

Disorder and the Quantum Hall Effect

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Anderson Localisation and IQHE

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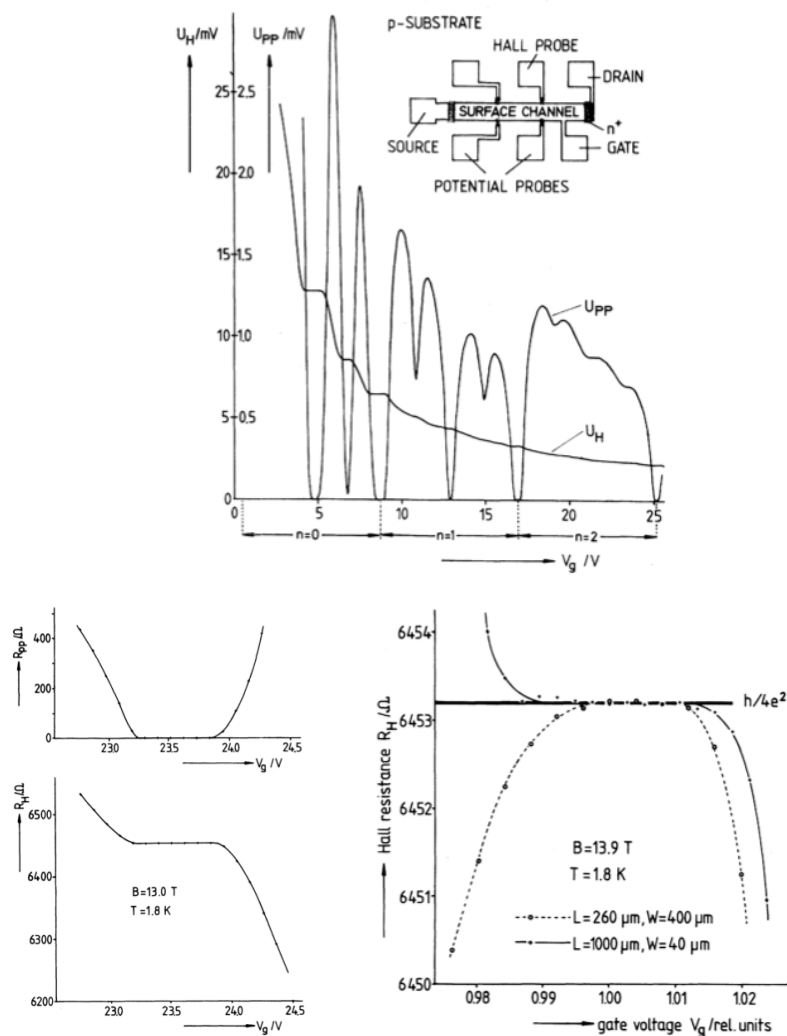
— additional symmetry classes and dirty superconductors

I. OVERVIEW OF IQHE

Why teach this material now? Answers: (i) Ideas that came into condensed matter physics via the IQHE (and FQHE) are now central - see applications in context of topological insulators. (ii) Experiments on the IQHE plateau transition are some of the clearest studies of an Anderson transition.

A. Experiment

Let's start by looking at the data in the original paper on the discovery of the IQHE



K von Klitzing, G. Dorda, and M. Pepper, PRL **45**, 494 (1980)

Note two aspects to phenomenon: (i) minima in R_{xx} and (ii) plateaus in R_{xy} . Questions: Why are there plateaus and minima? Why is R_{xy} so accurately quantised?

B. Basics

Before thinking about these questions in detail, we should recall some facts about the quantum mechanics of charged particles moving in a uniform magnetic field. We define the flux density B , the charge e , the effective mass m , the g -factor g^* and work in SI units. Two scales are important. One is the cyclotron frequency $\omega_c = eB/m$. Note that this is classical and material dependent. The other is the magnetic length $\ell_B = \sqrt{\hbar/(eB)}$, which is quantum and material-independent.

Consider the single-particle spectrum in the absence of disorder. It consists of a series of Landau levels at energies $E = (n + 1/2)\hbar\omega_c \pm \frac{1}{2}g^*\mu_B B$. The degeneracy of a Landau level is *one state per flux quantum*. In a system of area A with n electrons per unit area we have: $N_e = nA$ electrons and $N_\phi = BA/(h/e)$ flux quanta. The ratio of these is the filling factor

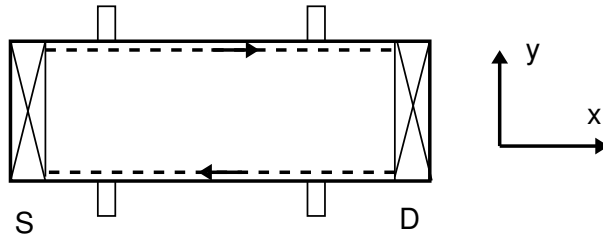
$$\nu = \frac{N_e}{N_\phi} = \frac{nh}{eB},$$

which is a key parameter.

To set a context for discussing the IQHE, we review the Hall effect in a clean system, remembering the drift velocity of charged particles in crossed magnetic and electric fields: $v_{\text{Drift}} = E/B$. The current in the set-up illustrated is $I = env_{\text{Drift}}w$, while the Hall voltage is $V_H = Bv_{\text{Drift}}w$. Hence the Hall resistance is

$$R_H = \frac{V_H}{I} = \frac{B}{ne} = \frac{1}{\nu} \times \frac{h}{e^2}.$$

At this point one might think we have a successful explanation of the experiment, but this is wrong: $R_H \propto 1/n$ is the envelope of the observed behaviour, but we will need Anderson localisation to understand the existence of the IQHE.



