Lecture 1

Fluids undergo laminar–turbulent transitions in several ways. For most of these lectures, we will be interested in how a laminar fluid becomes turbulent as we increase Re. But I want to start off by talking about the opposite case: how turbulence can become laminar. This is what happened in the sandstorm!

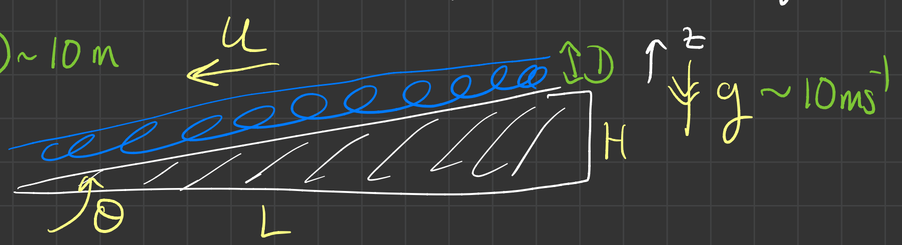
1.1 Speed of a river.

Kolmogorov was one of the first to ask how fast a river flows, but not the first. As far as I know, the French engineer Chézy around 1768, was the first to consider this scientifically. In modern terms, the argument is this. We will try to calculate the speed three ways.

$L = 10^6\ m$ $D \sim 10\ m$

$H = 10^2\ m$

$\theta = 10^{-4}$



Method 1: Conservation of energy.

$$\frac{1}{2}\rho U^2 = \rho g H \Rightarrow U = \sqrt{2gH}$$

$$= \sqrt{2}\sqrt{10 \cdot 10^2}$$

$$= 10 \times 1.4 \times 3$$

$$\boxed{\simeq 50 \text{ ms}^{-1}}$$

Method 2: Viscosity. Fluid flows in steady state so viscous force = gravitational force

i.e. $\nu \frac{d^2 u}{dz^2} = g\theta$ $\qquad \nu = \mu/\rho = 10^{-6} m^2 s^{-1}$

Boundary conditions $u = 0 \quad z = 0$

$\frac{du}{dz} = 0 \quad z = D$

$$u(z) = U_0 \frac{z(z - 2D)}{D^2}, \quad U_0 = \frac{g\theta D^2}{2\nu}$$

$$\boxed{U_0 \sim \frac{10 \cdot 10^{-4} \quad 10^2}{10^{-6}} = 10^5 \text{ ms}^{-1}}$$

Method 3: Use a basic fact about turbulence which was even known to Newton.

In steady state, rate of energy production = rate of energy dissipation.

The amazing guess by Newton, Chezy, Kolmogorov is that the drag force / unit mass

$$F_d \propto U^2 \quad \text{not } U \,!$$

Dimensions: $[F_d] = LT^{-2} \Rightarrow$ need U^2/length.

$$F_d = O(1)\, U^2/D \quad \text{hypothesis}$$

Rate of energy dissipation $\epsilon = F_d \cdot U = \dfrac{U^3}{D}$.

Note: Usual argument is $\exists$ cascade so $U^2 =$ energy, rate $= \frac{D}{U}$
$\Rightarrow U^3/D$.

The rate of energy production is

$$\underset{\text{force}}{g\Theta} \times \underset{\text{velocity}}{U}$$

I prefer my force-based argument to relate to Newton/Chézy. Both arguments are hypothesese + heuristic

Thus $\dfrac{U^3}{D} \sim g\Theta U \Rightarrow U^2 \sim g\Theta D$

$$\Rightarrow U \sim 10 \cdot 10^{-4} \cdot 10 = \boxed{10^{-2}\ \mathrm{m\,s^{-1}}}$$

Experimentally, mean river speed $\sim 10\,\mathrm{cm\,s^{-1}}$

So the turbulence guess for the friction is the closest, by far!

Conclusion: Generic flow state of river is turbulent.
True for air too,....

1.2 Suppression of turbulence.

The basic idea is that turbulent kinetic energy can be used up if there are particles suspended in the flow. Once this happens, and the details are very subtle and still being worked out, the flow will become more laminar and the mean speed will increase.

If the mean speed increases and the flow is over a loose soil, then more sand can be brought into the flow reducing the turbulence further, up to a limit.

1.3 Propagation of turbulence.

In a tornado, there is frequently dust and also water droplets. We might think of the interior of a tornado as somewhat laminar, the exterior as turbulent. The reason is that the suspension suppresses turbulence.

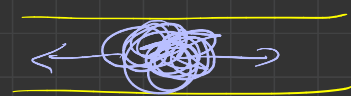
Q/ What controls the propagation of a laminar-turbulent boundary? Is it smooth or sharp?

