Disorder and the Quantum Hall Effect

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Outline

Lecture I

Overview of IQHE

Anderson Localisation and IQHE

Lecture II

Exactness of quantisation

— edge states, adiabatic flux insertion and Chern numbers

Lecture III

Plateau transition as critical point

— scaling theory, experiments, and network model

Lecture IV

Generalisations of the IQHE

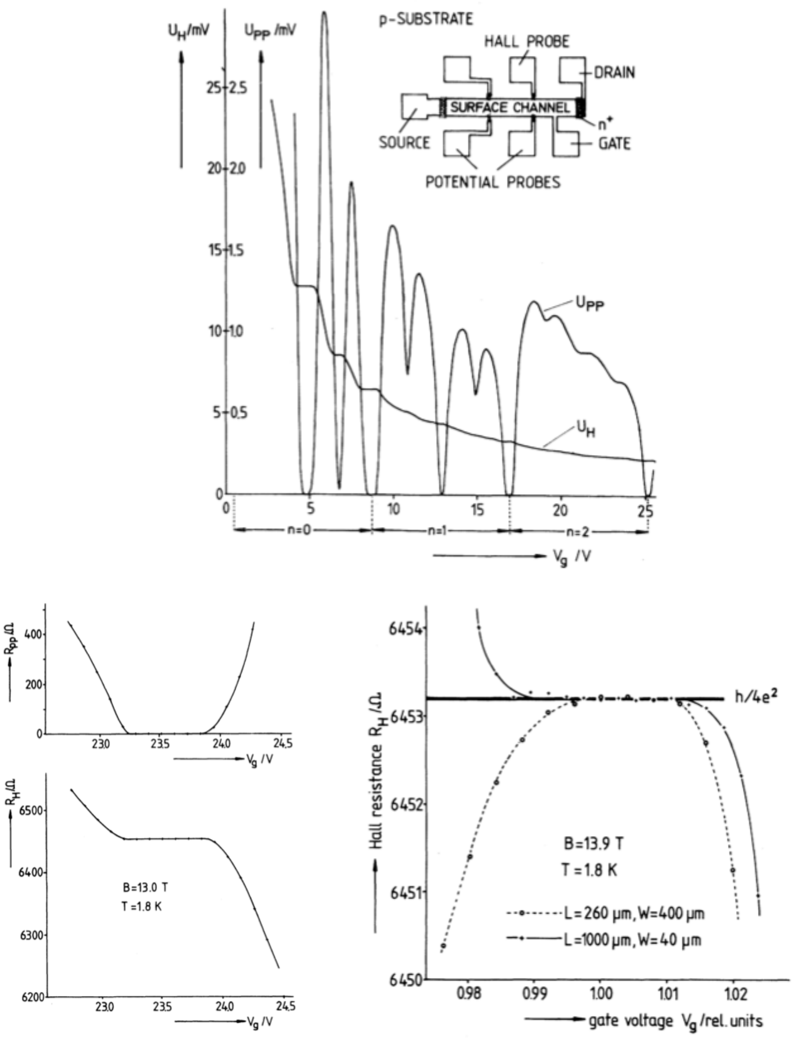
— additional symmetry classes and dirty superconductors

I. OVERVIEW OF IQHE

Why teach this material now? Answers: (i) Ideas that came into condensed matter physics via the IQHE (and FQHE) are now central - see applications in context of topological insulators. (ii) Experiments on the IQHE plateau transition are some of the clearest studies of an Anderson transition.

A. Experiment

Let's start by looking at the data in the original paper on the discovery of the IQHE



K von Klitzing, G. Dorda, and M. Pepper, PRL **45**, 494 (1980)

Note two aspects to phenomenon: (i) minima in R_{xx} and (ii) plateaus in R_{xy}. Questions: Why are there plateaus and minima? Why is R_{xy} so accurately quantised?

B. Basics

Before thinking about these questions in detail, we should recall some facts about the quantum mechanics of charged particles moving an a uniform magnetic field. We define the flux density B, the charge e, the effective mass m, the g-factor g^* and work in SI units. Two scales are important. One is the cyclotron frequency $\omega_c = eB/m$. Note that this is classical and material dependent. The other is the magnetic length $\ell_{\rm B} = \sqrt{\hbar/(eB)}$, which is quantum and material-independent.