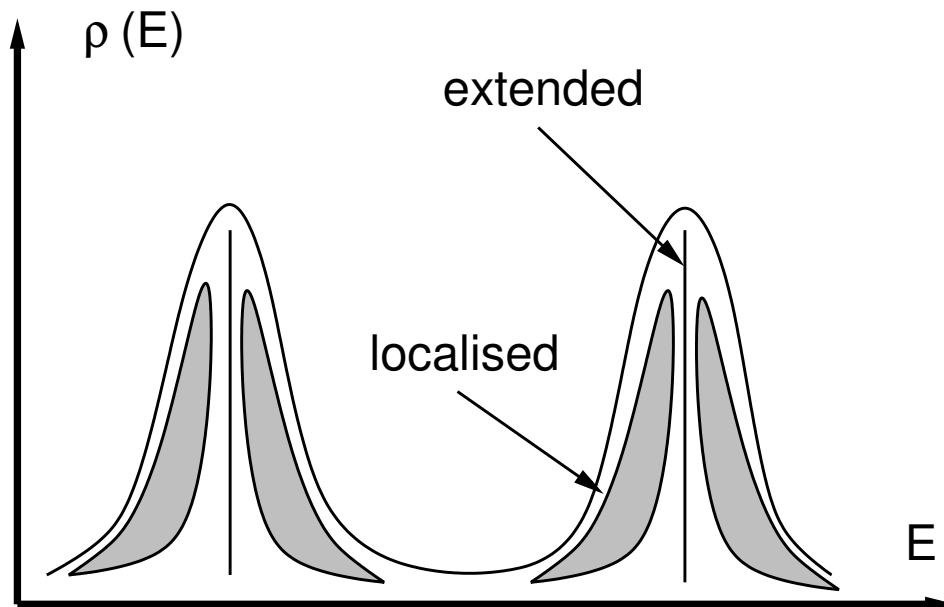
Aside on resistance, resistivity and conductivity: as we shall see when we discuss edge states, transport in QH systems cannot always be discussed in terms of a local resistivity or conductivity tensor. But when it can and the system is in a Hall plateau we have

$$\rho = \frac{1}{\nu} \times \frac{h}{e^2} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = \nu \times \frac{e^2}{h} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus *both* ρ_{xx} and σ_{xx} vanish, raising the question of how one should think about this situation. The key point is that it is dissipationless, since current density is transverse to the electric field.

C. Landau levels in the presence of disorder

To get a basic understanding of the experiment, we start by postulating that the nature of single-particle states for charged particles moving in two dimensions with a magnetic field and disorder that is not too strong is as illustrated. We will come to the justification for this later.



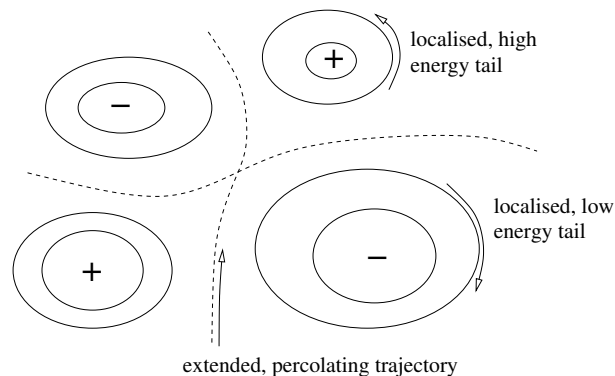
On this picture, disorder both broadens the Landau levels and gives states varying character as a function of energy within each Landau level: states in the Landau level tails are localised in space, while those at the Landau level centre extend through the sample.

How does localisation explain plateaus? If the chemical potential for electrons lies between Landau level centres, then we can change the filling factor by a small amount without changing the occupation of current-carrying states, hence leaving the Hall conductance constant. Also, dissipation requires excitation between occupied and empty current carrying states, so is thermally suppressed at low temperature.

What happens between plateaus? If the chemical potential moves through the energy of extended states, we can understand that σ_{xy} moves between quantised values, since the occupation of extended states has changed. Also, it is expected that $\sigma_{xx} > 0$ within this transition, since when the chemical potential lies close to the energy of extended states, dissipation is no longer suppressed.

D. Localisation in a smooth potential

Disorder enters the Hamiltonian for electrons via a random potential $V(x, y)$ that is characterised by an amplitude and a correlation length. The actual value of the amplitude is unimportant if it is much less than the cyclotron energy, since it then simply sets the energy scale for the problem. But the correlation length λ is an important parameter and its value has a big influence on the localisation problem. In particular, the smooth potential limit, $\lambda \gg \ell_B$ (which can be approached in modulation doped systems) leads to a simple picture for localisation. In this limit we can think locally about the effect of the potential on eigenstates using a Taylor expansion. The zeroth order term, the local value of the potential, is simply an off-set to the Landau level energies. The next (first-order) term, $\vec{\nabla}V(x, y)$ is equivalent to a local electric field: classically it will cause electrons to drift along equipotentials of $V(x, y)$, and quantum-mechanically we can expect the probability density of an eigenstate to be concentrated in a strip of width $\sim \ell_B$ around the equipotential. Hence we get the picture



and the question of quantum localisation translates at this level to a classical continuum percolation problem. Generically we expect there to be a single energy at which contours of $V(x, y)$ percolate, and at energies lower or higher than this contours are localised – around minima or maxima of $V(x, y)$ respectively. If the distribution $P[V(x, y)]$ is symmetric, this percolation energy is zero.

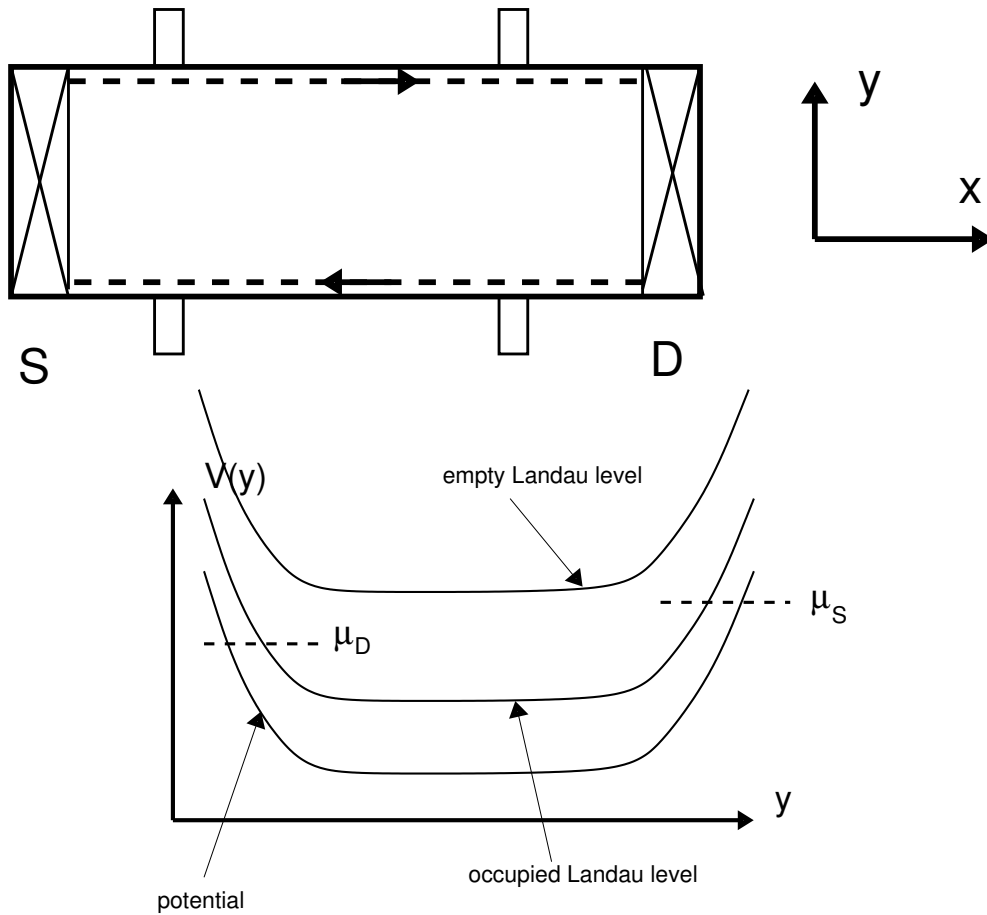
We will return to the possible corrections to this description in Lecture III.

