**Exotic Quantum Phases (III) [Senthil]**

- Mott transition: metal/superfluid/superconductor & insulator
  - Example: Bose-Hubbard $-t \sum (b_i^\dagger b_j + \text{h.c.}) + U \sum_i n_i(n_i-1)$
  - $t \gg U \Rightarrow \text{superfluid}, U \gg t \Rightarrow \text{Mott insulator}$
  - Approach from SF: $Z_{\text{sf}} \to 0$
  - Approach from Mott: $Z_{\text{M}} \to 0$ at $T=0$

\[
S = \int \frac{d\tau d^2k}{(2\pi)^2} \left( |\phi_{1\tau}^2| + 1|\phi_{2\tau}^2| + 1|\phi_{1\tau}^2| + u|\phi_{1\tau}^2| \right)
\]

Can approach by $e$-expansion, $\ln u$-expansion, Monte Carlo, etc.

- Example: one-band Hubbard on non-bipartite lattice
  - $H = -t \sum (c_{i\uparrow}^\dagger c_{j\downarrow} + \text{h.c.}) + U \sum_i n_i(n_i-1)/2$
  - AF Mott Insulator
    - (broken sym.)
    - (no Fermi surface)
  - Fermi liquid
    - (unbroken sym.)
    - (Fermi surface)

Case (1):
- 1st order

Case (2):
- AF metal
  - 2nd order

Case (3):
- spin liquid
  - insulator
  - 2nd order

Case (4): 2nd order?!
A similar problem occurs in heavy electron problem:

- Large Fermi surface
- Small Fermi surface

2nd order "non-Fermi"

Liquid critical point

\( a \) one band Mott transition + conduction bands

\( \uparrow \) f-electron

\( \downarrow \) d-electron

- Issues:
  1. Can Fermi surface disappear at same time when magnetic order appears?
  2. How to understand QCP where whole Fermi surface disappears?

- Possible Directions:
  1. Theory of "deconfined criticality."
  2. Concept of "critical Fermi surface."

- Fermi Surface Disappearance:
  - Quasiparticle weight may vanish completely and everywhere on Fermi surface. (Brinkman—Rice 1973)
  - Claim: at critical point, Fermi surface is still sharply defined ("critical Fermi surface"), even though \( Z = 0 \).

- In case of heavy fermion, both large/small Fermi surface must disappear.
Scaling Phenomenology near critical Fermi surface

- May expect $A_c(R, \omega, T) \sim \frac{1}{\lambda(\omega)} F\left(\frac{\omega}{\lambda(\omega)}, \frac{\omega}{\lambda(\omega)}\right)$
  But we can have $\alpha, \beta$ angle dependent, as long as it agrees with lattice symmetry.

- Expect scale invariant spectrum cut off at $k^* \sim \sqrt{\frac{\epsilon}{\omega}}$, $\omega \sim \frac{\epsilon}{\lambda(\omega)}$
  where $\frac{\epsilon}{\lambda(\omega)} \sim |g-g_c|^{-\nu}$ with $\nu$ depends on angle.

- Prediction $C_v \sim T \int_{FS} \frac{V_F}{\sqrt{\nu}} \sim T \int_{FS} \log |\lambda(\omega)|^{-\nu(1-2)}$
  For $\nu, \lambda(\omega)$ depends on $\Theta$, result dominated by portion with maximum $\nu(1-2)$

- Different portions of Fermi surface may emerge out of criticality at different energy scale
  $\Rightarrow$ richer finite-T crossover than usual.

- NOTE: we still assume only one critical point $g_c$ instead of multiple critical point.

Simultaneous disappearance of Fermi surface & appearance of magnetic order.

- Start with simpler situations where both side of critical points are well known & describe by Landau-Ginzburg-Wilson formalism.

- Example: $H = J \sum_{ij} \vec{S}_i \cdot \vec{S}_j$
  What's important is that Hilbert space is spin-$\frac{1}{2}$ moment with square lattice & spin rotation symmetry.

Consider Neel $\leftrightarrow$ VBS state

- VBS state has $\mathbb{Z}_4$ order parameter
- Ginzburg-Landau prediction:
  - Neel ≠ usual O(3) transition in d = 3
  - VBS ≠ usual Z_4 transition in d = 3
  - Reason: topological defects carry non-trivial quantum numbers
  - Consider approaching from VBS side. Topological defects are domain walls.

- These defects meet at Z_4 vortex:

- This lead to Z_4 transition, but with vortices carrying spin-1/2 quantum number.
  - When these vortices condense, we obtain Neel state.

- We want to transform such that vortices become primary variable (duality transformation).

- The vortices are described by XY-order.

- Near critical points, domain walls from vortices become "thick" & "soft". Domain wall thickness diverges near transition.

- Vortex core size \( \propto \xi \), \( \xi \) vs. diverges faster than \( \xi \).

Best numerical evidence from J-Q model:

- \( H = J \sum_{ij} \vec{S}_i \cdot \vec{S}_j - Q \sum_{i<j<k} \left( \vec{S}_i \cdot \vec{S}_j - \frac{1}{4} \right) (\vec{S}_k \cdot \vec{S}_2 - \frac{1}{4}) \)

  - \( Q/J \) small \( \Rightarrow \) Neel
  - \( Q/J \) large \( \Rightarrow \) VBS
Consider XY-model in (2+1) dimensions.

Recall \( E(-\epsilon_j^+) \sim \ln R \), so it can be interpreted as charges.

Formally, \( Z = \int \prod_i d\theta_i e^{-K \cos(\theta_i - \epsilon_j)} \approx \int \prod_i d\theta_i \sum_{g_3} e^{-\sum_{\text{links}} J_{ij}^2 + iJ_{ij} \theta} \).

Performing \( \theta \)-integral \( \Rightarrow \Delta \cdot \mathbf{j}^+ = 0 \).

\( Z \approx \sum_{g_3} S(\Delta \cdot \mathbf{j}) e^{-\sum_{\text{links}} J_{ij}^2} \).

Solve the constraint via \( \mathbf{j} = \Delta \times \mathbf{A} \) (\( \mathbf{A} \) lives on dual lattice), with \( \mathbf{A} \) an integer.

\( Z = \sum_{\Delta \mathbf{A}} e^{-\sum_{\text{links}} (\mathbf{A} \times \mathbf{X})^2} \).

\( Z \rightarrow \int \mathcal{D}\Delta \mathbf{A} \sum_{g_3} e^{-\sum_{\text{links}} (\mathbf{A} \times \mathbf{X})^2 + 2\pi i (J \mathbf{A} + e \mathbf{j} - \Delta \mathbf{A})} \).

\( \rightarrow \int \mathcal{D}A \mathcal{D}\phi e^{-\int u(\nabla \phi)^2 + \lambda \cos(2\pi(\phi - \Delta \phi))} \).