Consider the single-particle spectrum in the absence of disorder. It consists of a series of Landau levels at energies $E = (n + 1/2)\hbar\omega_c \pm \frac{1}{2}g^*\mu_{\rm B}B$. The degeneracy of a Landau level is *one state per flux quantum*. In a system of area A with n electrons per unit area we have: $N_e = nA$ electrons and $N_\phi = BA/(h/e)$ flux quanta. The ratio of these is the filling factor

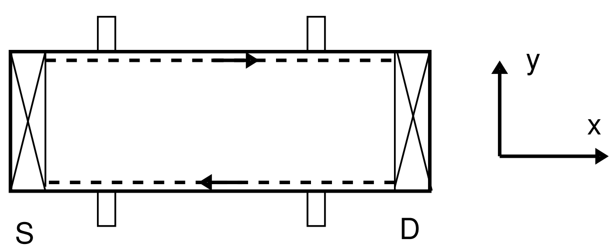
$$\nu = \frac{N_e}{N_\phi} = \frac{nh}{eB},$$

which is a key parameter.

To set a context for discussing the IQHE, we review the Hall effect in a clean system, remembering the drift velocity of charged particles in crossed magnetic and electric fields: $v_{\rm Drift} = E/B$. The current in the set-up illustrated is $I = env_{\rm Drift}w$, while the Hall voltage is $V_{\rm H} = Bv_{\rm Drift}w$. Hence the Hall resistance is

$$R_{\rm H} = \frac{V_{\rm H}}{I} = \frac{B}{ne} = \frac{1}{\nu} \times \frac{h}{e^2}\,.$$

At this point one might think we have a successful explanation of the experiment, but this is wrong: $R_{\rm H} \propto 1/n$ *is* the envelope of the observed behaviour, but we will need Anderson localisation to understand the existence of the IQHE.



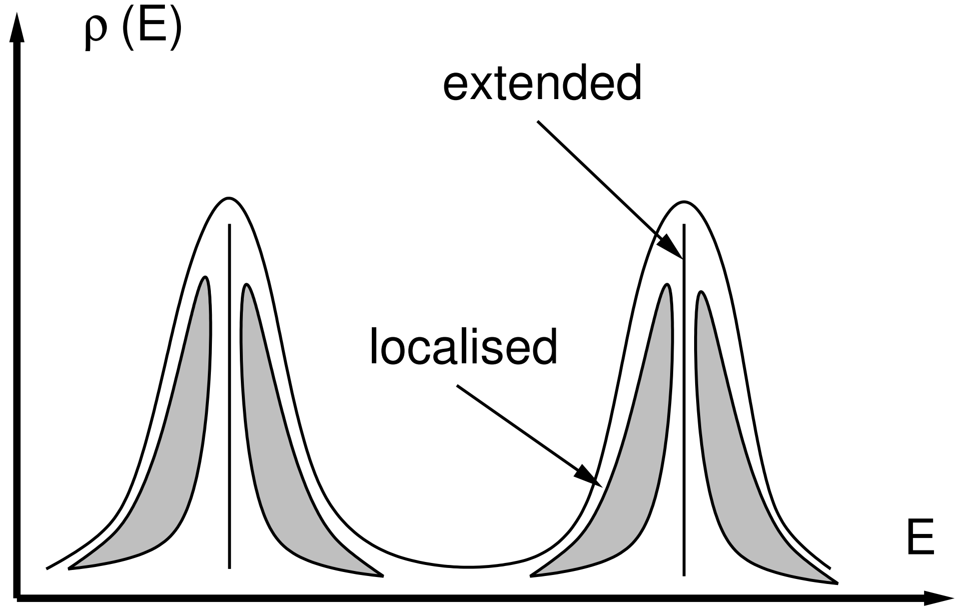
Aside on resistance, resistivity and conductivity: as we shall see when we discuss edge states, transport in QH systems cannot always be discussed in terms of a local resistivity or conductivity tensor. But when it can and the system is in a Hall plateau we have

$$\rho = \frac{1}{\nu} \times \frac{h}{e^2} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma = \nu \times \frac{e^2}{h} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

Thus *both* ρ_{xx} *and* σ_{xx} vanish, raising the question of how one should think about this situation. The key point is that it is dissipationless, since current density is transverse to the electric field.

C. Landau levels in the presence of disorder

To get a basic understanding of the experiment, we start by postulating that the nature of single-particle states for charged particles moving in two dimensions with a magnetic field and disorder that is not too strong is as illustrated. We will come to the justification for this later.



