- Lecture I
  - motivate/discuss celestial objects where rotation influences buoyancy driven flows
  - discuss energetics, waves, balance

- Lecture II
  - stability theory for rotating convection - what can we glean from it
  - motivate non-hydrostatic quasi-geostrophy
  - derive and investigate semi-analytic solutions (skeleton for strongly NL flows)

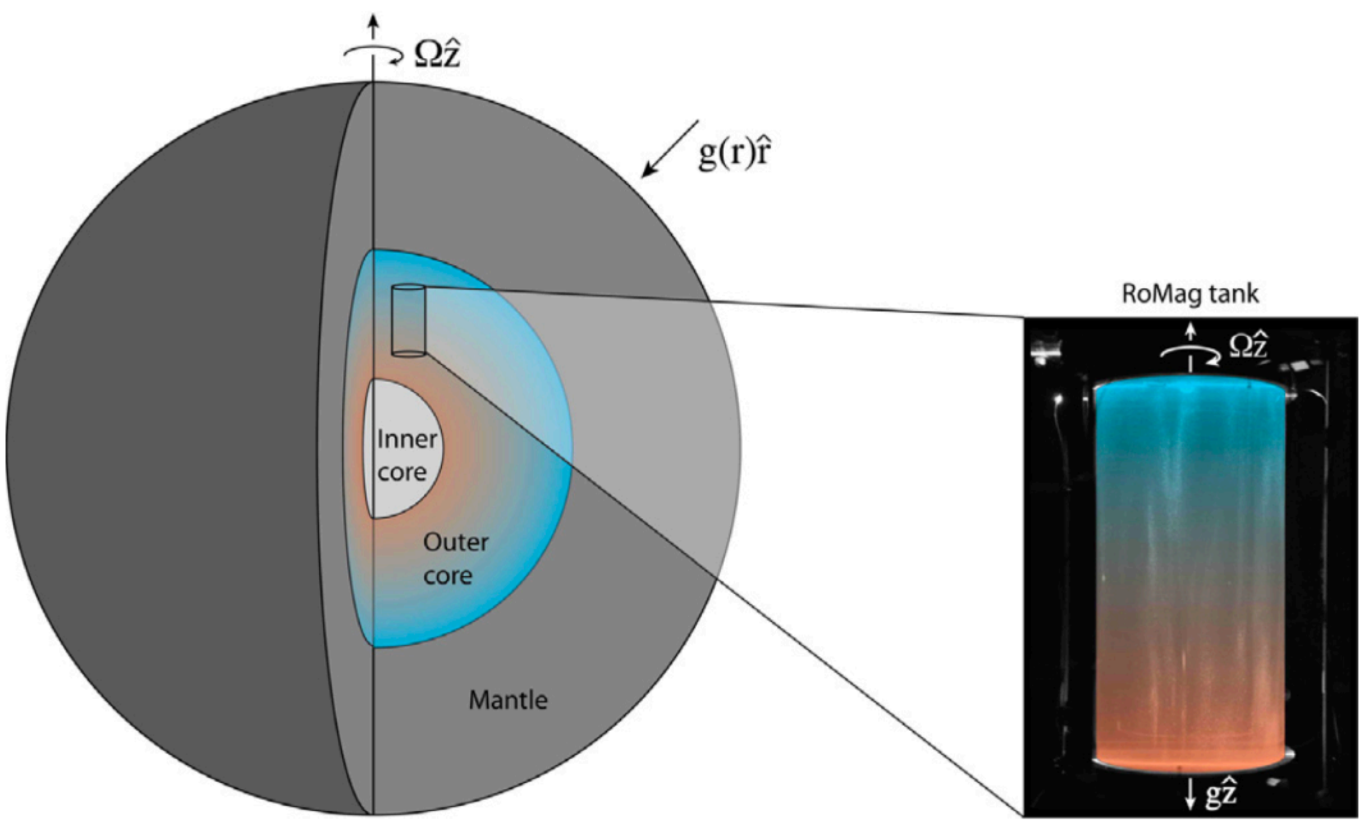
Lecture III

- investigate fully NL rotating convection from QG perspective
- assess how the theory holds up c.f. experiments and DNS
- comments on broader view and future outlook

# Local Area Investigations

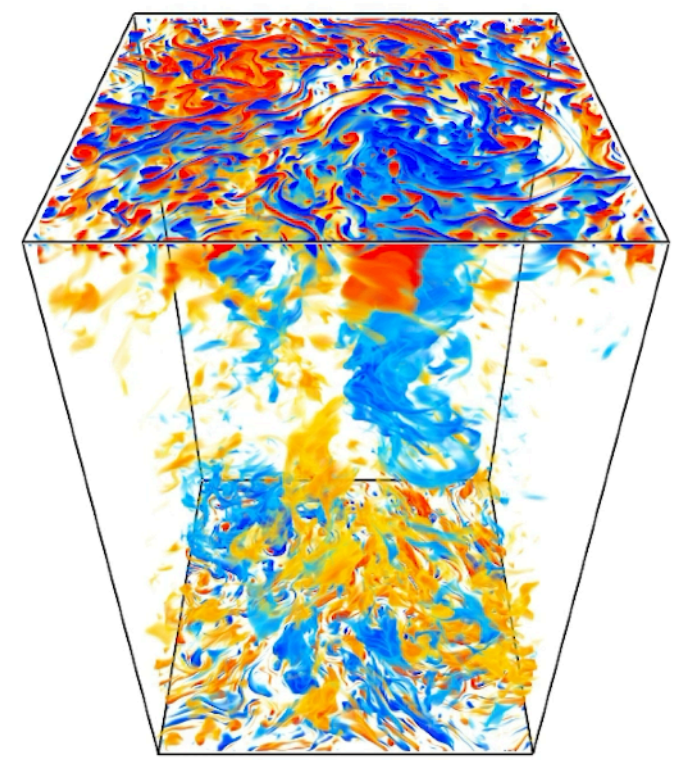
Rotating Rayleigh-Bénard convection (RRBC) - canonical paradigm for investigating influence of rotation on buoyantly-driven convection

Example: Convection in outer Core




Upright RRBC

Lab - bounded domains



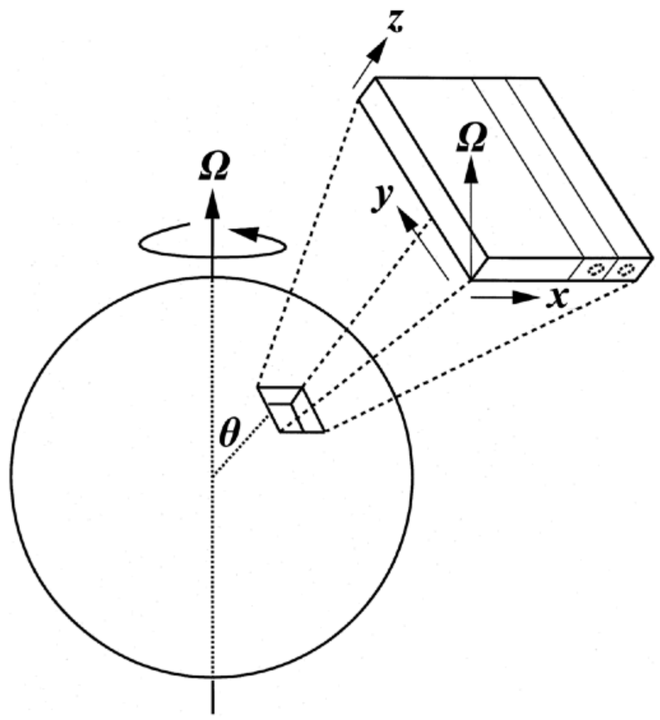
Upright RRBC

Theory - plane parallel layers

# Local Area Investigations

Rotating Rayleigh-Bénard convection (RRBC) - canonical paradigm for investigating influence of rotation on buoyantly-driven convection

Tilted RRBC - the *f*-plane

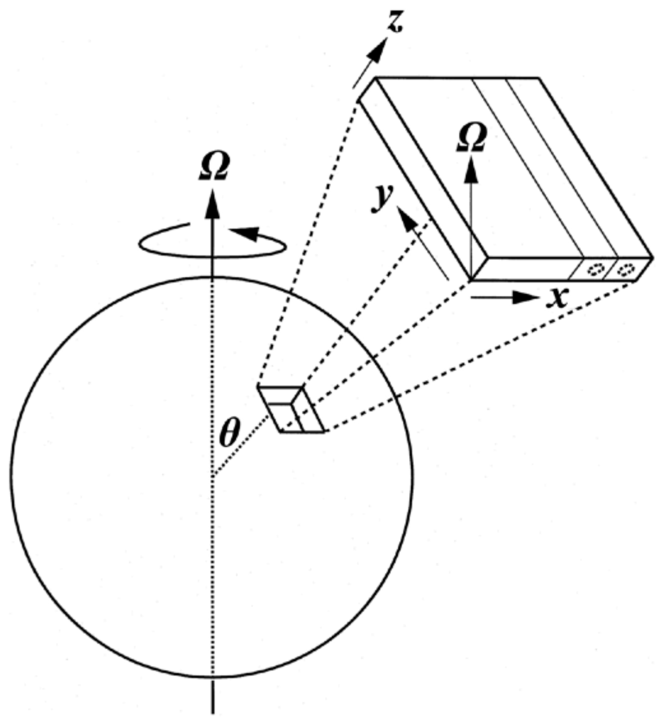



Theory - plane parallel layer

# Local Area Investigations - Governing Equations

Rotating Rayleigh-Bénard convection (RRBC) - canonical paradigm for investigating influence of rotation on buoyantly-driven convection

Tilted RRBC - *f*-plane




Body force - buoyancy under Boussinesq approx'n

$$\rho = \rho_0(1 - \alpha\Delta_T T), \quad \Delta_T T = T^* - T_0^*$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{Ro}\hat{\boldsymbol{e}}_{\boldsymbol{\Omega}} \times \boldsymbol{u} = -Eu\nabla p + \Gamma T\hat{\boldsymbol{z}} + \frac{1}{Re}\nabla^2\boldsymbol{u}$$

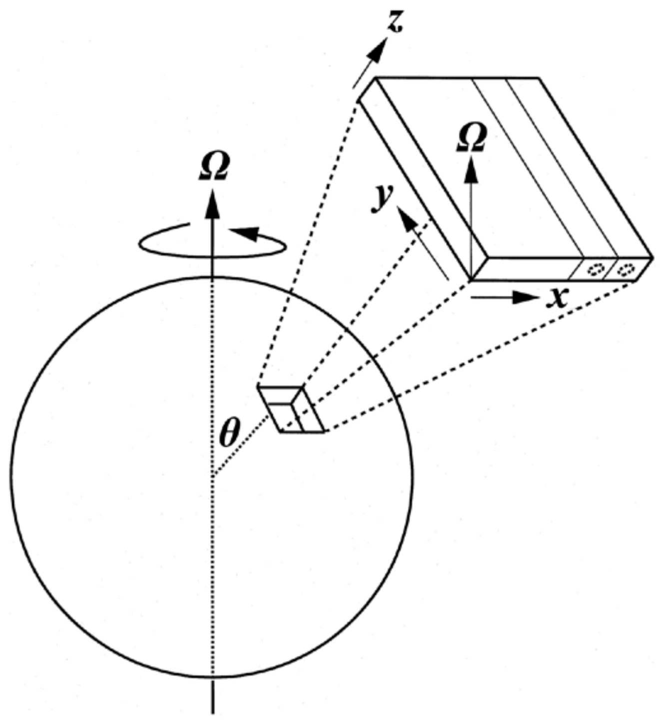
$$\nabla\cdot\boldsymbol{u} = 0$$

$$\partial_t T + \boldsymbol{u}\cdot\nabla T = \frac{1}{Pe}\nabla^2 T$$

# Local Area Investigations - Governing Equations

Rotating Rayleigh-Bénard convection (RRBC) - canonical paradigm for investigating influence of rotation on buoyantly-driven convection

Tilted RRBC - *f*-plane




Body force - buoyancy under Boussinesq approx'n

$$\rho = \rho_0(1 - \alpha\Delta_T T), \quad \Delta_T T = T^* - T_0^*$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{Ro}\hat{\boldsymbol{e}}_{\boldsymbol{\Omega}} \times \boldsymbol{u} = -Eu\nabla p + \Gamma T\hat{\boldsymbol{z}} + \frac{1}{Re}\nabla^2\boldsymbol{u}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