II. EXACTNESS OF QUANTISATION

The assumption that states in tails of disorder-broadened Landau levels are localised gives an understanding of existence of plateaus but leaves us with a new puzzle: since disorder (drastically) reduces the number of current-carrying states, why is the value of the Hall conductance unaffected? We will examine this question from three points of view.

A. Edge state transport

Consider states in a Hall bar within a single-particle description.



In this picture the chemical potential in the bulk of the Hall bar lies between Landau levels and the only states at the chemical potential are at the edge of the sample. These states carry a current along the edge of the sample because of electron drift in the potential gradient. If a potential V_{SD} is applied between the source (S) and drain (D) of the Hall bar, there is difference $\Delta\mu = \mu_S - \mu_D = eV_{SD}$ between the chemical potentials at the top and bottom edges. We want to calculate the source-drain current I in terms of this difference.

Consider the edge states in detail. With quantum number k their wave function has the form $\psi(x, y) = L^{-1/2} e^{ikx} \varphi_k(y)$. Writing their energy as $E(k)$ and their drift velocity as v_{Drift} , the current per state is ev_{Drift}/L . From the general connection between group velocity and the dispersion relation, we have $v_{\text{Drift}} = \hbar^{-1} dE(k)/dk$.

Next we need to calculate how many extra such states are occupied on the top edge compared to the bottom one. Since the extra k -range occupied is $\Delta\mu/(dE/dk)$, this number is $L\Delta\mu/(hv_{\text{Drift}})$.

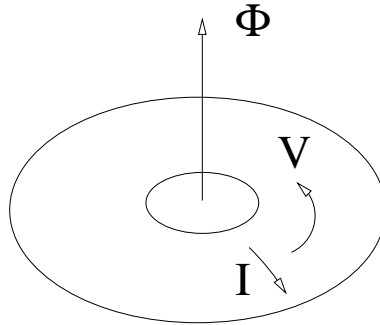
Putting everything together

$$I = \frac{ev_{\text{Drift}}}{L} \times \frac{L\Delta\mu}{hv_{\text{Drift}}} = \frac{e^2}{h} V_{\text{H}}$$

for the case when only edge states from a single Landau level are occupied. This is a reassuring result, but open to criticisms, in particular that it is not obvious current should be confined to the edge of the sample if we go to large interacting systems. We therefore discuss a second argument due to Laughlin [Phys. Rev. B **23**, 5632 (1981)].

B. Flux insertion

Consider the thought-experiment illustrated below, involving a quantum Hall sample in the form of an annulus. In addition to the magnetic field responsible for the quantum Hall effect, which pierces the surface of the annulus, we introduce a second magnetic flux Φ , threading through the hole at the centre of the annulus. Allowing this flux to vary as a function of time, we generate a voltage V around the circumference of the quantum Hall sample.



From Faraday's law, we have

$$V = -\frac{d\Phi}{dt}.$$

Within a Hall plateau, this produces a current flow

$$I = \sigma_{xy} V$$

in the perpendicular direction, which is radial. Integrating the rates of flux change and current flow over time, a given flux difference $\Delta\Phi$ corresponds to the transport of a certain charge Q between the inner and outer edges of the annulus. Now, we expect that a change in Φ of one flux quantum (h/e) will return the interior of the quantum Hall system to its initial state, implying that an integer number of electrons have then been transported across the annulus. We have

$$\text{integer} \times e = Q = \sigma_{xy} \cdot \Delta\Phi = \sigma_{xy} \frac{h}{e}$$

and hence the desired result

$$\sigma_{xy} = \text{integer} \times \frac{e^2}{h}.$$

Of course, it is important to examine this argument critically. Two questions are: (i) how it fails for a system that is not in a Hall plateau, and (ii) how slow the ‘adiabatic’ flux insertion should be.

C. Hall conductance Chern number

Finally we review a third important and appealing approach, which relates the Hall conductance to a topological quantity, the Chern number, that is guaranteed to be an integer. This was first put forward (in the context of Harper’s equation) by ‘TKNN’ [D J Thouless, M Kohmoto, M P Nightingale, and M den Nijs, Phys. Rev. Lett. **49**, 405 (1982)].

Consider a quantum Hall system on an $L_x \times L_y$ torus. As in the flux-insertion argument, we will use a time-dependent flux Φ threading the torus to drive a Hall current. We will also include a second flux Θ threading the torus in the alternative sense. Choosing the gauge $\mathbf{A} = (0, Bx, 0)$ the single-particle Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left\{ \left(-i\hbar\partial_x + \frac{e\Theta}{L_x} \right)^2 + \left(-i\hbar\partial_y + eBx + \frac{e\Phi}{L_y} \right)^2 \right\} + V(x, y).$$

The eigenstates satisfy the boundary conditions

$$\Psi(x, y + L_y) = \Psi(x, y) \quad \text{and} \quad \Psi(x + L_x, y) = e^{iyL_x/\ell_B^2} \Psi(x, y),$$

which are independent of Θ and Φ .