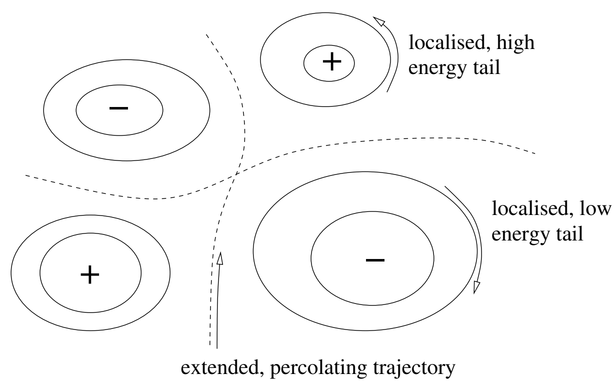
On this picture, disorder both broadens the Landau levels and gives states varying character as a function of energy within each Landau level: states in the Landau level tails are localised in space, while those at the Landau level centre extend through the sample.

How does localisation explain plateaus? If the chemical potential for electrons lies between Landau level centres, then we can change the filling factor by a small amount without changing the occupation of current-carrying states, hence leaving the Hall conductance constant. Also, dissipation requires excitation between occupied and empty current carrying states, so is thermally suppressed at low temperature.

What happens between plateaus? If the chemical potential moves through the energy of extended states, we can understand that σ_{xy} moves between quantised values, since the occupation of extended states has changed. Also, it is expected that $\sigma_{xx} > 0$ within this transition, since when the chemical potential lies close to the energy of extended states, dissipation is no longer suppressed.

D. Localisation in a smooth potential

Disorder enters the Hamiltonian for electrons via a random potential $V(x,y)$ that is characterised by an amplitude and a correlation length. The actual value of the amplitude is unimportant if it is much less than the cyclotron energy, since it then simply sets the energy scale for the problem. But the correlation length λ *is* an important parameter and its value has a big influence on the localisation problem. In particular, the smooth potential limit, $\lambda \gg \ell_{\rm B}$ (which can be approached in modulation doped systems) leads to a simple picture for localisation. In this limit we can think locally about the effect of the potential on eigenstates using a Taylor expansion. The zeroth order term, the local value of the potential, is simply an off-set to the Landau level energies. The next (first-order) term, $\vec{\nabla}V(x,y)$ is equivalent to a local electric field: classically it will cause electrons to drift along equipotentials of $V(x,y)$, and quantum-mechanically we can expect the probability density of an eigenstate to be concentrated in a strip of width $\sim \ell_{\rm B}$ around the equipotential. Hence we get the picture



and the question of quantum localisation translates at this level to a classical continuum percolation problem. Generically we expect there to be a single energy at which contours of $V(x,y)$ percolate, and at energies lower or higher than this contours are localised – around minima or maxima of $V(x,y)$ respectively. If the distribution $P[V(x,y)]$ is symmetric, this percolation energy is zero.

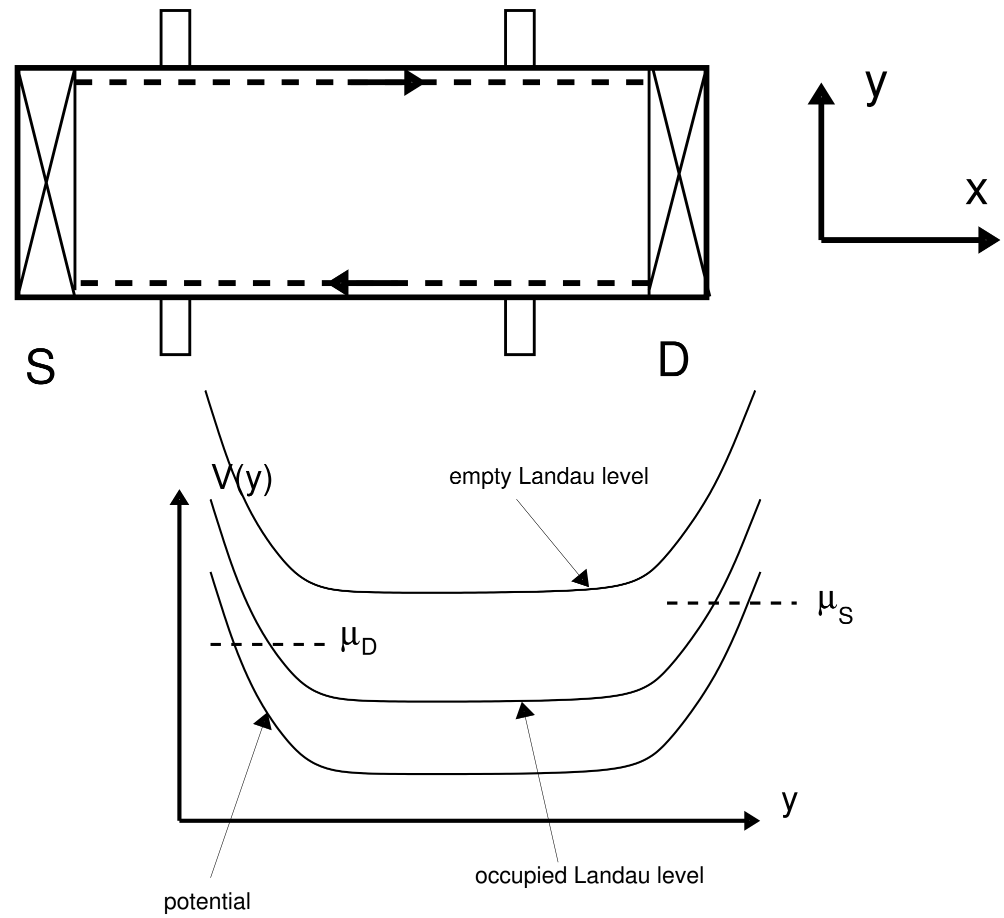
We will return to the possible correction to this description in Lecture III.