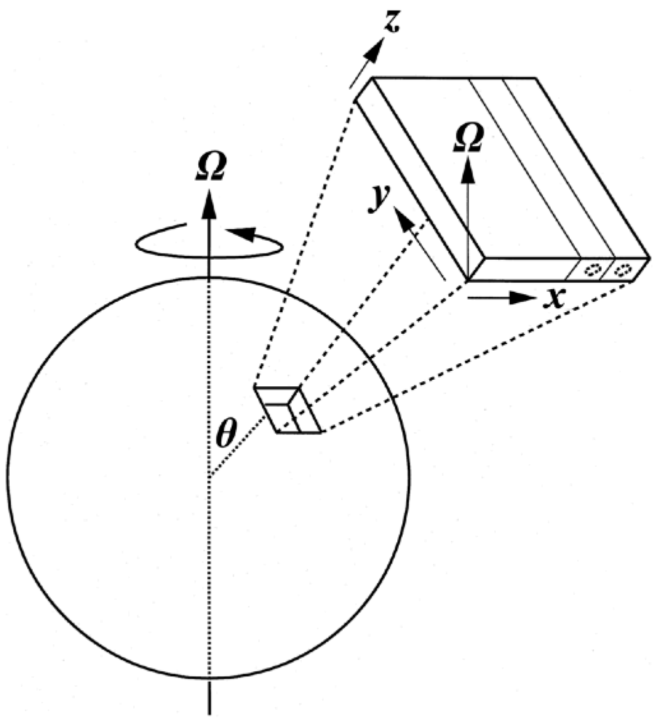
$$\partial_t T + \boldsymbol{u}\cdot\nabla T = \frac{1}{Pe}\nabla^2 T$$

$$\Gamma = \frac{g\alpha\Delta_T L}{U^2}, \quad Pe = \frac{UL}{\kappa} = PrRe$$

# Local Area Investigations - Governing Equations

Rotating Rayleigh-Bénard convection (RRBC) - canonical paradigm for investigating influence of rotation on buoyantly-driven convection

Tilted RRBC - *f*-plane




Body force - buoyancy under Boussinesq approx'n

$$\rho = \rho_0(1 - \alpha \Delta_T T), \quad \Delta_T T = T^* - T_0^*$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \hat{\boldsymbol{e}}_{\boldsymbol{\Omega}} \times \boldsymbol{u} = -Eu \nabla p + \Gamma T \hat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t T + \boldsymbol{u} \cdot \nabla T = \frac{1}{Pe} \nabla^2 T$$

$$\Gamma = \frac{g \alpha \Delta_T L}{U^2}, \quad Pe = \frac{UL}{\kappa} = Pr Re$$

Generic parameters - containing no characteristics of RBC $\Rightarrow$ $L = H, \quad U = \frac{\nu}{H}$

diffusive velocity scaling

# Local Area Investigations - Governing Equations

Rotating Rayleigh-Bénard convection (RRBC) - canonical paradigm for investigating influence of rotation on buoyantly-driven convection

$$L = H, \quad U = \frac{\nu}{H}$$

$$\Downarrow$$

$$Re = 1$$

$$Pe = Pr$$

$$\Gamma = \frac{Ra}{Pr}$$

$$Ro = E$$

$$\Rightarrow$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{E} \hat{\boldsymbol{e}}_{\boldsymbol{\Omega}} \times \boldsymbol{u} = -\nabla p + \frac{Ra}{Pr} T \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t T + \boldsymbol{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T$$

External parameters

$$Ra = \frac{g \alpha \Delta_T H^3}{\nu \kappa} = \frac{\tau_\nu \tau_\kappa}{\tau_B^2}$$

$$Pr = \frac{\nu}{\kappa} = \frac{\tau_\kappa}{\tau_\nu}$$

$$E = \frac{\nu}{2 \Omega H^2} = \frac{\tau_\Omega}{\tau_\nu}$$

Generic parameters - containing no characteristics of RBC $\Rightarrow$ $L = H, \quad U = \frac{\nu}{H}$

diffusive velocity scaling

Several objectives contribute to our understanding of GAFD

- Linear stability theory
- Patterns: formation and selection
- Turbulent scalings:

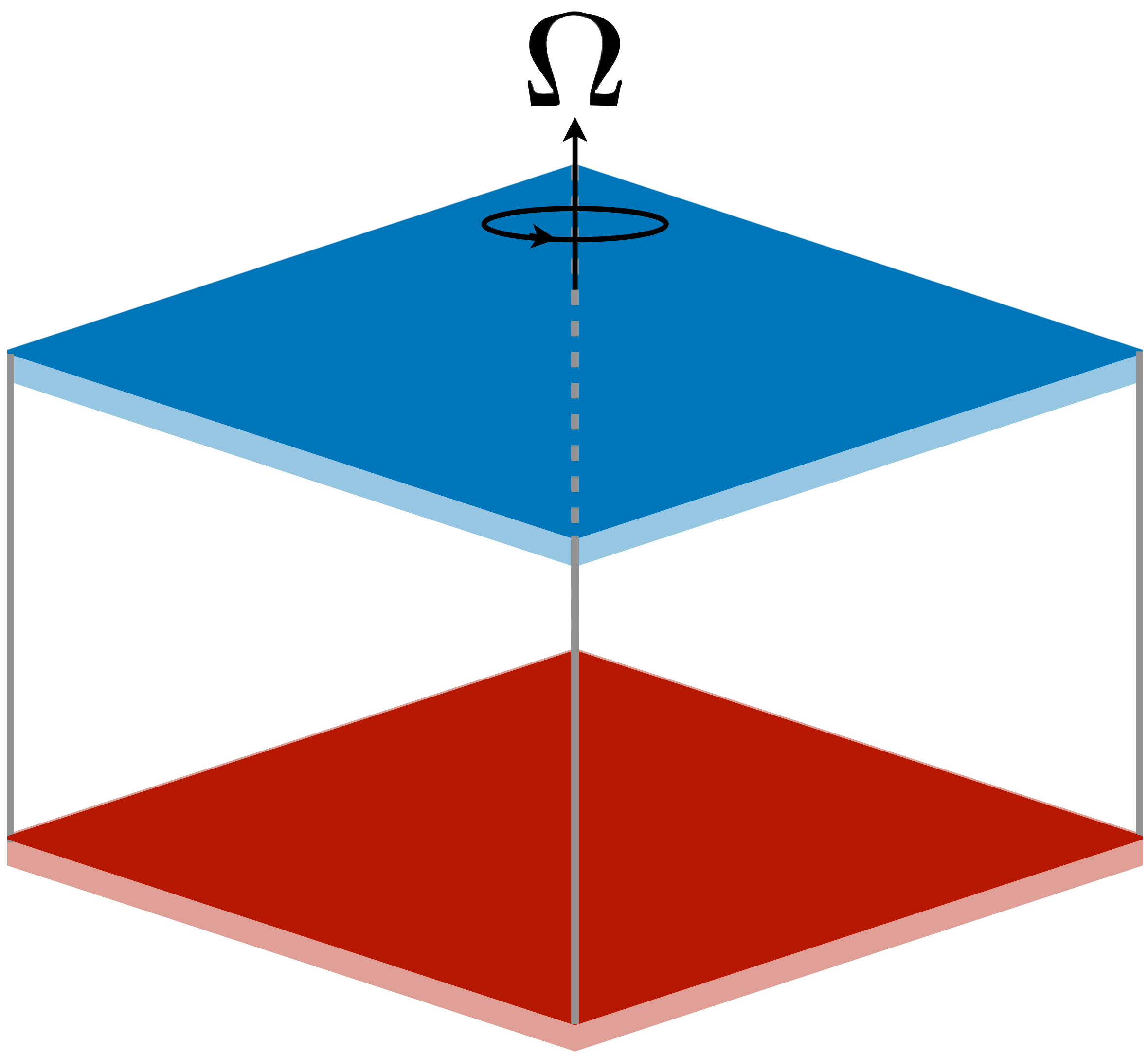
Efficiency of heat and momentum transport

$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta_T} \qquad Re \equiv \frac{UH}{\nu}$$

$$= \mathcal{F}(Ra, E, Pr) = Ra^{\alpha} Ek^{\beta} Pr^{\gamma}?$$

# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1$$




$$\boldsymbol{u}_b = 0$$

$$T_b(z) = 1 - z$$

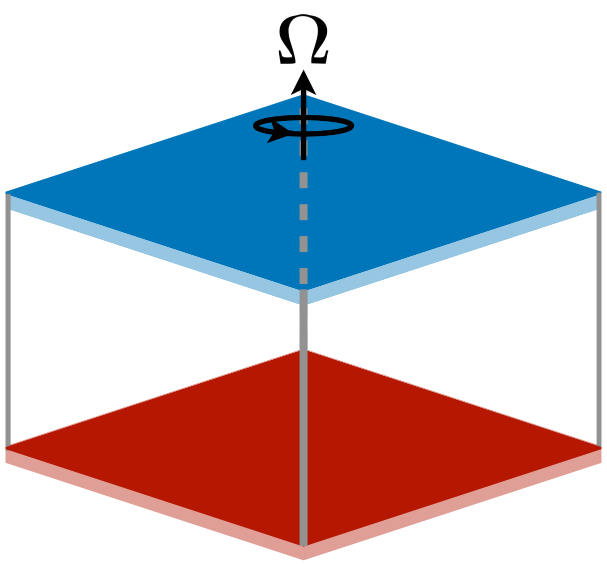
# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\partial_t \boldsymbol{u}' = -\frac{1}{E}\hat{\boldsymbol{z}} \times \boldsymbol{u}' - \nabla p' + \frac{Ra}{Pr} T' \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}'$$

$$0 = \nabla \cdot \boldsymbol{u}'$$

$$\partial_t T' = -w' + \frac{1}{Pr}\nabla^2 T'$$




$$\boldsymbol{u}_b = 0$$
$$T_b(z) = 1 - z$$

# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\partial_t \boldsymbol{u}' = -\frac{1}{E}\hat{\boldsymbol{z}} \times \boldsymbol{u}' - \nabla p' + \frac{Ra}{Pr} T' \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}'$$

$$0 = \nabla \cdot \boldsymbol{u}'$$

$$\partial_t T' = -w' + \frac{1}{Pr}\nabla^2 T'$$

**+**

boundary conditions @ *z=0,1*

Impenetrable: $w' = \boldsymbol{u'} \cdot \hat{\boldsymbol{z}} = 0$

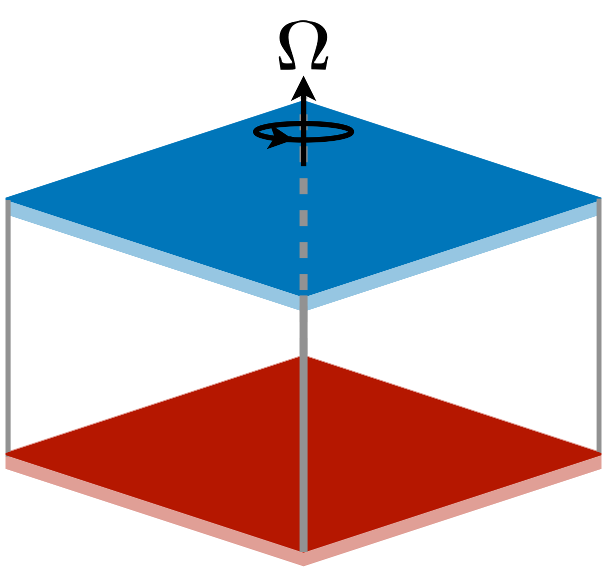
no-slip: $\boldsymbol{u'_\perp} = \boldsymbol{0}$ stress-free: $\partial_z \boldsymbol{u'_\perp} = \boldsymbol{0}$

fixed temp: $T' = 0$ fixed-flux: $\partial_z T' = 0$

$$\Uparrow$$

$$\boldsymbol{v}' = \boldsymbol{a}_{\boldsymbol{k}_\perp}(z) e^{(i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + st)} + c.c.$$

Normal mode ansatz




$$\boldsymbol{u}_b = 0$$
$$T_b(z) = 1 - z$$

# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\partial_t \boldsymbol{u}' = -\frac{1}{E}\hat{\boldsymbol{z}} \times \boldsymbol{u}' - \nabla p' + \frac{Ra}{Pr} T' \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}'$$

$$0 = \nabla \cdot \boldsymbol{u}'$$

$$\partial_t T' = -w' + \frac{1}{Pr}\nabla^2 T'$$

**+**

boundary conditions @ *z=0,1*

Impenetrable: $w' = \boldsymbol{u'} \cdot \hat{\boldsymbol{z}} = 0$

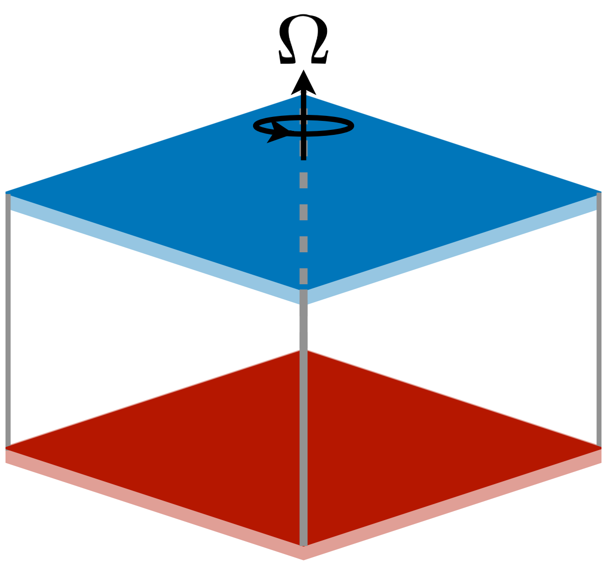
no-slip: $\boldsymbol{u'_\perp} = \boldsymbol{0}$ stress-free: $\partial_z \boldsymbol{u'_\perp} = \boldsymbol{0}$

fixed temp: $T' = 0$ fixed-flux: $\partial_z T' = 0$

$$\Uparrow$$

$$\boldsymbol{v}' = \boldsymbol{a}_{\boldsymbol{k}_\perp}(z) e^{(i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + st)} + c.c.$$

