2.2. Linear Stability Analysis + Transition to Turbulence

In the case of turbulent convection, the supercritical instability of the laminar state is separated from turbulence by a series of instabilities to different types of pattern.

(i) Pipe flow

Pressure gradient $\frac{dp}{dx} < 0$

$$\underline{v} = \left(1 - \frac{r^2}{a^2}\right) U_c \underline{\hat{x}}$$

$$\frac{dp}{dx} = -\nu \frac{4 U_c}{a^2}$$

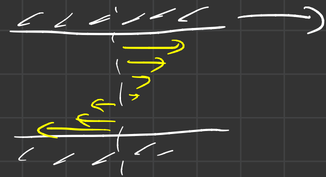
Stability about this solution: stable up to $Re = 10^7$

Belief that smooth pipes are linearly stable.

2021: Hall & Ozcakir: Roughness of scale ϵ $\Rightarrow$ $R_c \sim C \epsilon^{-3/2} |\log \epsilon|^{-3/4}$

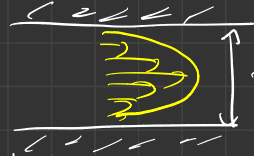
$C = O(30)$

(ii) Plane Couette Flow



Linearly stable (Romanov 1973) ←
but expt → turbulence $Re > 350$

(iii) Plane Poiseuille Flow



Pressure driven flow.
Linear instability: $R_c = 5772 = U_c h/\nu$
→ discovered by Heisenberg!

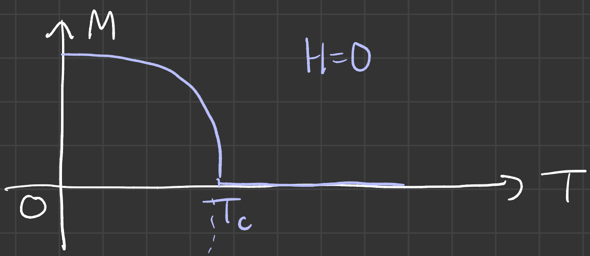
But: turbulence occurs at $R_c \sim 1500$, expt.

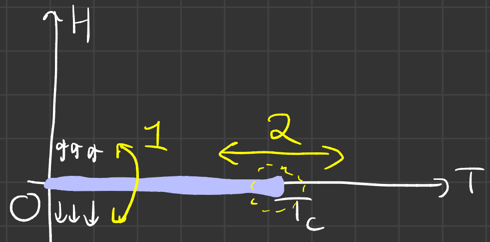
Conclusion: * Finite amplitude instabilities.
Critical amplitude of disturbance
$\epsilon_c \sim 1/Re$ (Hof)
PRL 2003

* We need to systematically study sub-critical transitions!

§2. Widom Scaling.

There are two stylized facts that were known about critical phenomena in the early 1960's.





Order parameter scaling

$$M = M_0 \left| \frac{T-T_c}{T_c} \right|^{\beta} \qquad T \to T_c^{\pm} \qquad (1)$$

$$t \equiv \frac{T-T_c}{T_c} \qquad h \equiv H/k_B T_c$$

Breakdown of linear response theory. (2)

$$M \sim H^{1/\delta} \qquad T = T_c$$

not $M \sim H/k_B T$ (Curie)

Widom and Kadanoff realised that together these results are equivalent to

$$M(t,h) = |t|^{\beta} F_m(h/t^{\Delta}) \qquad t<0$$

where Δ is a new exponent that we'll shortly calculate

Q/ What is the function F_M and the exponent Δ?

A/ • For (1) to hold, we need $F_M(z) = \text{const}$ for $z=0$

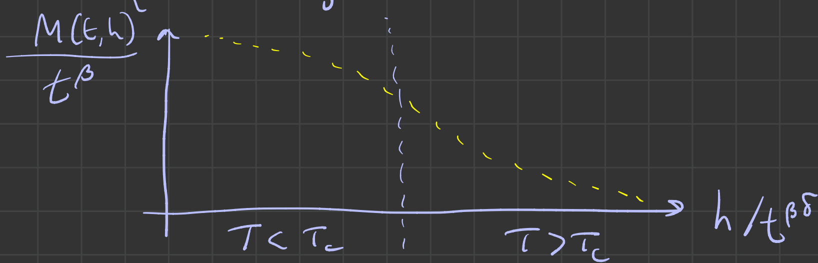
• For large z, i.e. $h \neq 0$ $t \to 0$ we need to recover (2) which means t must somehow cancel out. This can only happen if $F_M(z) \sim z^{1/\delta}$ as $z \to \infty$.

Then $t^{\beta - \Delta/\delta} = O(1) \Rightarrow \boxed{\beta\delta = \Delta}$

$$\Rightarrow \boxed{M(t,h) = t^{\beta} F_M\left(h/t^{\beta\delta}\right)}$$

$M(t,h)$ is ostensibly a function of two variables h, t, but near the critical point is actually a function of a "similarity" or "scaling" variable $h/t^{\beta\delta}$.