Now, crucially, the x -component of electric current density j_x is related to \mathcal{H} by

$$\frac{\partial \mathcal{H}}{\partial \Theta} = \frac{e}{mL_x} \left(-i\hbar\partial_x + \frac{e\Theta}{L_x} \right) = \frac{j_x}{L_x}$$

and so the current around the torus in the x -direction is

$$\langle I_x \rangle = \langle \Psi | \partial_\Theta \mathcal{H} | \Psi \rangle .$$

We re-write this as

$$\langle I_x \rangle = \partial_\Theta \langle \Psi | \mathcal{H} | \Psi \rangle - \{ \langle \partial_\Theta \Psi | \mathcal{H} | \Psi \rangle + \langle \Psi | \mathcal{H} | \partial_\Theta \Psi \rangle \}$$

and use the time-dependent Schrödinger equation to make the substitutions $\mathcal{H}|\Psi\rangle = i\hbar\partial_t|\Psi\rangle$ and $\langle\Psi|\mathcal{H} = -i\hbar\partial_t\langle\Psi|$. Now suppose that the flux Φ varies slowly in time. Let the system start in an instantaneous eigenstate with energy $E(t)$. The adiabatic theorem tells us that

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar} \int^t E(t') dt'} |\Psi_{\Theta, \Phi}\rangle$$

and so

$$i\hbar\partial_t|\Psi\rangle = E(t)|\Psi\rangle + i\hbar|\partial_\Phi\Psi\rangle \times \partial_t\Phi .$$

Putting things together, we have

$$\langle I_x \rangle = \partial_\Theta E - i\hbar[\langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Phi\Psi|\partial_\Theta\Psi\rangle] \times \partial_t\Phi .$$

The first term on the right is simply the ground state current (it flows even if $\partial_t\Phi = 0$) while the remaining two terms are the response current that we are concerned with. We can hence read off the Hall conductivity as

$$\sigma_{xy} = -i\hbar[\langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Phi\Psi|\partial_\Theta\Psi\rangle]$$

One key step is to say that we will focus not on σ_{xy} itself (for specific torus fluxes Θ and Φ) but rather on its average $\bar{\sigma}_{xy}$ over all fluxes. A second key step is to recognise that $\langle\partial_\Theta\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Phi\Psi|\partial_\Theta\Psi\rangle$ can be written as $\vec{\nabla} \times \vec{v}$ with the definitions

$$\vec{\nabla} \times \vec{v} = \partial_\Theta v_\Phi - \partial_\Phi v_\Theta \quad \text{and} \quad \vec{v} = (v_\Theta, v_\Phi) = \frac{1}{2}(\langle\Psi|\partial_\Theta\Psi\rangle - \langle\partial_\Theta\Psi|\Psi\rangle, \langle\Psi|\partial_\Phi\Psi\rangle - \langle\partial_\Phi\Psi|\Psi\rangle) .$$

Then we can use Stokes' theorem to re-write the area integral arising from an average over fluxes as a line integral:

$$\bar{\sigma}_{xy} = -\frac{i\hbar}{(h/e)^2} \int_0^{h/e} \int_0^{h/e} d\Theta d\Phi \vec{\nabla} \times \vec{v} = \frac{e^2}{h} \times \frac{1}{2\pi i} \times \oint \vec{v} \cdot d\vec{l} .$$

Since the components of \vec{v} give $i \times$ (the rate of change of the phase of Ψ) as the fluxes are changed, this line integral is an integer multiple of $2\pi i$, getting us for the third time to our desired conclusion.

III. PLATEAU TRANSITION AS CRITICAL POINT

The IQHE plateau transitions are examples of quantum critical points. What sort of theoretical description should we look for? Recall Anton Andreev's lectures, in which we saw how the localisation problem of a quantum particle moving in a random potential can be represented by a suitable sigma model. So what is required is 'just' to include a magnetic field in this description. Here we will instead take a phenomenological approach.

A. Scaling flow diagram

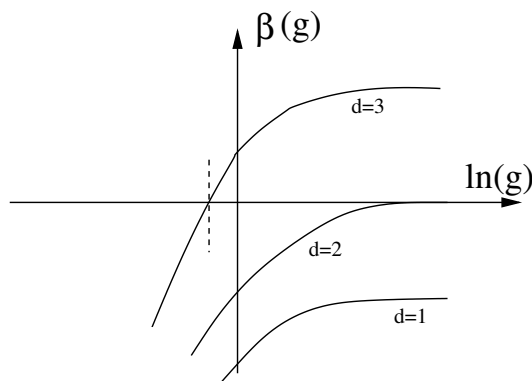
To start, we review the renormalisation group (RG) treatment of the localisation transition in the absence of magnetic field. The coupling constant is the dimensionless conductance $g(L)$ of a system at length scale L , related to the conductivity $\sigma_{xx}(L)$ in d dimensions by

$$g(L) = \frac{h}{e^2} \sigma_{xx}(L) L^{d-2}.$$

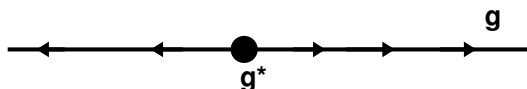
The central quantity in an RG description of the transition is

$$\beta(g) = \frac{\partial \ln g}{\partial \ln L}.$$

In a good metal $\sigma_{xx}(L)$ is independent of L and so $g(L) \propto L^{d-2}$ implying $\beta(g) = d - 2$, while in an Anderson insulator $g(L) \sim e^{-L/\xi}$ and so $\beta(g)$ is negative. These considerations lead to a form for $\beta(g)$

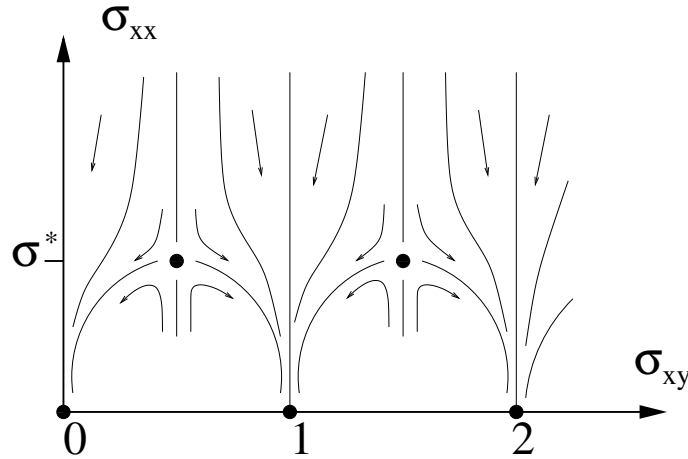


which can also be represented for $d > 2$ as



Since two is the lower critical dimension for the localisation transition in the simplest symmetry class, the critical conductance g^* diverges as $d \rightarrow 2$. For $d = 2$, flow in the absence of magnetic field (or special symmetries) is towards the insulator from all starting points g .

The question we face is how things change in the presence of a magnetic field. The answer is first that σ_{xy} appears as a second coupling constant in the sigma model [see: H. Levine, S. B. Libby, and A. M. M. Pruisken, Phys. Rev. Lett. **51**, 1915 (1983)], and second that the scaling flow (now in the σ_{xx} — σ_{xy} plane) is believed to have the form [D. E. Khmel'nitzkii, JETP Lett. **38**, 552 (1983)]



Note that this scaling flow diagram contains fixed points of two kinds: stable fixed points at $(\sigma_{xx}, \sigma_{xy}) = (0, ne^2/h)$, with $n = 0, 1, 2, \dots$, which represent Hall plateaus, and unstable fixed points at $\sigma_{xx}, \sigma_{xy} = (\sigma^*, [n + 1/2]e^2/h)$, which represent plateau transitions. This form for the scaling flow diagram implies that the Hall conductance should change at transitions only in single steps of e^2/h . It also implies that the width of plateau transitions should shrink to zero as they are probed on longer length scales – for example, by going to lower temperature in experiment.