Normal mode ansatz




$\boldsymbol{u}_b = 0$

$T_b(z) = 1 - z$

| | |
|---|---|
| $s = \mu + i\omega, \quad \mu(Ra_c) = 0$ | onset of convection |
| Steady bif'n | exp. growth monotonic |
| Hopf bif'n | exp. growth oscillatory |

# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\partial_t \boldsymbol{u}' = -\frac{1}{E}\hat{\boldsymbol{z}} \times \boldsymbol{u}' - \nabla p' + \frac{Ra}{Pr} T' \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}'$$

$$0 = \nabla \cdot \boldsymbol{u}'$$

$$\partial_t T' = -w' + \frac{1}{Pr}\nabla^2 T'$$

**+**

boundary conditions @ *z=0,1*

Impenetrable: $w' = \boldsymbol{u}' \cdot \hat{\boldsymbol{z}} = 0$

no-slip: $\boldsymbol{u}'_\perp = \boldsymbol{0}$ stress-free: $\partial_z \boldsymbol{u}'_\perp = \boldsymbol{0}$

fixed temp: $T' = 0$ fixed-flux: $\partial_z T' = 0$

$\Uparrow$

$$\boldsymbol{v}' = \boldsymbol{a}_{\boldsymbol{k}_\perp}(z) e^{(i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + st)} + c.c.$$

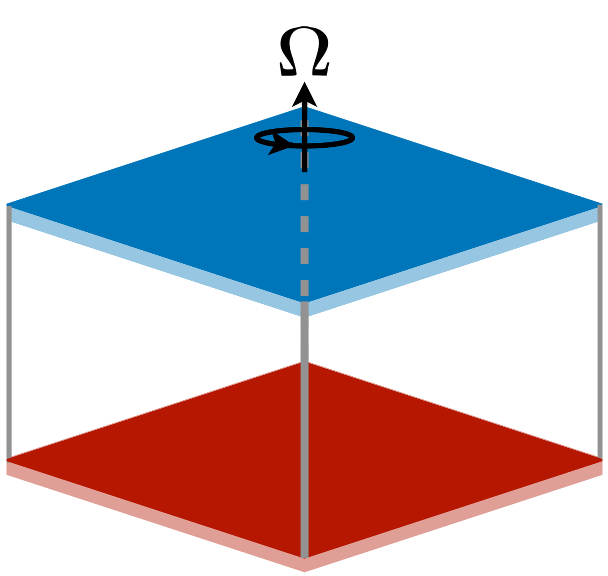
Normal mode ansatz

$\Downarrow$

$$s\boldsymbol{\mathcal{M}}\boldsymbol{a}_{\boldsymbol{k}_\perp} = \boldsymbol{\mathcal{L}}(\partial_z; \boldsymbol{k}_\perp)\boldsymbol{a}_{\boldsymbol{k}_\perp}$$

$\Downarrow$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}, \qquad \boldsymbol{\mathcal{L}} = \begin{pmatrix} \partial_{zz} - k_\perp^2 & E^{-1} & & ik_x & \\ -E^{-1} & \partial_{zz} - k_\perp^2 & & ik_y & \\ & & \partial_{zz} - k_\perp^2 & \partial_z & Ra/Pr \\ ik_x & ik_y & \partial_z & & \\ & & & -1 & Pr^{-1}(\partial_{zz} - k_\perp^2) \end{pmatrix}$$




$\boldsymbol{u}_b = 0$

$T_b(z) = 1 - z$

$s = \mu + i\omega, \quad \mu(Ra_c) = 0$ onset of convection

Steady bif'n exp. growth monotonic

Hopf bif'n exp. growth oscillatory

# Linear Stability Theory

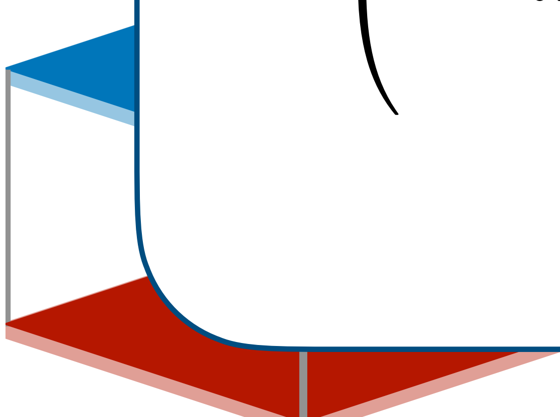
$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix},$$

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} \partial_{zz} - k_\perp^2 & E^{-1} & & ik_x & \\ -E^{-1} & \partial_{zz} - k_\perp^2 & & ik_y & \\ & & \partial_{zz} - k_\perp^2 & \partial_z & Ra/Pr \\ ik_x & ik_y & \partial_z & & \\ & & & -1 & Pr^{-1}(\partial_{zz} - k_\perp^2) \end{pmatrix}$$

$\partial_z \boldsymbol{u}'_\perp = \boldsymbol{0}$

$\partial_z T' = 0$



of convection

Steady bif'n — exp. growth monotonic

Hopf bif'n — exp. growth oscillatory

# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\partial_t \boldsymbol{u}' = -\frac{1}{E}\hat{\boldsymbol{z}} \times \boldsymbol{u}' - \nabla p' + \frac{Ra}{Pr} T' \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}'$$

$$0 = \nabla \cdot \boldsymbol{u}'$$

$$\partial_t T' = -w' + \frac{1}{Pr}\nabla^2 T'$$

**+**

boundary conditions @ *z=0,1*

Impenetrable: $w' = \boldsymbol{u}' \cdot \hat{\boldsymbol{z}} = 0$

no-slip: $\boldsymbol{u}'_\perp = \boldsymbol{0}$ stress-free: $\partial_z \boldsymbol{u}'_\perp = \boldsymbol{0}$

fixed temp: $T' = 0$ fixed-flux: $\partial_z T' = 0$

$\Uparrow$

$$\boldsymbol{v}' = \boldsymbol{a}_{\boldsymbol{k}_\perp}(z) e^{(i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + st)} + c.c.$$

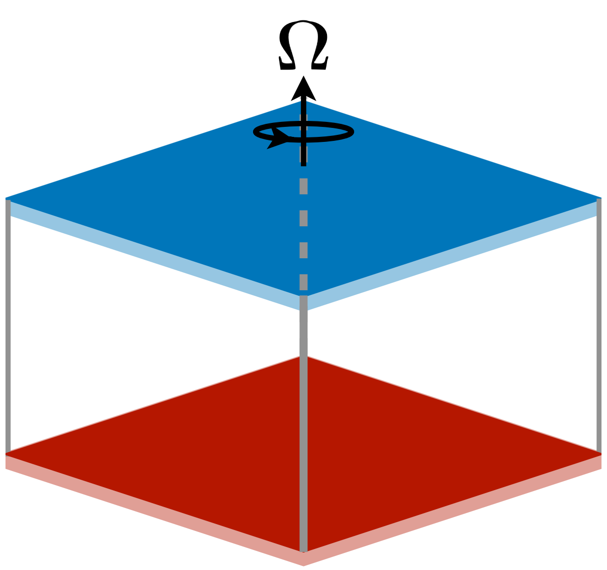
Normal mode ansatz

$\Downarrow$

$$s\boldsymbol{\mathcal{M}}\boldsymbol{a}_{\boldsymbol{k}_\perp} = \boldsymbol{\mathcal{L}}(\partial_z; \boldsymbol{k}_\perp)\boldsymbol{a}_{\boldsymbol{k}_\perp}$$

$\Downarrow$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}, \quad \boldsymbol{\mathcal{L}} = \begin{pmatrix} \partial_{zz} - k_\perp^2 & E^{-1} & & ik_x & \\ -E^{-1} & \partial_{zz} - k_\perp^2 & & ik_y & \\ & & \partial_{zz} - k_\perp^2 & \partial_z & Ra/Pr \\ ik_x & ik_y & \partial_z & & \\ & & & -1 & Pr^{-1}(\partial_{zz} - k_\perp^2) \end{pmatrix}$$




$$\boldsymbol{u}_b = 0$$
$$T_b(z) = 1 - z$$

| | |
|---|---|
| $s = \mu + i\omega, \quad \mu(Ra_c) = 0$ | onset of convection |
| Steady bif'n | exp. growth monotonic |
| Hopf bif'n | exp. growth oscillatory |

# Linear Stability Theory

$$\boldsymbol{v} = (\boldsymbol{u}, p, T), \quad \boldsymbol{v} = \boldsymbol{v}_b + \boldsymbol{v}', \quad |\boldsymbol{v}'| \ll 1 \qquad \Rightarrow$$

$$\partial_t \boldsymbol{u}' = -\frac{1}{E}\hat{\boldsymbol{z}} \times \boldsymbol{u}' - \nabla p' + \frac{Ra}{Pr} T' \hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}'$$

$$0 = \nabla \cdot \boldsymbol{u}'$$

$$\partial_t T' = -w' + \frac{1}{Pr}\nabla^2 T'$$

**+**

boundary conditions @ *z=0,1*

Impenetrable: $w' = \boldsymbol{u}' \cdot \hat{\boldsymbol{z}} = 0$

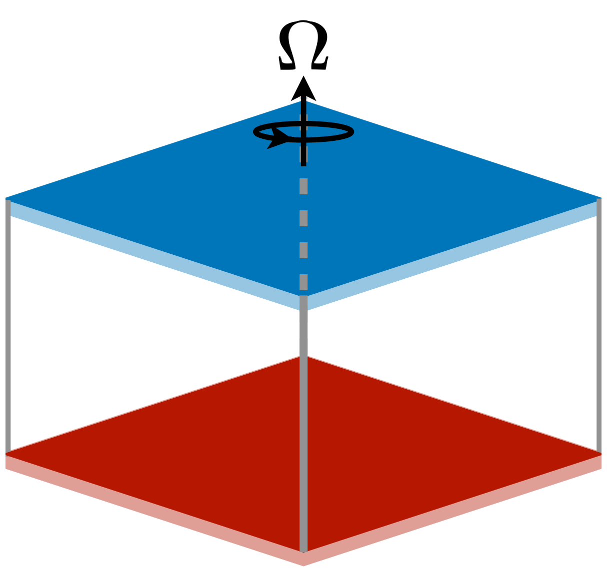
no-slip: $\boldsymbol{u}'_\perp = \boldsymbol{0}$ stress-free: $\partial_z \boldsymbol{u}'_\perp = \boldsymbol{0}$

fixed temp: $T' = 0$ fixed-flux: $\partial_z T' = 0$

$\Uparrow$

$$\boldsymbol{v}' = \boldsymbol{a}_{\boldsymbol{k}_\perp}(z) e^{(i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + st)} + c.c.$$

Normal mode ansatz




$$\boldsymbol{u}_b = 0$$
$$T_b(z) = 1 - z$$

$\Downarrow$

$$s\boldsymbol{\mathcal{M}}\boldsymbol{a}_{\boldsymbol{k}_\perp} = \boldsymbol{\mathcal{L}}(\partial_z; \boldsymbol{k}_\perp)\boldsymbol{a}_{\boldsymbol{k}_\perp}$$

$\Downarrow$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}, \quad \boldsymbol{\mathcal{L}} = \begin{pmatrix} \partial_{zz} - k_\perp^2 & E^{-1} & & ik_x & \\ -E^{-1} & \partial_{zz} - k_\perp^2 & & ik_y & \\ & & \partial_{zz} - k_\perp^2 & \partial_z & Ra/Pr \\ ik_x & ik_y & \partial_z & & \\ & & & -1 & Pr^{-1}(\partial_{zz} - k_\perp^2) \end{pmatrix}$$