We can test this as follows. Take sets of data for $t = t_1, t_2, t_3, \ldots$ $h = h_1, h_2, \ldots$ and plot $\{M(t_i, h_i)\}$ as



→ Back to slides

§ 3. Predator-Prey Model

3.1 Centers and neutral cycles

$A =$ predator density $\qquad B =$ prey density.

$$\dot{A} = p\,AB - d\,A$$
$$\dot{B} = b\,B - p\,AB$$

$A + B \xrightarrow{p} A + A$

$A \xrightarrow{d} \emptyset$

$B \xrightarrow{b} B + B$

We can easily calculate steady states and phase portrait.

$\dot{A} = 0 \Rightarrow A(pB - d) = 0 \qquad A^* = 0$ or $B^* = \frac{d}{p}$

$\dot{B} = 0 \rightarrow B(b - pA) = 0 \qquad B^* = 0$ or $A^* = \frac{b}{p}$

Coexistence

Steady states: $(A^*, B^*) = (0,0)$; $\boxed{(A^*, B^*) = (b/p, d/p)}$

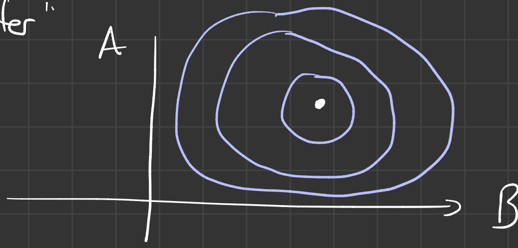
Long-time $A^* = 0$, $B \to \infty$ as e^{bt} as $t \to \infty$.

Linear stability: $A = A^* + \delta A e^{\omega t} \quad B = B^* + \delta B e^{\omega t}$

$(0,0)$ unstable $\qquad (b/p, d/p) \rightarrow \omega = \pm i\omega_0$

$(0,\infty)$ unstable $\qquad \omega_0 = \sqrt{bd}$

These oscillations about the coexistence fixed point describe a "center"



The phase space is obtained from.

$$\frac{dA}{dB} = \dot{A}/\dot{B} = \frac{pAB - dA}{bB - pAB} = \frac{A(pB-d)}{B(b-pA)}$$

$$\Rightarrow \int_0^t \frac{dA}{A} \cdot (b-pA) = \int_0^t \frac{dB}{B} \cdot (pB-d)$$

$$\Rightarrow b \ln A(t) - pA(t) + pA(0) - b \ln A(0) = pB(t) - d \ln B(t) + d \ln B(0) - pB(0)$$

$$\Rightarrow b \ln A(t) + d \ln B(t) - p(A(t)+B(t)) = b \ln A(0) + d \ln B(0) - p(A(0)+B(0))$$

So $C(t) = b \ln A(t) + d \ln B(t) - p(A(t)+B(t))$

$= C(0)$ i.e. a conserved integral of the motion.

This model predicts periodic oscillations, with trajectory (amplitude / phase) determined by $C(0)$.

However, this solution is unphysical because it is structurally unstable.

3.2 Finite carrying capacity.

The FP $A=0$, $B = B_0 e^{bt}$ is unphysical because in reality there is a finite amount of food for the prey. So B is bounded above by what is called "carrying capacity".

We model this as

$$\dot{B} = bB(1 - B/K) - pAB.$$

Fixed Points:

$A^* = B^* = 0.$ — Extinction

$A^* = 0 \quad B^* = K$ — Predator death, Prey saturate.

$A^* = \left(1 - \frac{d}{Kp}\right)\frac{b}{p}\,; \quad B^* = d/p$ — Coexistence

Stability:

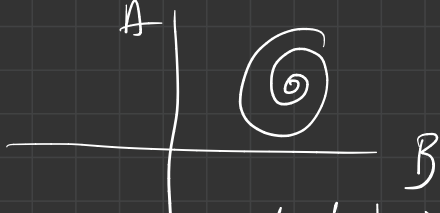
Extinction: unstable.

Prey saturation: stable $\quad p < p_c = d/K$

Coexistence: $p > p_c$ and linearly stable with

$$\omega = -\frac{db}{2Kp}\left(1 \pm \sqrt{1 - \frac{4Kp}{b}\left(\frac{Kp}{d} - 1\right)}\right)$$

Summary: No persistent population cycles. The phase portrait has stable spiral



To see what has gone wrong, go back to individual level model.

Q/ How did ecologists deal with this embarrasment?

A/ Let's change the physical picture.

The predation term pAB only applies if the concentration of predator and prey is small.

But suppose prey concentration is large. Predator does not need to look for to find a prey to eat. In other words, prey concentration is not a limiting factor. So

$$pAB \longrightarrow \frac{pAB}{C+B}$$

where C is a constant.

For $B \ll C$, pAB is recovered.

For $B \gg C$ $\quad pAB \longrightarrow pA$ indep of B.

Thus

$$\mathring{A} = \frac{pAB}{C+B} - dA$$

$$\hat{B} = bB(1-B/K) - \frac{pAB}{C+B}$$

This non-linear system does have limit cycles

Back to slides!