx 6$, the functions G(z) and K(z) are identical to those for triangular nets
(2) G(z) is essentially independent of initial z
(3) The behavior of K is richer than G. It weakly depends upon z but stays finite as we get closer to transition point.

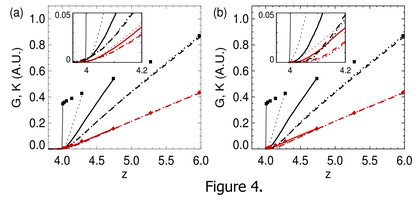

Figure 4.

Quantifying the self organization

The response of a network to addition or removal of bonds at the marginal state is a good measure of its self-organization. To quantify this measure we introduce two new indices:
***h* index:** Defined by removing one bond randomly from the marginal state, counting the number of new green hinges, averaging over every bond in the network, and dividing by the number of sites so that $0<h<1$.
***s* index:** Defined by adding one bond randomly to the marginal state, counting the number of new stressed bonds, averaging over all bonds and dividing by the number of bonds so that $0<s<1$. High values of h and s mean strong self-organization of the network.

We find that the h and s indices for three types of networks we have been studying are as follows:
Jamming: h=0.97, s=0.98
SR: h=0.28, s=0.47
RP: $h\approx 0.0003$, $s\approx 0.001$

Future Work

According to **Hilbert** stability condition any site in a globally stable network must have at least three connected neighbors distributed such that no empty semi circles can be found in an imaginary circle around the site. This condition of course can be violated when we remove bonds randomly from a network and this is the main reason the average s index was smaller than unity for our 50 jammed networks. It turn out that imposing Hilbert condition as we dilute a random network can rise its s index to 1.

This opens up a number of new questions: What would happen if we impose Hilbert stability condition as we dilute an amorphous network such as one shown in Figure 5.(a)? Would it allow us to build an isostatic state that behaves like a jammed packing derived network?

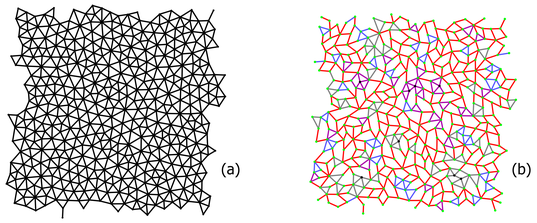

Figure 5. (a) An amorphous stressed network (b) The rigid sub graphs of the the same network at its isostatic state.

Our next step is to investigating these questions by calculating the s and h indices of networks in Fig. 5 and measuring their shear and bulk moduli.

Acknowledgements & References

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