II. EXACTNESS OF QUANTISATION

The assumption that states in tails of disorder-broadened Landau levels are localised gives an understanding of existence of plateaus but leaves us with a new puzzle: since disorder (drastically) reduces the number of current-carrying states, why is the value of the Hall conductance unaffected? We will examine this question from three points of view.

A. Edge state transport

Consider states in a Hall bar within a single-particle description.



In this picture the chemical potential in the bulk of the Hall bar lies between Landau levels and the only states at the chemical potential are at the edge of the sample. These states carry a current along the edge of the sample because of electron drift on the potential gradient. If a potential $V_{\rm SD}$ is applied between the source (S) and drain (D) of the Hall bar, there is difference $\Delta\mu = \mu_{\rm S} - \mu_{\rm D} = eV_{\rm SD}$ between the chemical potentials at the top and bottom edges. We want to calculate the source-drain current I in terms of this difference.

Consider the edge states in detail. With quantum number k their wave function has the form $\psi(x, y) = L^{-1/2}e^{ikx}\varphi_k(y)$. Writing their energy as $E(k)$ and their drift velocity as $v_{\rm Drift}$, the current per state is $ev_{\rm Drift}/L$. From the general connection between group velocity and the dispersion relation, we have $v_{\rm Drift} = \hbar^{-1}dE(k)/dk$.

Next we need to calculate how many extra such states are occupied on the top edge compared to the bottom one. Since the extra k-range occupied is $\Delta\mu/(dE/dk)$, this number is $L\Delta\mu/(hv_{\rm Drfit})$.

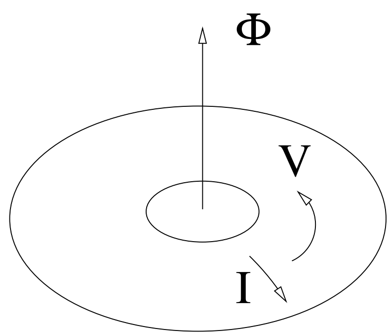
Putting everything together

$$I = \frac{ev_{\rm Drfit}}{L} \times \frac{L\Delta\mu}{hv_{\rm Drfit}} = \frac{e^2}{h}V_{\rm H}$$

for the case when only edge states from a single Landau level are occupied. This is a reassuring result, but open to criticisms, in particular that it is not obvious current should be confined to the edge of the sample if we go to large interacting systems. We therefore discuss a second argument due to Laughlin [Phys. Rev. B **23**, 5632 (1981)].

B. Flux insertion

Consider the thought-experiment illustrated below, involving a quantum Hall sample in the form of an annulus. In addition to the magnetic field responsible for the quantum Hall effect, which pierces the surface of the annulus, we introduce a second magnetic flux Φ, threading through the hole at the centre of the annulus. Allowing this flux to vary as a function of time, we generate a voltage V around the circumference of the quantum Hall sample.



From Faraday's law, we have

$$V = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}\,.$$

Within a Hall plateau, this produces a current flow

$$I = \sigma_{xy}V$$

in the perpendicular direction, which is radial. Integrating the rates of flux change and current flow over time, a given flux difference $\Delta\Phi$ corresponds to the transport of a certain charge Q between the inner and outer edges of the annulus. Now, we expect that a change in Φ of one flux quantum (h/e) will return the interior of the quantum Hall system to its initial state, implying that an integer number of electrons have then been transported across the annulus. We have

$$\text{integer} \times e = Q = \sigma_{xy} \cdot \Delta\Phi = \sigma_{xy}\frac{h}{e}$$

and hence the desired result

$$\sigma_{xy} = \text{integer} \times \frac{e^2}{h}\,.$$

Of course, it is important to examine this argument critically. Two questions are: (i) how it fails for a system that is not in a Hall plateau, and (ii) how slow the 'adiabatic' flux insertion should be.