B. Critical behaviour

Let's examine this critical behaviour. Suppose the behaviour of the localisation length $\xi(E)$ as a function of energy E has the form $\xi(E) \propto |E - E^*|^{-\nu}$ or at fixed Fermi energy as a function of magnetic field $\xi \propto |\Delta B|^{-\nu}$. If we allow for inelastic scattering with a dephasing time $\tau_\varphi \sim T^{-p}$ and hence (assuming diffusive motion) a dephasing length $l_\varphi \sim (D\tau_\varphi)^{1/2} \sim T^{-p/2}$, then the transition width, set by $\xi \sim l_\varphi$, will vary as

$$|\Delta E| \sim |\Delta B| \sim T^{p/2\nu}.$$

This view of the transition is closely tied to a single-particle description, with dephasing as an added extra idea. It is more appropriate to describe the transition using the general scaling framework for a zero-temperature quantum critical point. Then associated with the divergent correlation length ξ there is in general a correlation time ξ_τ related via the dynamical scaling exponent z by $\xi_\tau \sim \xi^z$. For a zero temperature quantum critical point we expect $\xi_\tau = \hbar/k_B T$. Taking $\xi \sim |\Delta B|^{-\nu}$ we have scaling for the elements of the resistivity tensor in the form

$$\rho_{ij}(\Delta B, T) = f_{ij}(|\Delta B|/T^{1/z\nu})$$

and this has been tested in a series of experiments, as we now illustrate.

C. Experiments

Scaling with temperature [H P Wei, D C Tsui, M A Paalanen and A M M Pruisken, PRL **61**, 1294 (1988)]

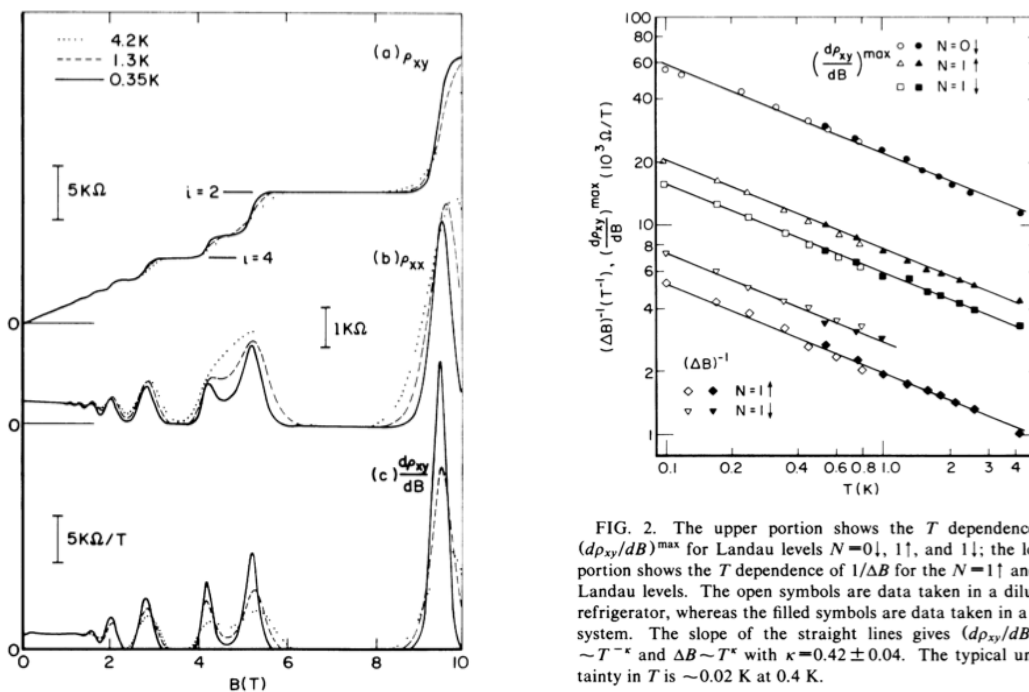
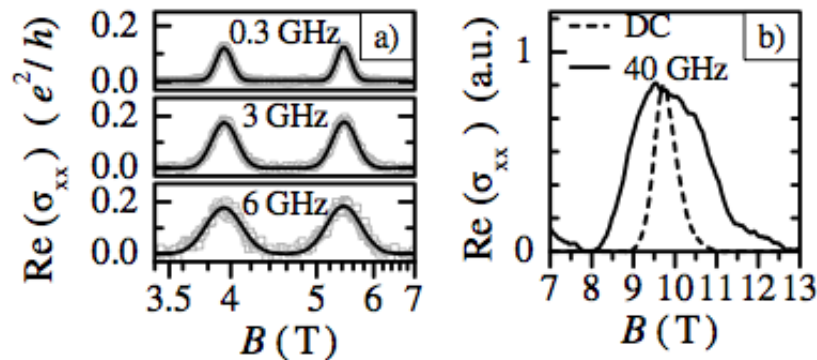


FIG. 2. The upper portion shows the T dependence of $(d\rho_{xy}/dB)^{\max}$ for Landau levels $N=0\downarrow$, $1\uparrow$, and $1\downarrow$; the lower portion shows the T dependence of $1/\Delta B$ for the $N=1\uparrow$ and $1\downarrow$ Landau levels. The open symbols are data taken in a dilution refrigerator, whereas the filled symbols are data taken in a ^3He system. The slope of the straight lines gives $(d\rho_{xy}/dB)^{\max} \sim T^{-\kappa}$ and $\Delta B \sim T^\kappa$ with $\kappa=0.42 \pm 0.04$. The typical uncertainty in T is ~ 0.02 K at 0.4 K.