$s = \mu + i\omega, \quad \mu(Ra_c) = 0$ onset of convection

Steady bif'n exp. growth monotonic

Hopf bif'n exp. growth oscillatory

# Linear Stability Theory: Non-Rotating Convection

Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

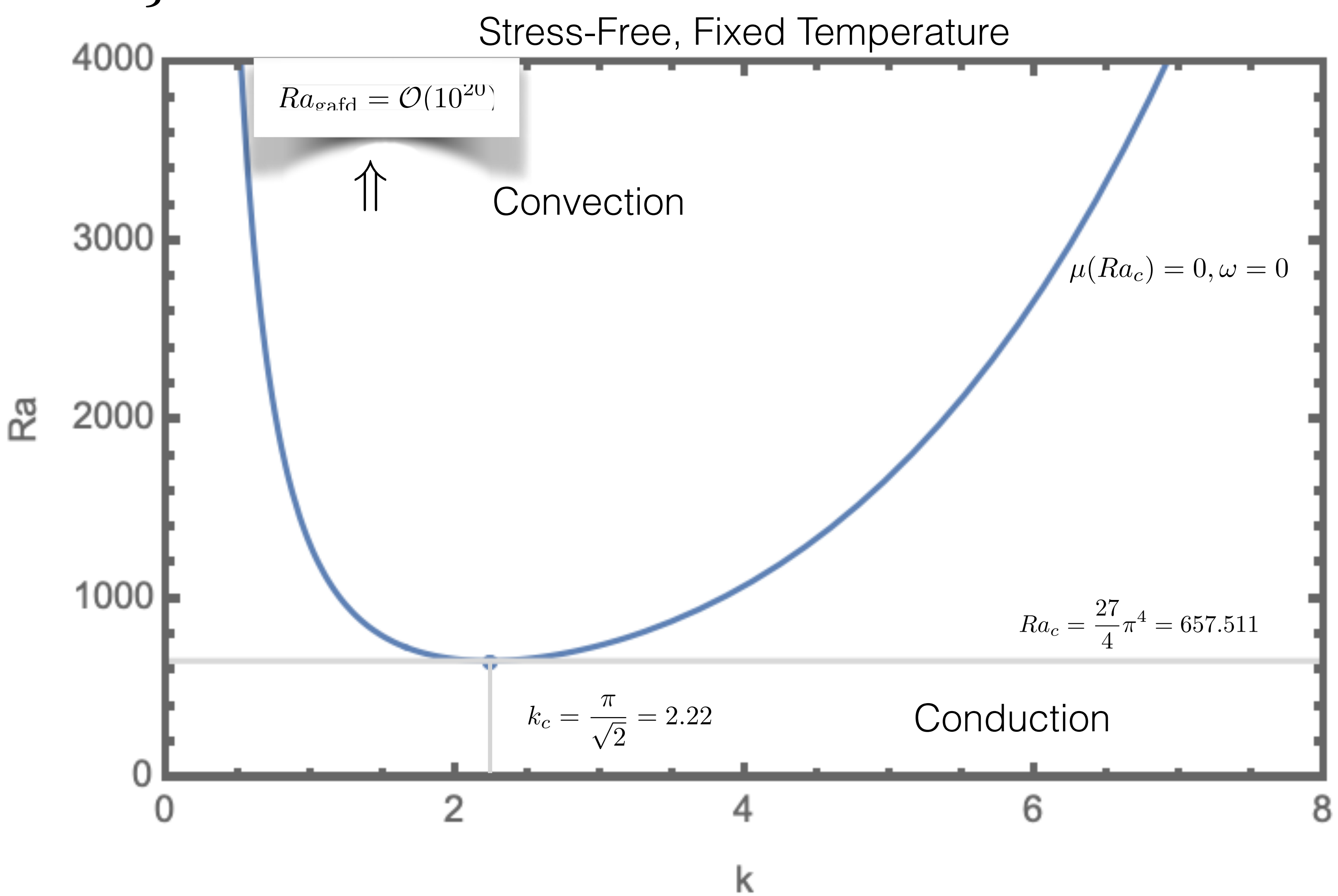
# Linear Stability Theory: Non-Rotating Convection

Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

Marginal Stability

$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3}{k_\perp^2}$$



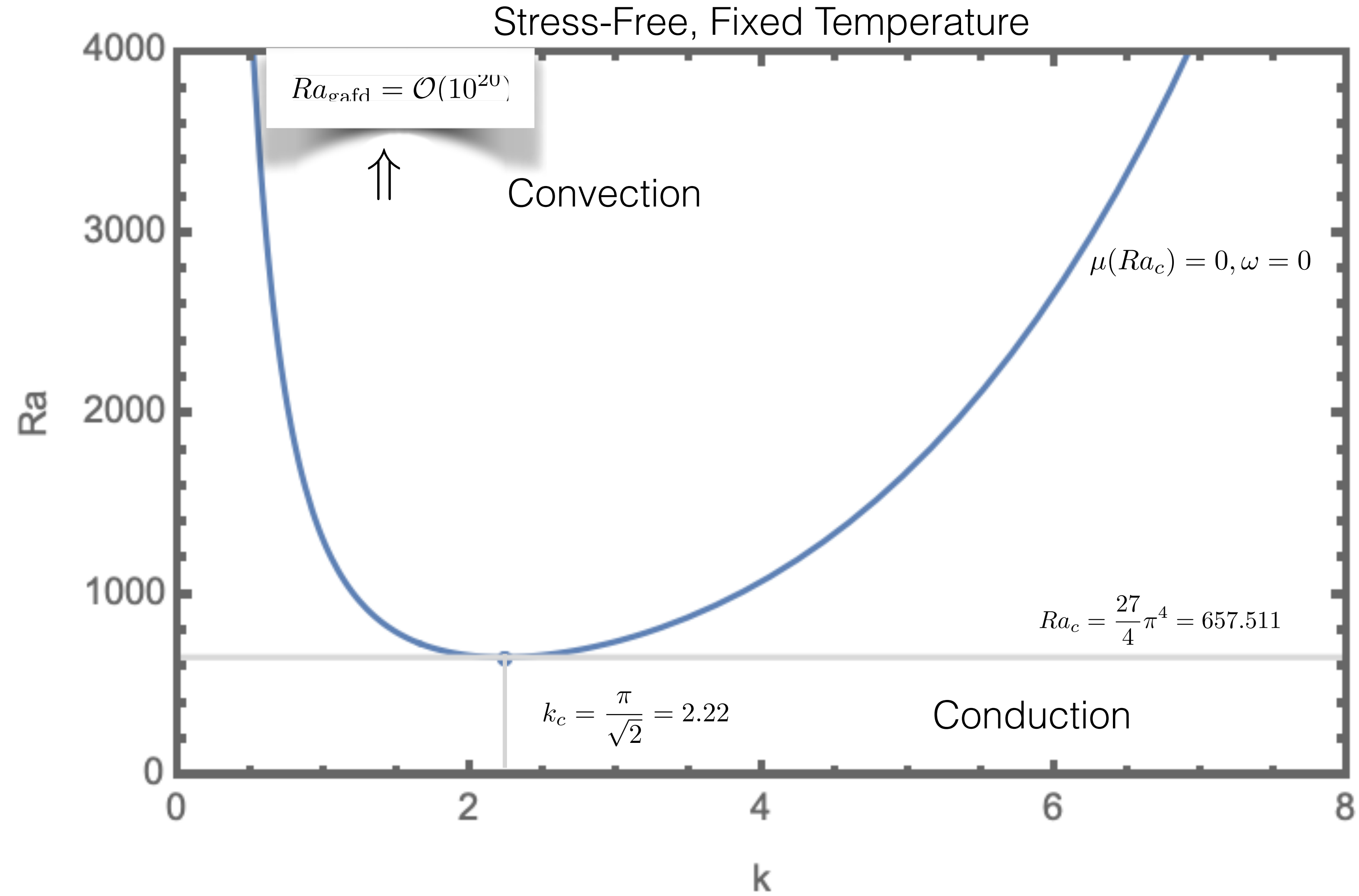
# Linear Stability Theory: Non-Rotating Convection

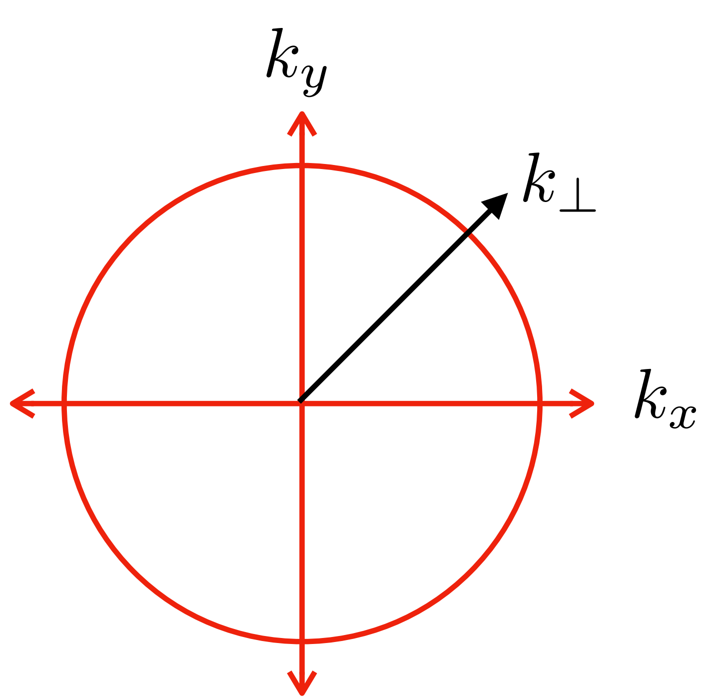
Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

Marginal Stability

$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3}{k_\perp^2}$$



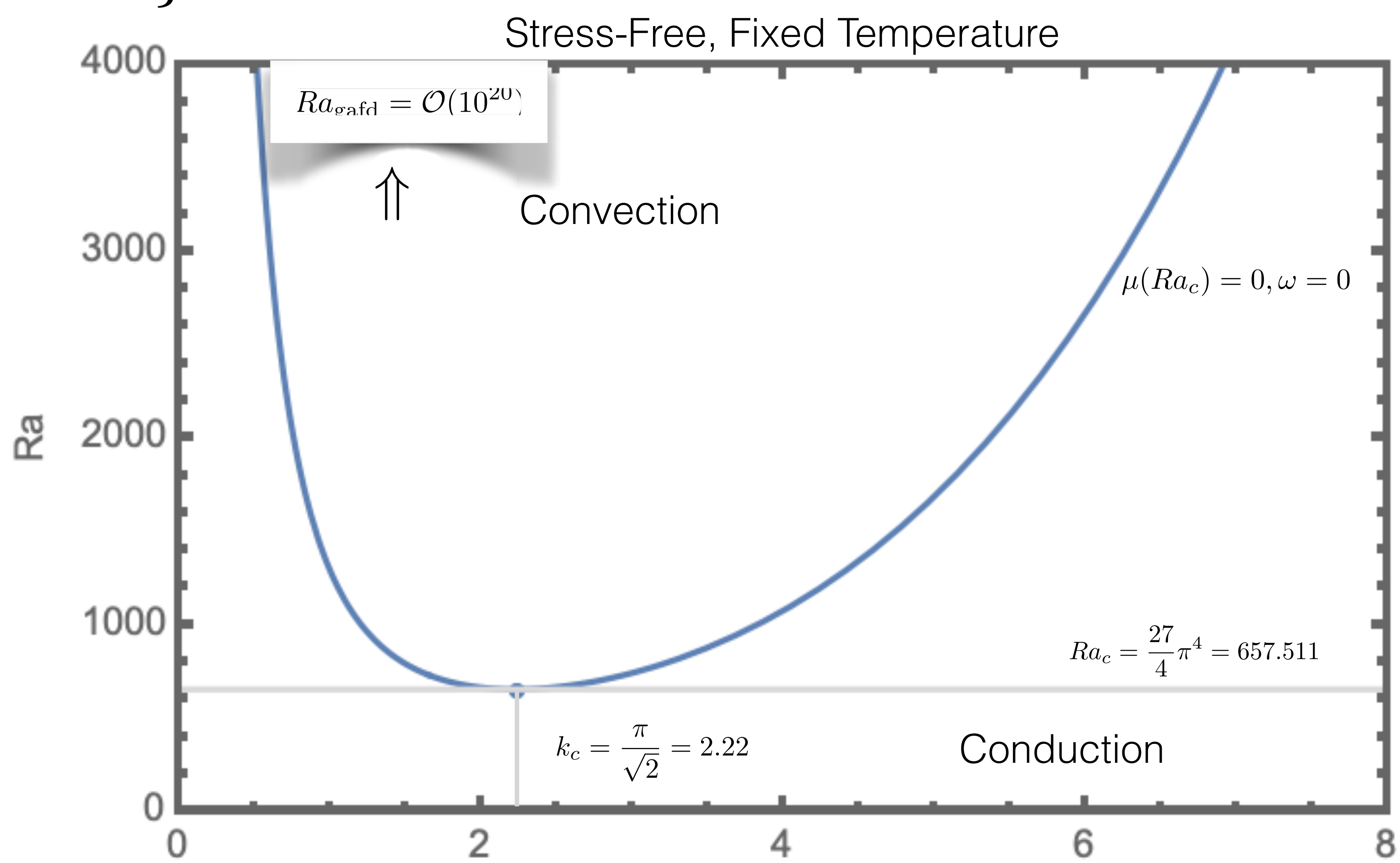




Unstable to all orientations: Rolls

# Linear Stability Theory: Non-Rotating Convection

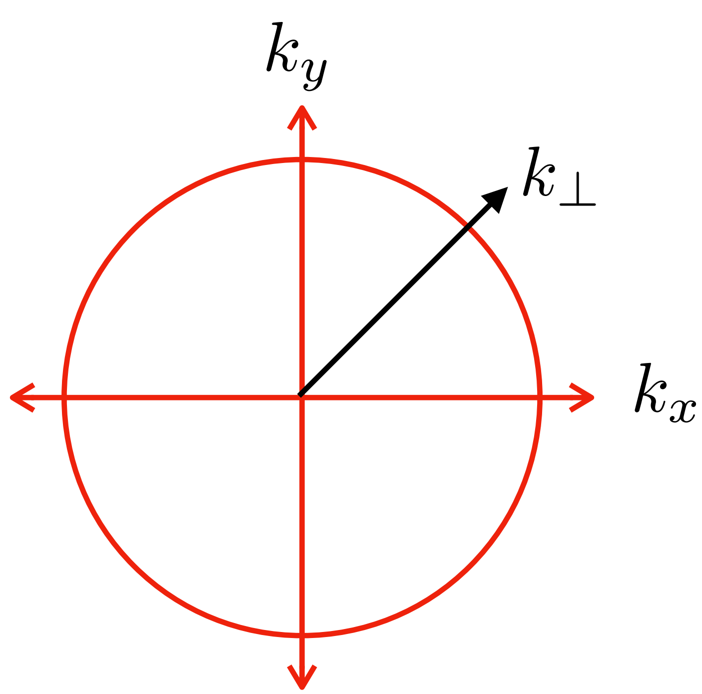
Stress-Free, Fixed Temperature: full analytic representation