C. Hall conductance Chern number

Finally we review a third important and appealing approach, which relates the Hall conductance to a topological quantity, the Chern number, that is guaranteed to be an integer. This was first put forward (in the context of Harper's equation) by 'TKNN' [D J Thouless, M Kohmoto, M P Nightingale, and M den Nijs, Phys. Rev. Lett. **49**, 405 (1982)].

Consider a quantum Hall system on an $L_x \times L_y$ torus. As in the flux-insertion argument, we will use a time-dependent flux Φ threading the torus to drive a Hall current. We will also include a second flux Θ threading the torus in the alternative sense. Choosing the gauge $\mathbf{A} = (0, Bx, 0)$ the single-particle Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left\{\left(-i\hbar\partial_x + \frac{e\Theta}{L_x}\right)^2 + \left(-i\hbar\partial_y + eBx + \frac{e\Phi}{L_y}\right)^2\right\} + V(x,y)\,.$$

The eigenstates satisfy the boundary conditions

$$\Psi(x, y + L_y) = \Psi(x,y) \qquad \text{and} \qquad \Psi(x + L_x, y) = e^{iyL_x/\ell_{\mathrm{B}}^2}\Psi(x,y)\,,$$

which are independent of Θ and Φ.

Now, crucially, the x-component of electric current density j_x is related to $\mathcal{H}$ by

$$\frac{\partial\mathcal{H}}{\partial\Theta} = \frac{e}{mL_x}\left(-i\hbar\partial_x + \frac{e\Theta}{L_x}\right) = \frac{j_x}{L_x}$$

and so the current around the torus in the x-direction is

$$\langle I_x \rangle = \langle \Psi \, | \partial_\Theta \mathcal{H} | \, \Psi \rangle \ .$$

We re-write this as

$$\langle I_x \rangle = \partial_\Theta \, \langle \Psi \, | \mathcal{H} | \, \Psi \rangle \ - \{ \langle \partial_\Theta \Psi | \mathcal{H} | \Psi \rangle + \langle \Psi | \mathcal{H} | \partial_\Theta \Psi \rangle \}$$

and use the time-dependent Schrödingier equation to make the substitutions $\mathcal{H}|\Psi\rangle = i\hbar\partial_t|\Psi\rangle$ and $\langle\Psi|\mathcal{H} = -i\hbar\partial_t\langle\Psi|$. Now suppose that the flux Φ varies slowly in time. Let the system start in an instantaneous eigenstate with energy $E(t)$. The adiabatic theorem tells us that

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\int^t E(t')dt'} |\Psi_{\Theta,\Phi}\rangle$$

and so

$$i\hbar\partial_t|\Psi\rangle = E(t)|\Psi\rangle + i\hbar|\partial_\Phi\Psi\rangle \times \partial_t\Phi \, .$$

Putting things together, we have

$$\langle I_x \rangle = \partial_\Theta E - i\hbar[\langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle] \times \partial_t\Phi.$$

The first term on the right is simply the ground state current (it flows even if $\partial_t\Phi = 0$) while the remaining two terms are the response current that we are concerned with. We can hence read off the Hall conductivity as

$$\sigma_{xy} = -i\hbar[\langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle]$$

One key step is to say that we will focus not on σ_{xy} itself (for specific torus fluxes Θ and Φ) but rather on its average $\overline{\sigma}_{xy}$ over all fluxes. A second key step is to recognise that $\langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle$ can be written as $\vec{\nabla}\times\vec{v}$ with the definitions

$$\vec{\nabla}\times\vec{v} = \partial_\Theta v_\Phi - \partial_\Phi v_\Theta \quad \text{and} \quad \vec{v} = (v_\Theta, v_\Phi) = \frac{1}{2}(\langle\Psi|\partial_\Theta\Psi\rangle - \langle\partial_\Theta\Psi|\Psi\rangle, \, \langle\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Phi\Psi|\Psi\rangle) \, .$$

Then we can use Stokes' theorem to re-write the area integral arising from an average over fluxes as a line integral:

$$\overline{\sigma}_{xy} = -\frac{i\hbar}{(h/e)^2}\int_0^{h/e}\int_0^{h/e} d\Theta d\Phi \; \vec{\nabla}\times\vec{v} = \frac{e^2}{h} \times \frac{1}{2\pi i} \times \oint \vec{v}\cdot d\vec{l}.$$

Since the components of $\vec{v}$ give $i\,\times$(the rate of change of the phase of Ψ) as the fluxes are changed, this line integral is an integer multiple of $2\pi i$, getting us for the third time to our desired conclusion.