Scaling with frequency [F Hohls, U Zeitler, R J Haug, R. Meisels, K. Dybko, and F. Kuchar, PRL **89**, 276801 (2002)]



Scaling with sample size [S Koch, R J Haug, K v Klitzing and K. Ploog, PRL **67**, 883 (1991)]

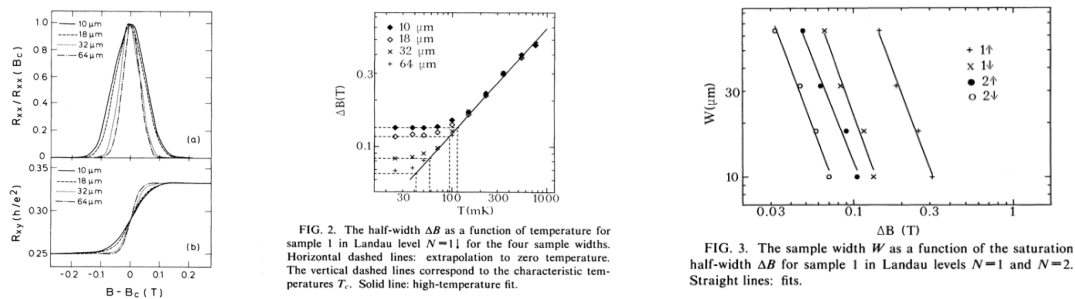
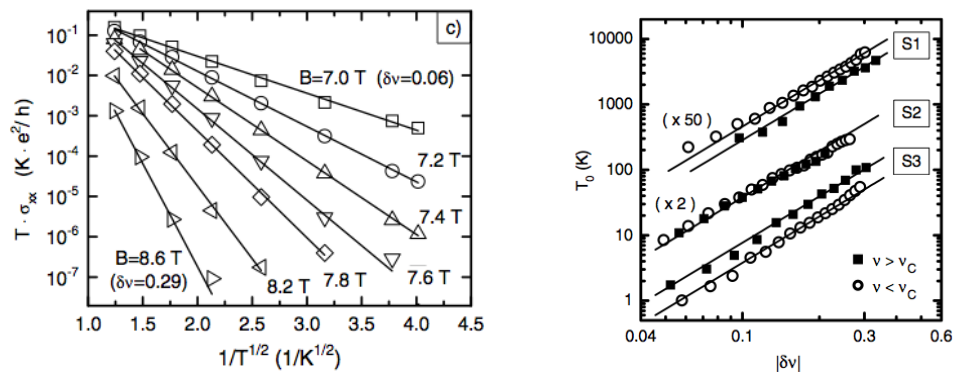


FIG. 2. The half-width ΔB as a function of temperature for sample 1 in Landau level $N=1$ for the four sample widths. Horizontal dashed lines: extrapolation to zero temperature. The vertical dashed lines correspond to the characteristic temperatures T_c . Solid line: high-temperature fit.

FIG. 3. The sample width W as a function of the saturation half-width ΔB for sample 1 in Landau levels $N=1$ and $N=2$. Straight lines: fits.

Direct measurement of localisation length via study of hopping transport [F Hols, U Zeitler and R J Haug, PRL **88**, 036802 (2002)]



Note that in the right hand graph the data is compared to the result from the theory of hopping conduction: $\sigma(T) = \sigma_0 \exp(-\sqrt{T_0/T})$ where T_0 is related to the localisation length via

$$k_B T_0 = C \frac{e^2}{4\pi\epsilon\epsilon_0\xi}.$$

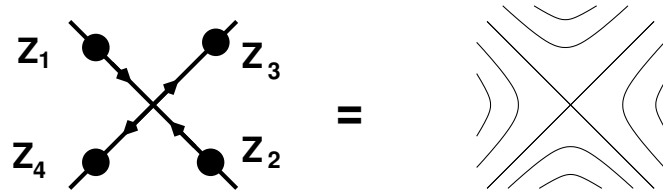
D. Network model

We now turn to a theoretical description of the localisation problem in the IQHE. Ideally one would like to discuss the effects of both disorder and interactions but we will restrict ourselves to the single-particle problem.

One obvious approach would be to consider the full Hamiltonian $\mathcal{H} = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 + V(x, y)$, but instead we will build on the classical percolation picture discussed in Lecture I.

Suppose we carry out a semi-classical quantisation of the guiding centre motion along contours of the potential. This gives us quantum eigenstates associated with closed classical orbits. One of the main corrections to the semiclassical picture comes from mixing of spatially separated semi-classical states associated with different minima of the potential. Such mixing happens because of quantum tunnelling between disjoint classical contours at the same (or nearly the same) energy, which is important near saddle points in the potential.

The network model aims to encode this physics. First consider tunnelling at a single saddle-point and represent the incident and outgoing states by current amplitudes z_i as illustrated.



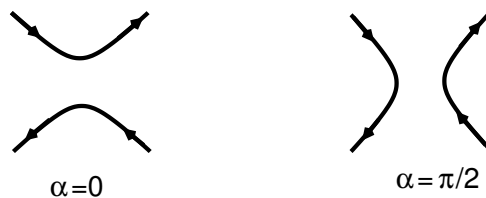
Then scattering theory leads us to write

$$\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = U \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and current conservation requires that the 2×2 scattering matrix U must obey $U^\dagger U = \mathbf{1}$. Note that we can factorize U as

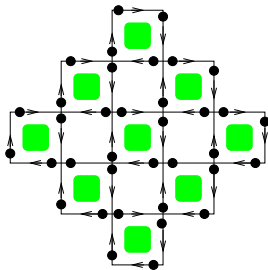
$$U = \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}$$

Here the parameter $\alpha \in [0, \pi/2]$ is the crucial one, with limiting cases



To describe a complete sample we connect up a large number of such scattering nodes. In principle there are three sources of randomness: (i) in the distances along equipotentials between saddle-points, which will lead to randomness in the phase accumulated; (ii) in the values of the tunnelling parameter α at each node; and (iii) in the topology of the resulting network. For simplicity we retain only (i), expecting that this will be enough to access universal behaviour.

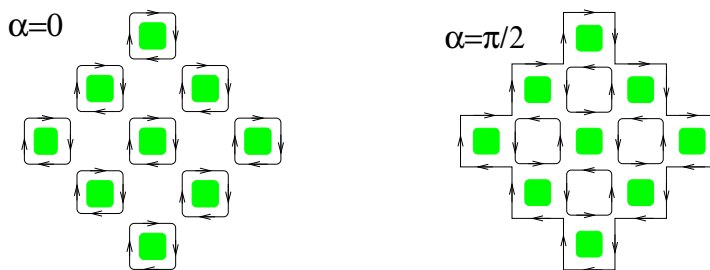
Hence we arrive at a model of the form



in which every node is characterised by the same parameter α and the phases associated with links are independent, uniformly distributed random variables. The model can be represented by a unitary matrix, which is a time evolution operator \mathcal{U} for a time-step in which electrons propagate from the start of one link, along a link and through a node, to the start of the next link. For a system of \mathcal{N} links the matrix \mathcal{U} is $\mathcal{N} \times \mathcal{N}$. It acts on the wave function of the system, represented by the \mathcal{N} -component vector $(z_1, z_2, \dots, z_{\mathcal{N}})^T$. It can be broken down into two factors: $\mathcal{U} = \mathcal{U}_2 \mathcal{U}_1$. The factor \mathcal{U}_1 represents the effect of links, and is a diagonal matrix with random phases on the diagonal. The factor \mathcal{U}_2 represents the nodes and is composed of 2×2 blocks, each of the form

$$\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$

Varying the parameter α between $\alpha = 0$ and $\alpha = \pi/2$ is equivalent to sweeping in energy across a Landau level, as we see by considering the limiting cases



The transition has so far been studied only numerically: the localisation length ξ diverges on approaching $\alpha = \pi/4$ as $\xi \sim |\alpha - \pi/4|^{-\nu}$ with an exponent $\nu \approx 2.5$.