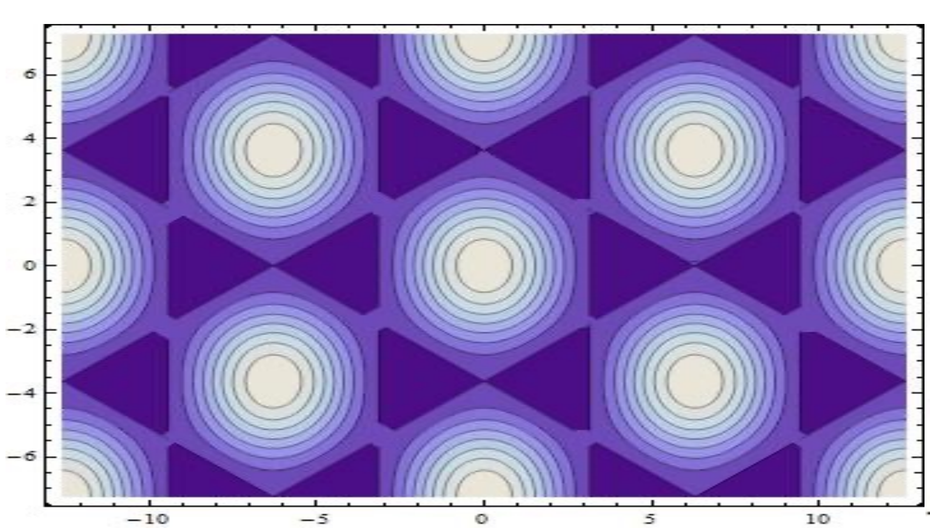
$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

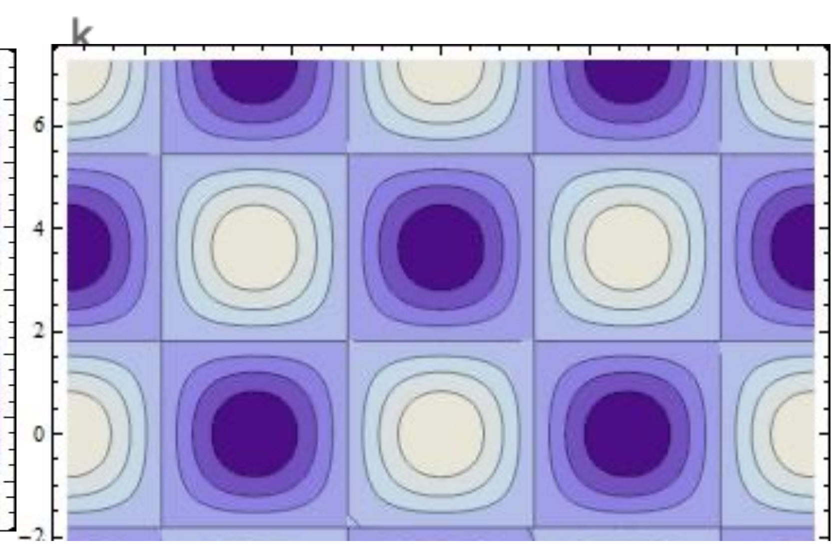



Marginal Stability

$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3}{k_\perp^2}$$





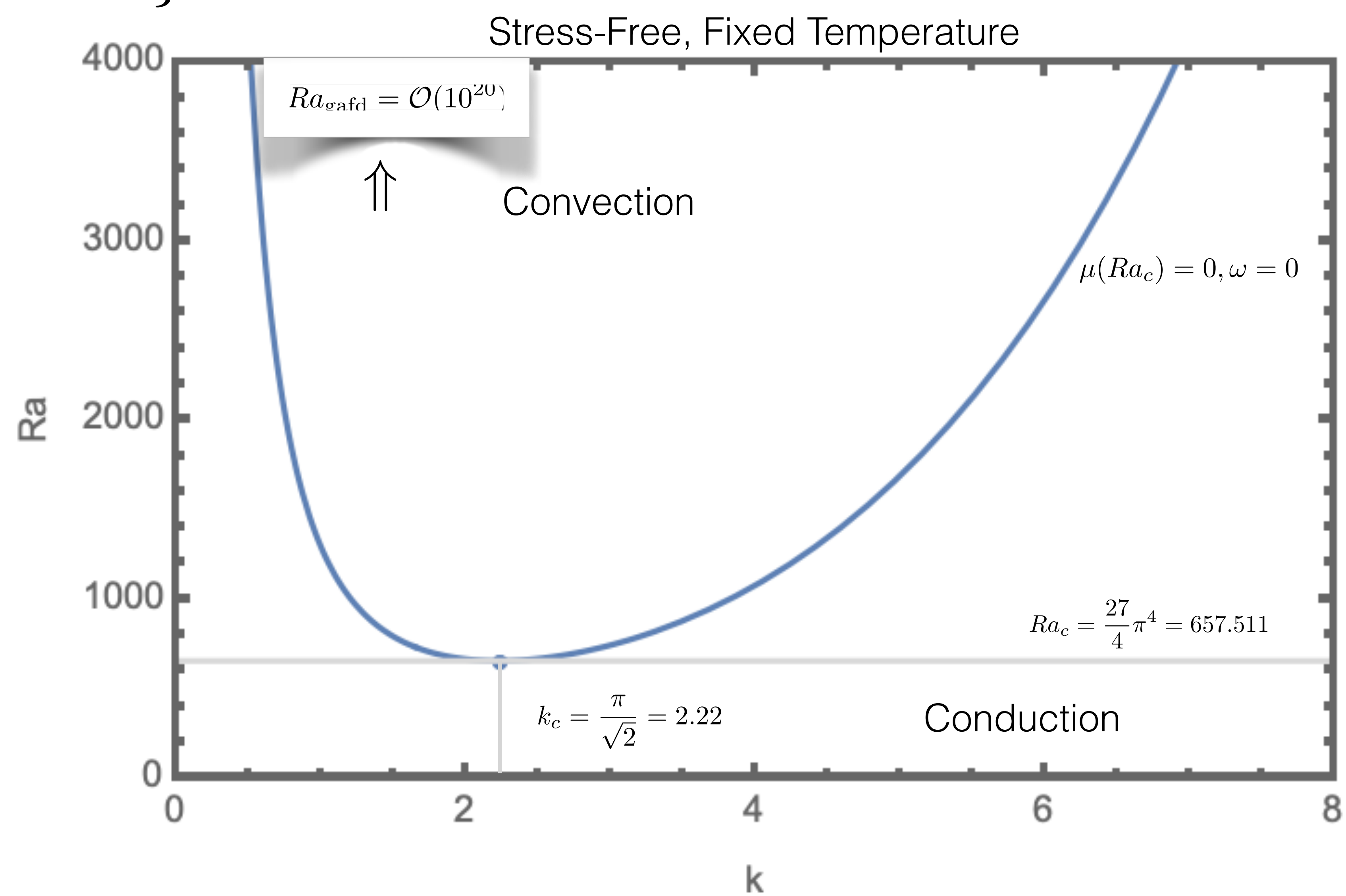
Unstable to all orientations: Rolls

Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

Marginal Stability

$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3}{k_\perp^2}$$




| $T$ | $\boldsymbol{u}$ | $Ra_c$ | $k_c$ |
|---|---|---|---|
| FT | SF | 657.11 | 2.221 |
| FT | NS | 1707.7 | 3.117 |
| FT | SF/NS | 1100.65 | 2.682 |

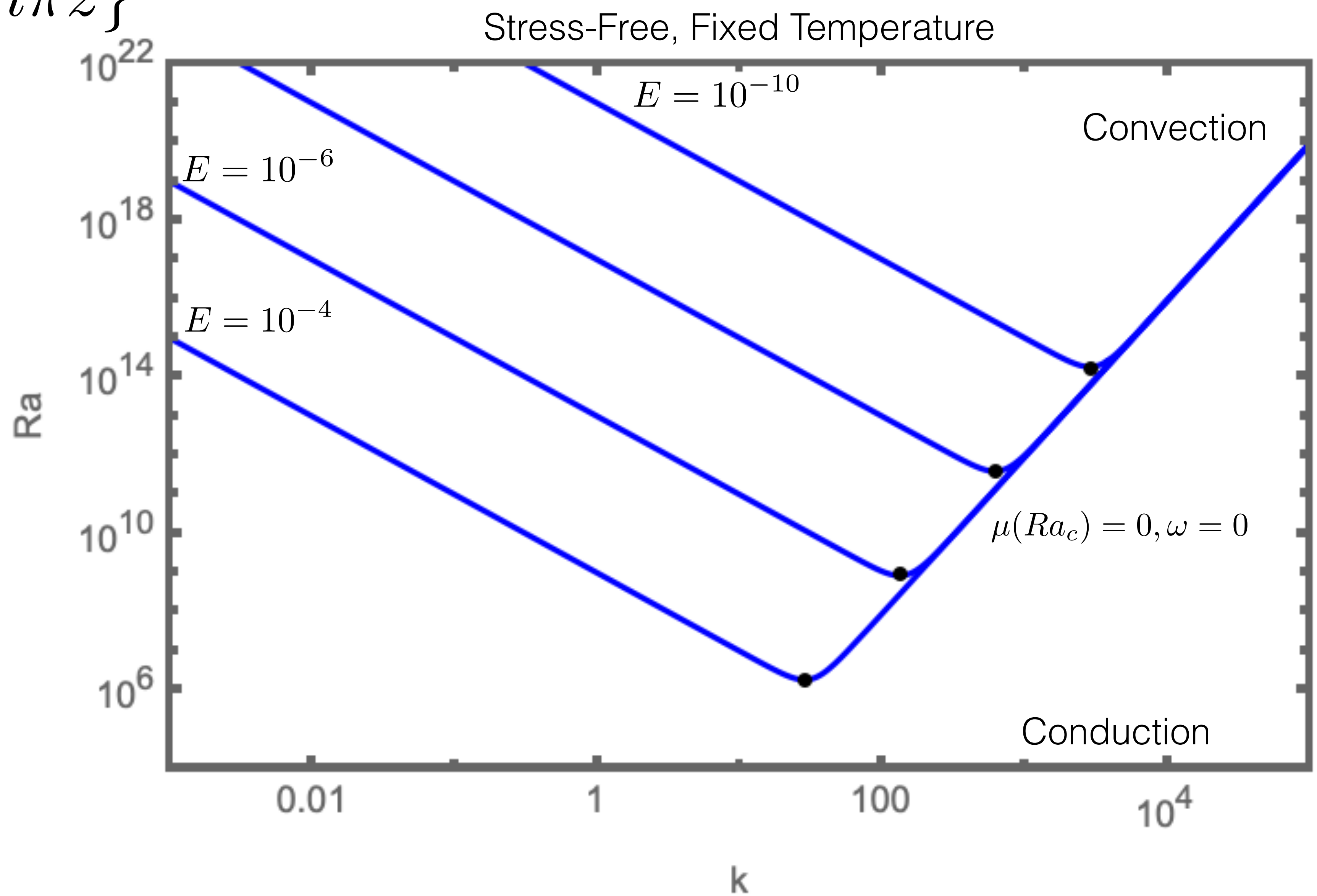
Viscous boundaries delay convective onset