A/ Let $q(z,t)$ = turbulent kinetic energy at (z,t)

$$\frac{dq}{dt} = -\epsilon_0 \frac{u^3}{L} = -\epsilon_0 \frac{q^{3/2}}{L}$$

rate of energy dissipation

if there is no spatial variation.



Now put in space by assuming that there is a diffusion of energy: (in $d=1$ for simplicity)

$$\partial_t q = \partial_z \left(D_q \partial_z q \right) - \epsilon_0 \frac{q^{3/2}}{L}$$

energy diffusion constant

The crux of the argument is due to Kolmogorov 1942

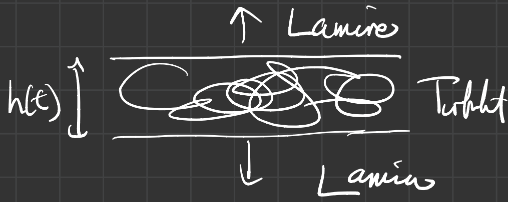
$$D_q = O(1) \times L\sqrt{q}$$

since by K41 hypothesis, the turbulent dissipation should not involve ν.

We will set constant = 1.

We assume L is proportional to the size of the turbulent patch.

$$L = \alpha h(t)$$



This gives:

$$\partial_t q = \alpha \partial_z \left(h(t) q^{\frac{1}{2}} \partial_z q \right) - \frac{\epsilon q^{3/2}}{\alpha h(t)} \qquad |z| \leq h(t)$$

$$q = 0 \qquad |z| > h(t)$$

We assume initial turbulent energy is in $-a < z < a$

$$q(z,0) = \frac{Q}{a} u_0\left(\frac{z}{a}\right) \qquad Q = \int_{-a}^{a} q(z,0)\,dz$$

$$\int_{-1}^{1} u_0(s)\,ds = 1$$

We can solve in the case $\varepsilon = 0$.

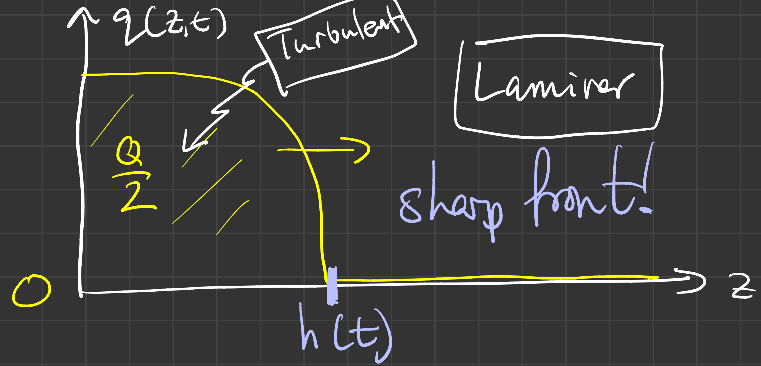
The solution is $\quad q(z,t) = \dfrac{Q \xi_0^3}{(6\alpha)^3 h(t)} \left(1 - \dfrac{z^2}{h(t)^2}\right)_+^2$

where $\xi_0^3 = 135 \alpha^2 / 4$

$$h(t) = \left(\xi_0^{2/3} Q^{1/2} t + h(0)^{3/2} \right)^{2/3} \sim t^{2/3}$$

and the function $(1-x^2)_+$ means $(1-x^2)$ as long as $(1-x^2)$ is +ve.

Thus



Now let's look at effect of $\epsilon \neq 0$. The right way to do this is by RG, but we can do this in a quick and dirty way as follows.

If $\epsilon \ll 1$, the energy is not conserved but is being removed slowly with respect to the spreading of the turbulence. So we will make an adiabatic assumption and say that Q is no longer constant but slowly decays with time.

Since $Q \equiv \int_{-h(t)}^{h(t)} q(z,t)\,dz$

$$\dot{Q} = -\frac{\epsilon}{\alpha h(t)} \int_{-h}^{h} q^{3/2} \, dz$$

Our adiabatic approximation is to replace Q by $Q(t)$ in the zeroth order solution for $q(z,t)$.

i.e. $$q(z,t) = \frac{Q(t)\,\xi_0^3}{(6\alpha)^3 h(t)} \left(1 - \frac{z^2}{h(t)^2}\right)_+^2$$

Thus
$$\dot{Q} = -\frac{\epsilon}{\alpha} \frac{Q^{3/2}}{(6\alpha)^3} \frac{\xi_0^{9/2}}{h^{3/2}} \int_{-1}^{1} (1-z^2)^3 \, dz$$

For long times, in the zeroth order solution
$$h(t) \sim \xi_0 \, Q(t)^{1/3} t^{2/3}$$
$$\Rightarrow h^{3/2} \sim \xi_0^{3/2} Q^{1/2} t$$

$$\Rightarrow \frac{dQ}{dt} = -(\text{stuff}) \times \frac{Q}{t}$$

i.e. $Q(t)$ is a power law in t!