IV. GENERALISATIONS OF IQHE

To discuss generalisations of the IQHE we need first to consider symmetry classes for disordered conductors. In the following we aim to give a simple-minded account of how the classes arise.

A. Symmetry Classes

The notion of symmetry classes for random Hamiltonians first arose in nuclear physics when random matrices were studied as models for the statistical properties of highly excited states in nuclei. At that time three ‘Wigner-Dyson’ symmetry classes were identified, according to whether a system has no time-reversal symmetry, or has time-reversal symmetry without or with Kramers degeneracy. These are known respectively as the unitary, orthogonal and symplectic classes. Recall that Kramers degeneracy arises in systems with half odd-integer spin, and that time-reversal of spin vectors may be written in terms of the conventional Pauli matrices as $\tau_y \vec{\tau}^* \tau_y = -\vec{\tau}$. Then for a Hamiltonian \mathcal{H} the time-reversal operations are

$$\text{orthogonal : } \mathcal{H}^* = \mathcal{H}, \quad \text{symplectic : } \tau_y \mathcal{H}^* \tau_y = \mathcal{H}.$$

It is a feature of the Wigner-Dyson classes that no energy is special, and that statistical properties are unchanged under a shift in energy.

The so-called additional symmetry classes arise in systems that do have a special energy, which we take to be zero. They were first classified by Altland and Zirnbauer [Phys. Rev. B **55**, 1142 (1997)]. The significance of the special energy is that eigenvalues occur in $\pm E$ pairs for each realisation of the disordered Hamiltonian. In consequence, there is an operator X that transforms between states in the pair. Suppose

$$\mathcal{H}\Psi = E\Psi.$$

Then there are two possibilities. Either

$$\mathcal{H}(X\Psi) = -E(X\Psi), \quad \text{implying} \quad X^{-1}\mathcal{H}X = -\mathcal{H},$$

or we need to use complex conjugation to generate the second state in the pair, and

$$\mathcal{H}(X\Psi^*) = -E(X\Psi^*), \quad \text{implying} \quad X^{-1}\mathcal{H}^*X = -\mathcal{H}.$$

A simple example is the one-dimensional chain with disordered nearest neighbour hopping. With site labels n and eigenfunction amplitudes ψ_n , the eigenvalue equation is

$$E\psi_n = t_{n,n+1}\psi_{n+1} + t_{n,n-1}\psi_{n-1}$$

and the transformation X takes the form $(X\psi)_n = (-1)^n\psi_n$.

1. Chiral classes

We would now like to enumerate the additional symmetry classes. Instead of following the original route via Cartan's classification of symmetric spaces, I present some informal arguments that I first heard from T. Senthil. In general, because eigenvalues appear in pairs, we expect the Hamiltonians we are concerned with to have a 2×2 structure. Let σ_α be the Pauli matrices acting in this space (the σ 's are distinct from the τ 's). We can write an arbitrary Hamiltonian in the form

$$\mathcal{H} = h_0\mathbb{1} + \vec{h} \cdot \vec{\sigma},$$

where h_0, h_x, h_y and h_z are Hermitian matrices.

We want to understand what the restrictions are on these matrices. We will deal separately with the two possibilities for transforming between states in the pair. Consider first $X^{-1}\mathcal{H}X = -\mathcal{H}$. This implies that X is an element of $SU(2)$ so that we can parameterise it as $X = e^{i\alpha\hat{n}\cdot\vec{\sigma}}$. Moreover, since $X^2 \propto \mathbb{1}$ we can set $\alpha = \pi/2$. The direction of \hat{n} is now a matter of convention. We pick $\hat{n} = \hat{z}$ giving $X = i\sigma_z$. The symmetry relation then reads

$$\sigma_z\mathcal{H}\sigma_z = -\mathcal{H},$$

implying that $h_0 = h_z = 0$. In this way we arrive at the chiral ensembles, with

$$\mathcal{H} = \begin{pmatrix} 0 & h_x - ih_y \\ h_x + ih_y & 0 \end{pmatrix}.$$

The same questions about time-reversal symmetry arise for the chiral classes as in the Wigner-Dyson cases, and so we have chiral orthogonal, unitary and symplectic classes.

2. BdG classes

Consider next the alternative transformation $X^{-1}\mathcal{H}^*X = -\mathcal{H}$. As before we use the parameterisation $X = e^{i\alpha\hat{n}\cdot\vec{\sigma}}$. We also have $XX^* \propto \mathbb{1}$, and we treat separately the two cases $XX^* = +\mathbb{1}$ and $XX^* = -\mathbb{1}$.

In the first case we have $X^* = X^{-1}$, which implies that $n_y = 0$. We pick $\hat{n} = \hat{x}$ and $\alpha = \pi/2$, getting the condition

$$\sigma_x \mathcal{H}^* \sigma_x = -\mathcal{H}, \quad (1)$$

which has the solution $h_x^* = -h_x$, $h_y^* = -h_y$, $h_z^* = h_z$ and $h_0^* = -h_0$, so that

$$\mathcal{H} = \begin{pmatrix} h_0 + h_z & h_x - ih_y \\ h_x + ih_y & h_0 - h_z \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & -a^T \end{pmatrix}. \quad (2)$$

Next we treat the second of our alternatives, $XX^* = -\mathbb{1}$, which implies $X = i\sigma_y$ and hence

$$\sigma_y \mathcal{H}^* \sigma_y = -\mathcal{H},$$

with the solution $h_0^* = -h_0$ and $\vec{h}^* = \vec{h}$. Then

$$\mathcal{H} = \begin{pmatrix} h_0 + h_z & h_x - ih_y \\ h_x + ih_y & h_0 - h_z \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & -a^T \end{pmatrix}. \quad (3)$$

We can find realisations of the Hamiltonians of Eqns (2) and (3) as Bogoliubov de-Gennes (BdG) Hamiltonians for superconductors, for spinless and spin-singlet systems respectively. To see this, let's review the form taken by such superconductor Hamiltonians.