# Linear Stability Theory: Rotating Convection

Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3 + \pi^2 E^{-2}}{k_\perp^2}$$

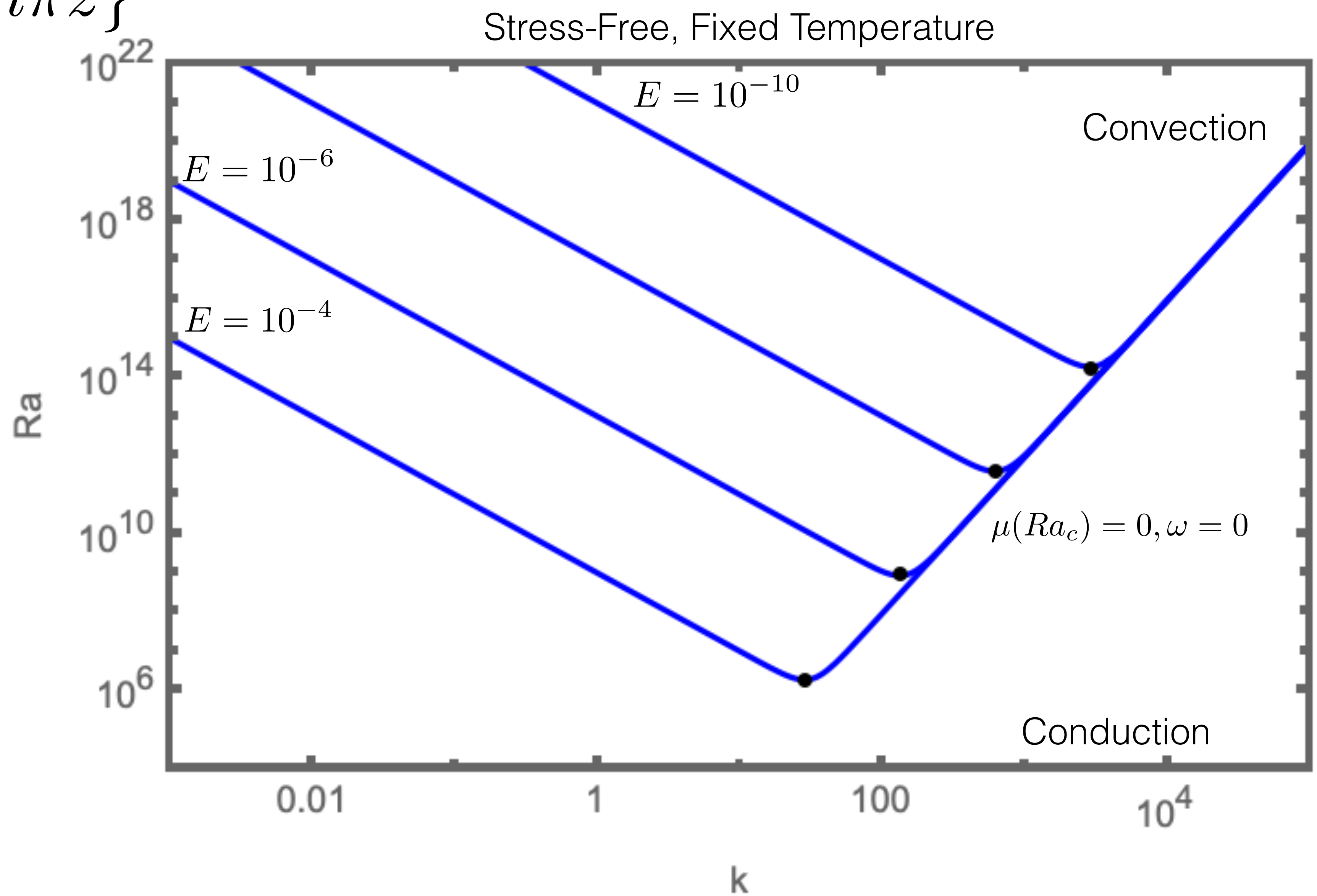


# Linear Stability Theory: Rotating Convection

Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$

$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3 + \pi^2 E^{-2}}{k_\perp^2}$$



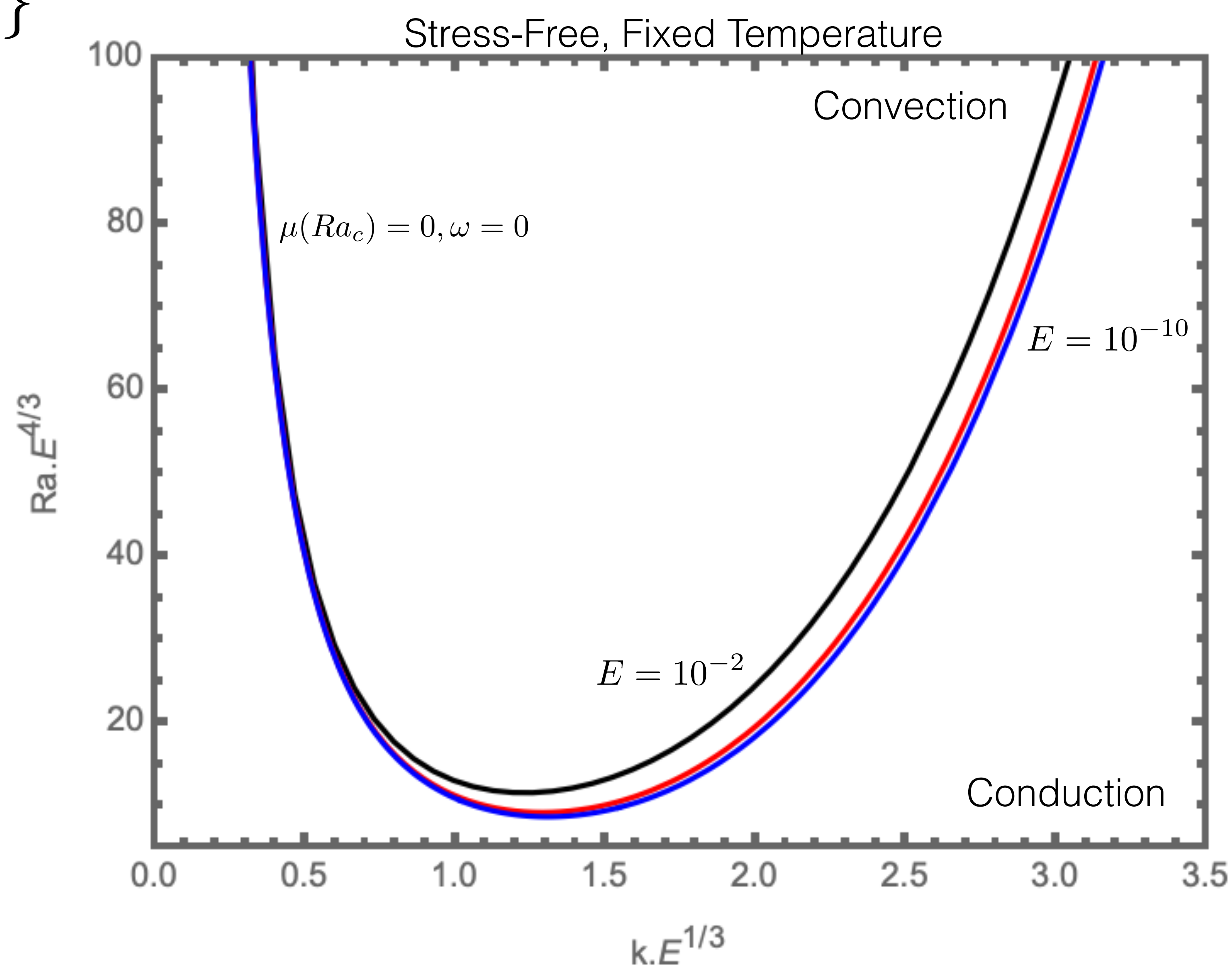

As $\lim E \to 0$

- Rotation imparts rigidity $Ra_{c,s}$ increases
- Onset Length scale of convection. $\propto k_{c,\perp}^{-1}$ decreases
- Asymptotic?

# Linear Stability Theory: Rotating Convection

Stress-Free, Fixed Temperature: full analytic representation

$$\boldsymbol{a}_{\boldsymbol{k}_\perp}(z) \propto \{\sin n\pi z, \cos n\pi z\}$$




$$Ra_{c,s} = \frac{(k_\perp^2 + \pi^2)^3 + \pi^2 E^{-2}}{k_\perp^2}$$

$$k = \widetilde{k} E^{-1/3}, \quad Ra_{c,s} = \widetilde{Ra}_{c,s} E^{-4/3}$$

Asymptotic convergence for SF-FT as $\lim E \to 0$

*Chandrasekhar H2S1961*

# Linear Stability Theory: Rotating Convection

Steady:

$$\widetilde{Ra}_{c,s} = \frac{\pi^2}{\widetilde{k}^2} + \widetilde{k}^4$$

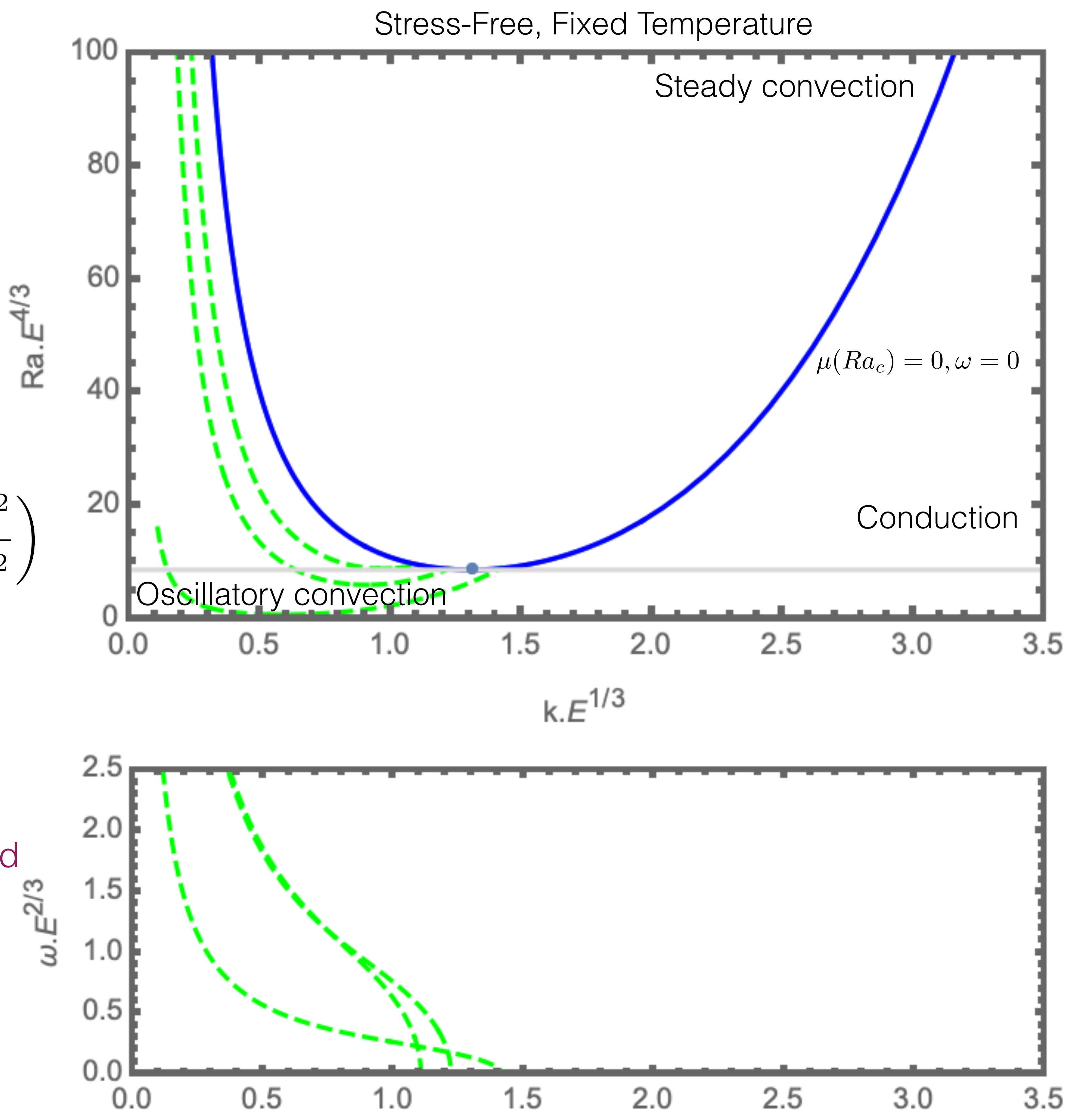
$$\widetilde{Ra}_{c,min} = \frac{3}{2}(2\pi^4)^{1/3}, \quad \widetilde{k}_{c,min} = \left(\frac{\pi^2}{2}\right)^{1/3}$$

Oscillatory:

$$\widetilde{Ra}_{c,o} = 2(1+Pr)\left(\widetilde{k}^4 + \left(\frac{Pr}{1+Pr}\right)\frac{\pi^2}{\widetilde{k}^2}\right)$$

$$\widetilde{\omega}^2_{c,o} = Pr^2\left[\left(\frac{1-Pr}{1+Pr}\right)\frac{\pi^2}{\widetilde{k}^2} - \widetilde{k}^4\right]$$

Asymptotic linear theory can be understood from a force-balance analysis




*Chandrasekhar H2S1961*

# Linear Stability Theory: RRBC - Force Balance Analysis

Asymptotic scalings can be understood as a VAC (Viscous-Archimedean-Coriolis) balance.