Specifically: $$Q(t) = Q(t_0) \left(\frac{t}{t_0}\right)^{-\epsilon/7\alpha^2}$$

Thus we conclude that the propagation is
$$h(t) \sim t^{2/3 - \epsilon/21\alpha^2 + O(\epsilon^2)}$$

This calculation shows that a turbulent front spreads with an anomalous scaling in time.

Notes:

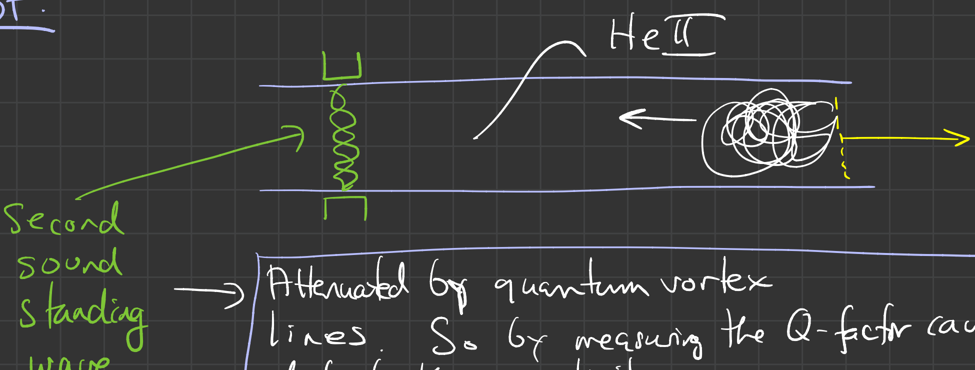
1. The RG theory was worked out by Chen and me in 1992, based on Barenblatt's original description.

2. The sharp front predicted by Barenblatt was observed in experiments on Helium 2. See the paper by Mike Smith.

The sharp front not a Gaussian decay of turbulence may explain the extreme localization in a tornado

3. Later we will discuss turbulence propagation at low Re. The calculation given here is for high Re where K41 scaling holds.

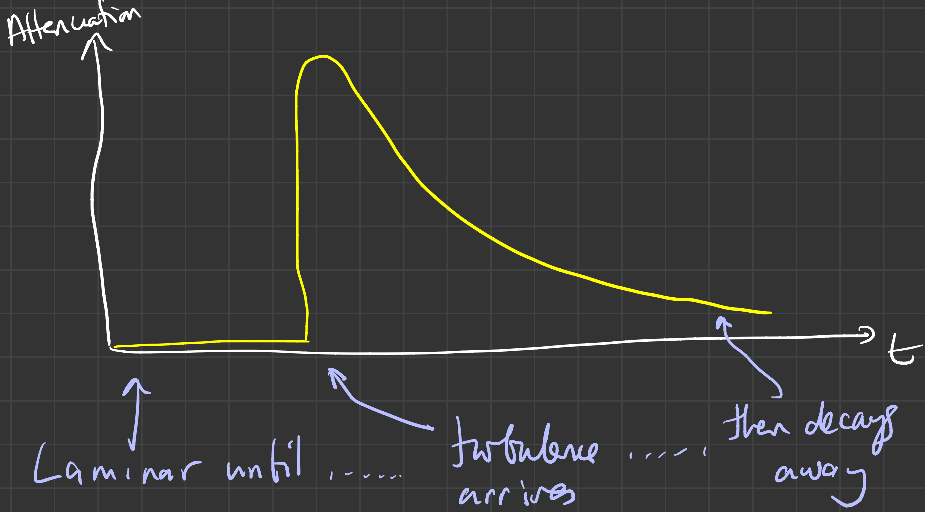
Expt:



Attenuated by quantum vortex lines. So by measuring the Q-factor can detect the vorticity.

Q/ What do we see at standing wave?

A/



1.4 Transition to turbulence

Now we start to examine how fluids become turbulent as a control parameter is varied. I will start by explaining the set-up for Rayleigh-Bénard convection, where we vary the temperature difference between two plates containing a fluid. Then we will spend the rest of the lectures talking about pipe flow mainly.