In the spinless case, in terms of fermion creation and annihilations operators c_α and c_α^\dagger for orbitals $\alpha, \beta \dots$, we have

$$\mathcal{H} = \sum_{\alpha, \beta} \left[h_{\alpha, \beta} c_\alpha^\dagger c_\beta + \frac{1}{2} \left(\Delta_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger + \Delta_{\alpha\beta}^* c_\beta c_\alpha \right) \right].$$

Note that Hermiticity and fermion anticommutation relations imply $h^\dagger = h$ and $\Delta^T = -\Delta$. We can re-write this Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2} (\mathbf{c}^\dagger, \mathbf{c}) \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{c}^\dagger \end{pmatrix} + \text{const}, \quad (4)$$

which matches that of Eq. (2).

In the singlet case, a generic form for the BdG Hamiltonian is

$$\mathcal{H} = \sum_{\alpha\beta} \left[h_{\alpha\beta} \left(c_{\alpha\uparrow}^\dagger c_{\beta\uparrow} + c_{\alpha\downarrow}^\dagger c_{\beta\downarrow} \right) + \left(\Delta_{\alpha\beta} c_{\alpha\uparrow}^\dagger c_{\beta\downarrow}^\dagger + \Delta_{\alpha\beta}^* c_{\beta\downarrow} c_{\alpha\uparrow} \right) \right].$$

Spin rotation symmetry requires $\Delta^T = \Delta$. We can make conservation of spin explicit by doing a particle-hole transformation on one spin direction, taking $\gamma_{\alpha\uparrow}^\dagger = c_{\alpha\uparrow}^\dagger$ and $\gamma_{\alpha\downarrow}^\dagger = c_{\alpha\downarrow}$. Then the

singlet BdG Hamiltonian takes the form

$$\begin{aligned} \mathcal{H} &= \sum_{\alpha\beta} \left[h_{\alpha\beta} \left(\gamma_{\alpha\uparrow}^\dagger \gamma_{\beta\uparrow} - \gamma_{\beta\downarrow}^\dagger \gamma_{\alpha\downarrow} \right) + \left(\Delta_{\alpha\beta} \gamma_{\alpha\uparrow}^\dagger \gamma_{\beta\downarrow} + \Delta_{\alpha\beta}^* \gamma_{\beta\downarrow}^\dagger \gamma_{\alpha\uparrow} \right) \right] . \\ &= \left(\gamma_{\uparrow}^\dagger, \gamma_{\downarrow}^\dagger \right) \begin{pmatrix} h & \Delta \\ \Delta^* & -h^T \end{pmatrix} \begin{pmatrix} \gamma_{\uparrow} \\ \gamma_{\downarrow} \end{pmatrix} + \text{const}, \end{aligned}$$

which matches that of Eq. (3). Each of these BdG Hamiltonians (spinless or spin singlet) gives rise to two symmetry classes – with or without time-reversal symmetry.

3. Summary

In summary, we have a grand total of 10 classes: the 3 Wigner-Dyson ones, the 3 chiral ones, 2 for spinless superconductors, and 2 for spin singlet superconductors. The 10 classes are listed using standard nomenclature in the table, and the three cases showing versions of the quantum Hall effect are picked out. One of the criteria for showing a QHE is of course that the symmetry class should not have time reversal symmetry. This selects one class from each of the three groupings termed Wigner-Dyson, spineless BdG and singlet BdG. A further criterion is that RG flow at zero generalised Hall conductance should be to an Anderson insulator, which is true for these three examples but not in the chiral unitary class. Hence we end up with two generalisations of the IQHE — in classes C and D.

Wigner-Dyson	Orthogonal	AI	IQHE
	Unitary	A	
	Symplectic	AII	
Chiral	Orthogonal	BDI	
	Unitary	AIII	
	Symplectic	CII	
Spinless BdG	with TRS	DIII	Majorana QHE
	no TRS	D	
Singlet BdG	with TRS	CI	SQHE
	no TRS	C	

Some remarks about the BdG versions of the QHE are in order. First we should ask what transport effect will show a Hall plateau. We are not concerned with charge transport, since the BdG Hamiltonians do not conserve quasiparticle number, and anyway quasiparticle transport

will be short-circuited by the superconducting condensate. Instead, we consider the quantities that are conserved, and so expect plateaus in the thermal Hall conductivity for both class C and class D, and additionally in the spin conductivity for class C.

To make contact with other problems of current interest, it is useful to re-visit our discussion of spinless superconductors, picking up the discussion from Eq. (1), which reads $\sigma_x \mathcal{H}^* \sigma_x = -\mathcal{H}$. Making the transformation $H = s^\dagger \mathcal{H} s$, where $s^2 = \sigma_x$, we have an alternative representation satisfying $H^* = -H$, or (since $H^\dagger = H$), $H^T = -H$. This is exactly the condition arising for Majorana Hamiltonians of the form $\frac{i}{2} \sum_{\alpha\beta} H_{\alpha\beta} b_\alpha b_\beta$ with Majorana operators b_α, b_β satisfying the standard relations: $b_\alpha^\dagger = b_\alpha$, $\{b_\alpha, b_\beta\} = 0$ for $\alpha \neq \beta$ and $(b_\alpha)^2 = 1$.

B. Network models

Two main ingredients in the definition of the network model are the amplitudes z_l on links and the phases $e^{i\phi_l}$ acquired in propagation along a link. An obvious extension is to allow the amplitudes to be N -component vectors. Then the link phases are replaced by $N \times N$ matrices W .

Since the link phases arise during propagation, we can think of them as given in terms of a Hamiltonian \mathcal{H} via $W = e^{i\mathcal{H}}$. Then the Wigner-Dyson unitary symmetry class for $N \times N$ matrices \mathcal{H} generates $U(N)$ link phases. This generalisation does not change the symmetry. It therefore gives N copies of the IQHE transition as the node parameter varies between $\alpha = 0$ and $\alpha = \pi/2$, with each transition occurring at a distinct value of α .

Alternatively, we can consider the BdG symmetry classes. For class C the condition $\sigma_y \mathcal{H}^* \sigma_y = -\mathcal{H}$ implies that W belongs to $Sp(2n)$, and in the simplest case $Sp(2) \sim SU(2)$. Thus we can model a plateau transition in symmetry class C using a network model with two-component amplitudes on the links and $SU(2)$ link phases. This does indeed give a transition in a different universality class from the standard IQHE.

For class D, using the representation $H^* = -H$ we have link phases $W = e^{iH}$ that are $O(N)$ matrices. In the simplest case, $N = 1$, they are just real phases $W = \pm 1$. Again, this gives behaviour distinct from the standard IQHE (and from class C), with both a plateau transition and a metallic phase, depending on details of the model.