# Linear Stability Theory: RRBC - Force Balance Analysis

Asymptotic scalings can be understood as a VAC (Viscous-Archimedean-Coriolis) balance.

Geostrophy: $$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u} \sim \nabla p \quad \Longrightarrow \quad P \sim 2\Omega U \ell_\perp$$

# Linear Stability Theory: RRBC - Force Balance Analysis

Asymptotic scalings can be understood as a VAC (Viscous-Archimedean-Coriolis) balance.

Geostrophy: $$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u} \sim \nabla p \quad \Longrightarrow \quad P \sim 2\Omega U \ell_\perp$$

T-P operates on scale: $\ell_\perp$ $$\nabla_\perp \cdot \boldsymbol{u}_\perp \approx 0 \Longrightarrow \partial_z \sim \nabla_\perp \approx 0$$ 3D must imply $\ell_\perp \ll H$?

# Linear Stability Theory: RRBC - Force Balance Analysis

Asymptotic scalings can be understood as a VAC (Viscous-Archimedean-Coriolis) balance.

Geostrophy: $2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u} \sim \nabla p \implies P \sim 2\Omega U \ell_\perp$

T-P operates on scale: $\ell_\perp$ $\quad \nabla_\perp \cdot \boldsymbol{u}_\perp \approx 0 \implies \partial_z \sim \nabla_\perp \approx 0$ $\quad$ 3D must imply $\ell_\perp \ll H$?

Viscous-Coriolis (VC):

$$2\Omega \partial_z w \sim \nu \nabla^2 \zeta \implies \frac{2\Omega U}{H} \sim \nu \frac{U}{\ell_\perp^3}$$

$$\implies \frac{\ell_\perp}{H} = \left(\frac{\nu}{2\Omega H^2}\right)^{1/3} = E^{1/3}$$

- Viscosity is crucial in setting characteristics length scale
- Importantly, anisotropy enables 3D dynamics to overcome T-P constraint

# Linear Stability Theory: RRBC - Force Balance Analysis

Asymptotic scalings can be understood as a VAC (Viscous-Archimedean-Coriolis) balance.

Geostrophy:

$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u} \sim \nabla p \quad \Longrightarrow \quad P \sim 2\Omega U \ell_\perp$$

T-P operates on scale: $\ell_\perp$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp \approx 0 \Longrightarrow \partial_z \sim \nabla_\perp \approx 0$$

3D must imply $\ell_\perp \ll H$?

Viscous-Coriolis (VC):

$$2\Omega\partial_z w \sim \nu\nabla^2\zeta \quad \Longrightarrow \quad \frac{2\Omega U}{H} \sim \nu\frac{U}{\ell_\perp^3}$$

$$\Longrightarrow \quad \frac{\ell_\perp}{H} = \left(\frac{\nu}{2\Omega H^2}\right)^{1/3} = E^{1/3}$$

- Viscosity is crucial in setting characteristics length scale
- Importantly, anisotropy enables 3D dynamics to overcome T-P constraint

Coriolis-Archimedean (AC) or Viscous-Archimedean (VA):

$$\partial_z p \sim g\alpha T' \sim \nu\nabla^2 w \qquad \qquad \frac{P}{H} \sim g\alpha\Delta_T\frac{T'}{\Delta_T} \sim \nu\frac{U}{\ell_\perp^2}$$

$$\Longrightarrow$$

$$w\partial_z\overline{T} \sim \kappa\nabla^2 T' \qquad \qquad U\frac{\Delta_T}{H} \sim \kappa\frac{T'}{\ell_\perp^3}$$

$$\Longrightarrow \quad RaE^{4/3} \sim 1$$

- Buoyant driving must be sufficiently large  to overcome T-P constraint

# RRBC is in the QG Regime

Asymptotic scalings can be understood as a VAC (Viscous-Archimedean-Coriolis) balance.

Geostrophy: $$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u} \sim \nabla p \implies P \sim 2\Omega U \ell_\perp$$

T-P operates on scale: $\ell_\perp$ $$\nabla_\perp \cdot \boldsymbol{u}_\perp \approx 0 \implies \partial_z \sim \nabla_\perp \approx 0$$ 3D must imply $\ell_\perp \ll H$?

Viscous-Coriolis (VC):

$$2\Omega\partial_z w \sim \nu\nabla^2\zeta \implies \frac{2\Omega U}{H} \sim \nu\frac{U}{\ell_\perp^3}$$

$$\implies \frac{\ell_\perp}{H} = \left(\frac{\nu}{2\Omega H^2}\right)^{1/3} = E^{1/3}$$

- Viscosity is crucial in setting characteristics length scale
- Importantly, anisotropy enables 3D dynamics to overcome T-P constraint

Coriolis-Archimedean (AC) or Viscous-Archimedean (VA):

$$\partial_z p \sim g\alpha T' \sim \nu\nabla^2 w \qquad \frac{P}{H} \sim g\alpha\Delta_T\frac{T'}{\Delta_T} \sim \nu\frac{U}{\ell_\perp^2}$$

$$\implies$$

$$w\partial_z\overline{T} \sim \kappa\nabla^2 T' \qquad U\frac{\Delta_T}{H} \sim \kappa\frac{T'}{\ell_\perp^3}$$

$$\implies RaE^{4/3} \sim 1$$

- Buoyant driving must be sufficiently large to overcome T-P constraint

Oscillations: $$\partial_t\zeta \sim 2\Omega\partial_z w \implies s^* \sim 2\Omega\frac{\ell_\perp}{H} \implies s \sim E^{-2/3}$$

# Linear Stability Theory: Rotating Convection

Appears there is an asymptotic linear reduction

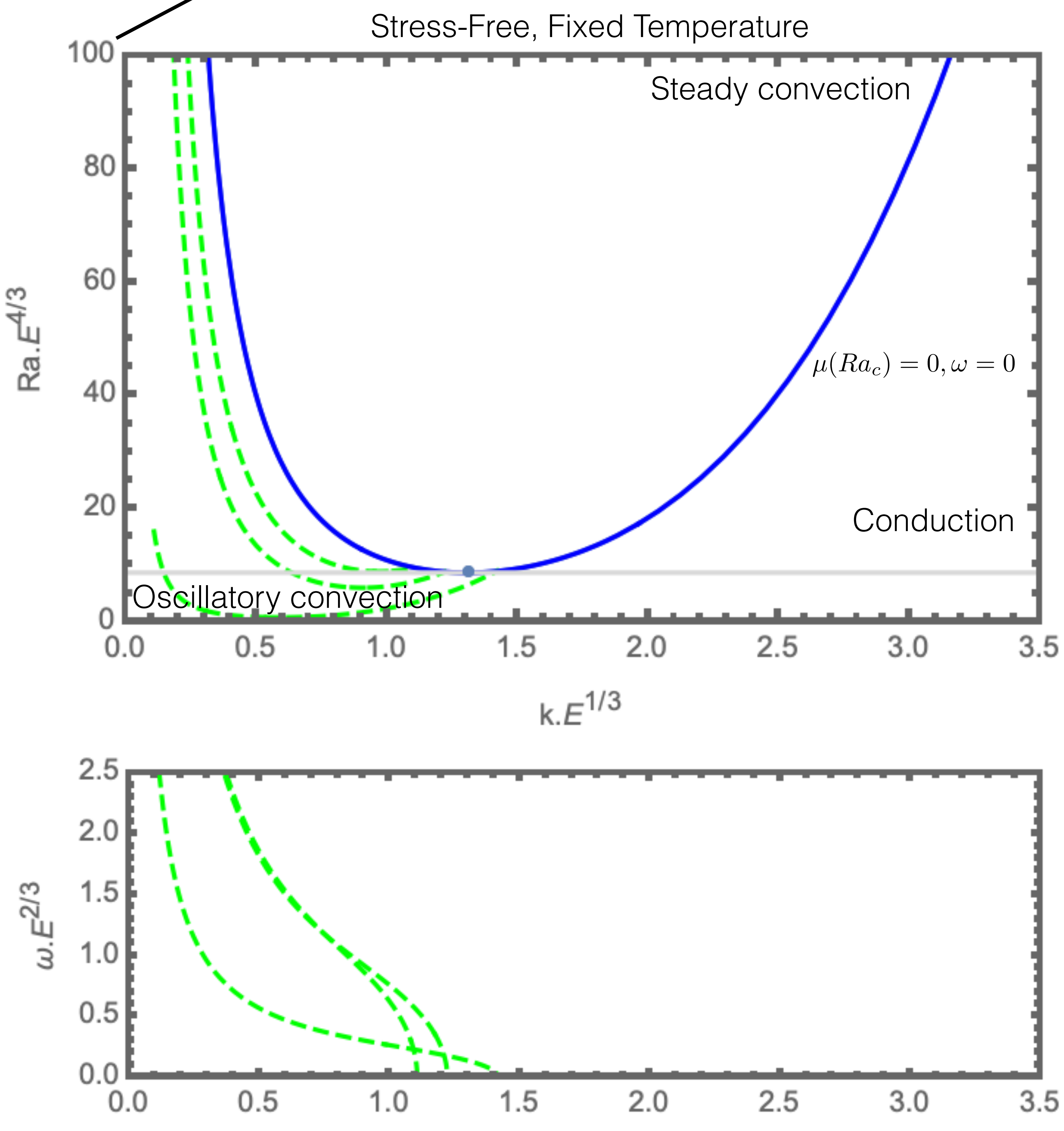
$@\ E = 10^{-15} \implies Ra = 10^{22}$

Can we go beyond linear theory?

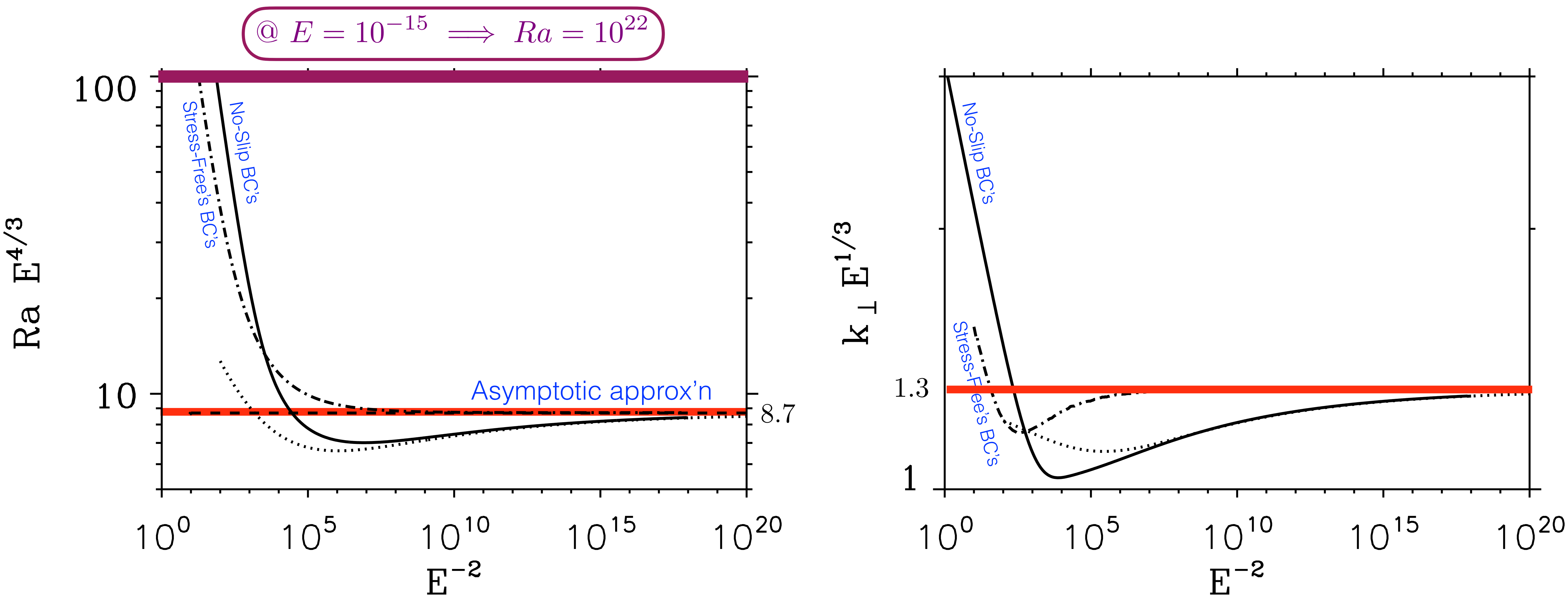
Potential for fully nonlinear model

in GAFD regimes! At least for SF-FT

What about sensitivity to bc's



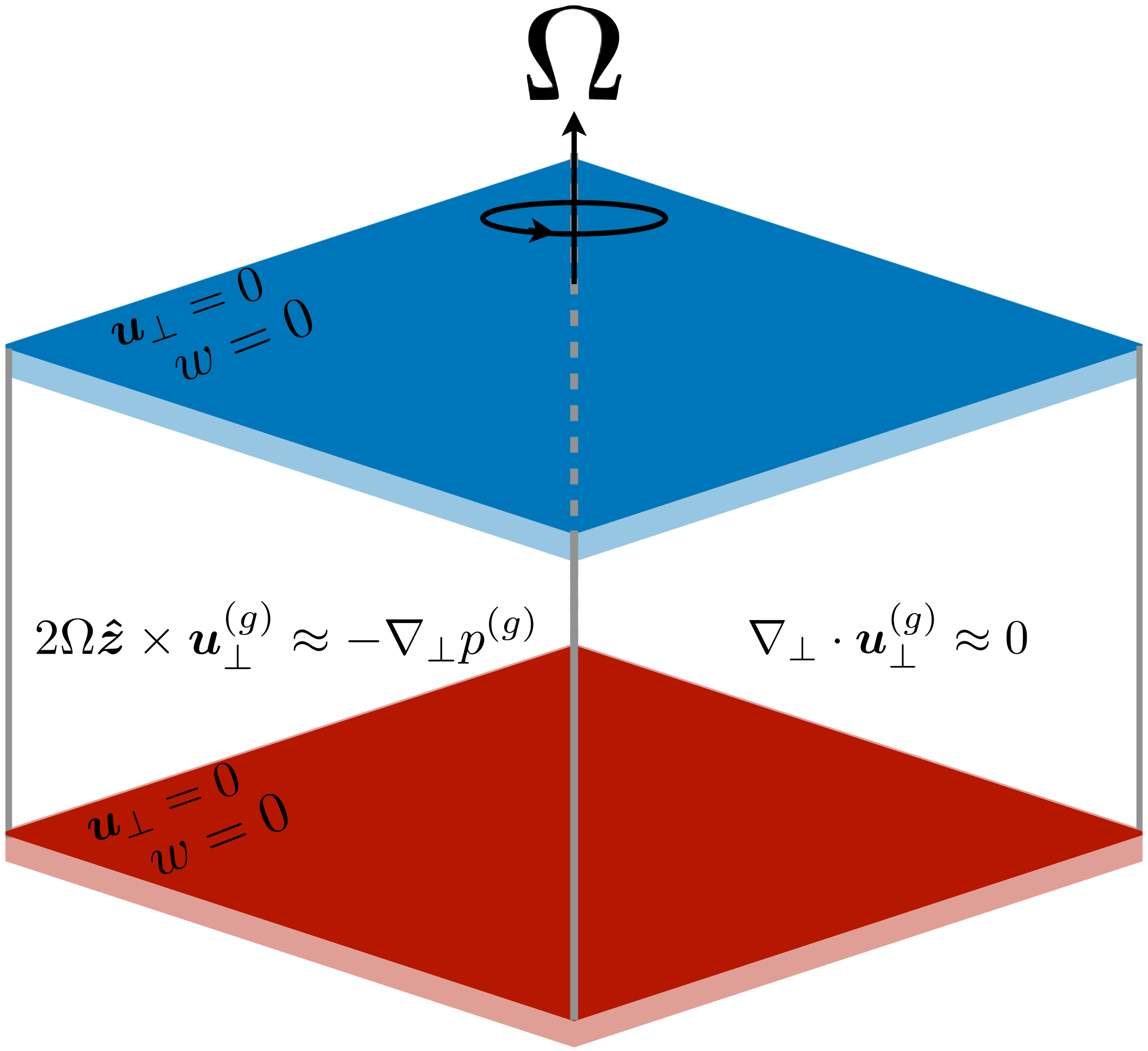
# Convergence to Asymptotic Limit?



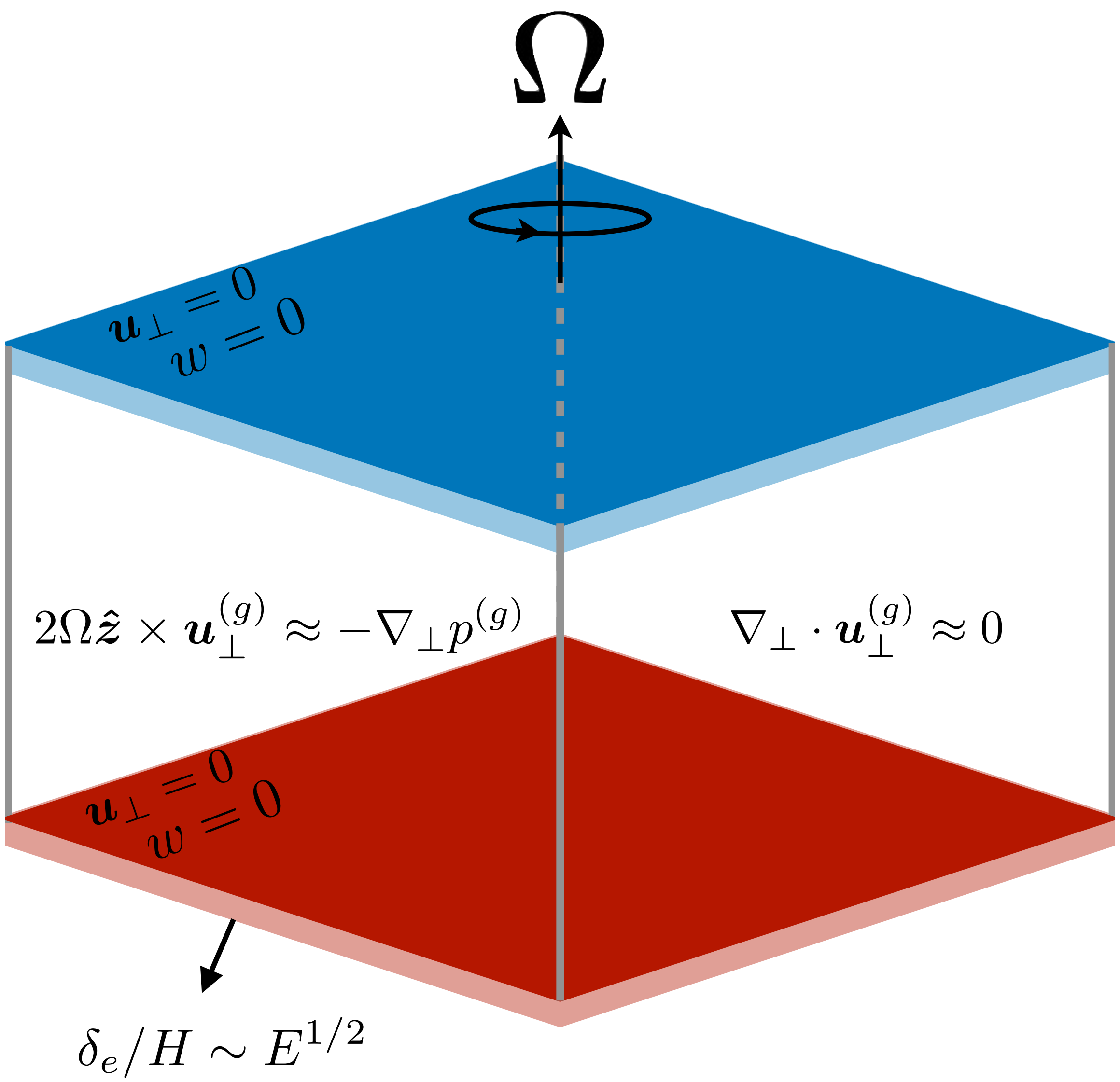

- SF-FT: rapid convergence
- NS-FT: by comparison, slowly convergent, less stable (enhanced viscous drag relaxes T-P constraint. How?
- NS bc's are most destabilizing (opposite from non-rotating).

*KJ et al JFM 2016*

# Linear Stability Theory: Sensitivity btw NS & SF BC's



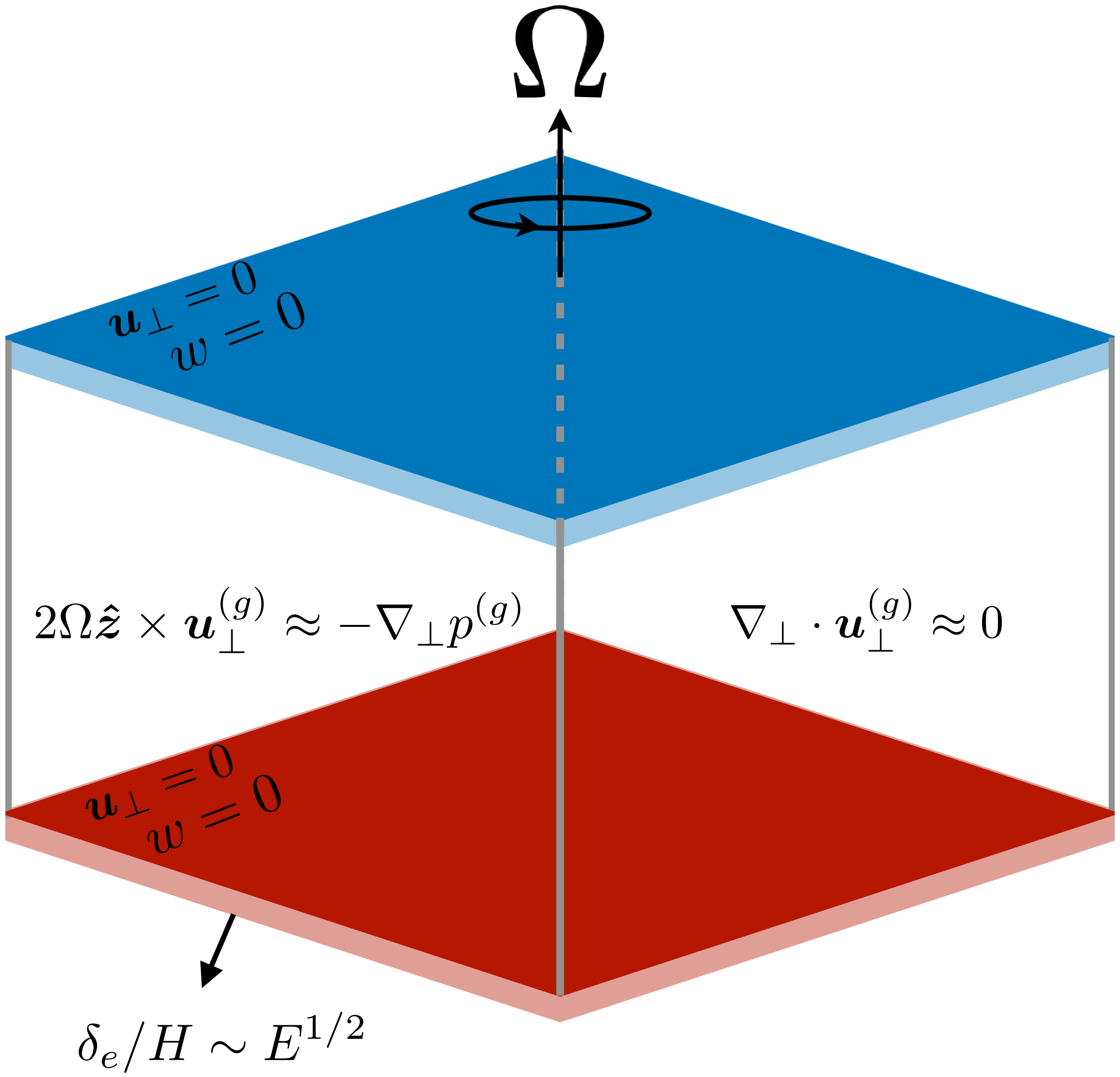

- Geostrophic solution incompatible with no-slip boundaries.




$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(e)} \approx \nu\partial_{\mu\mu}\boldsymbol{u}_\perp^{(e)}$$

- Geostrophic solution incompatible with no-slip boundaries.
- Evidence for (Ekman) boundary layer solution s.t.

# Linear Stability Theory: Sensitivity btw NS & SF BC's




$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(e)} \approx \nu \partial_{\mu\mu} \boldsymbol{u}_\perp^{(e)}$$

- Geostrophic solution incompatible with no-slip boundaries.
- Evidence for (Ekman) boundary layer solution s.t.

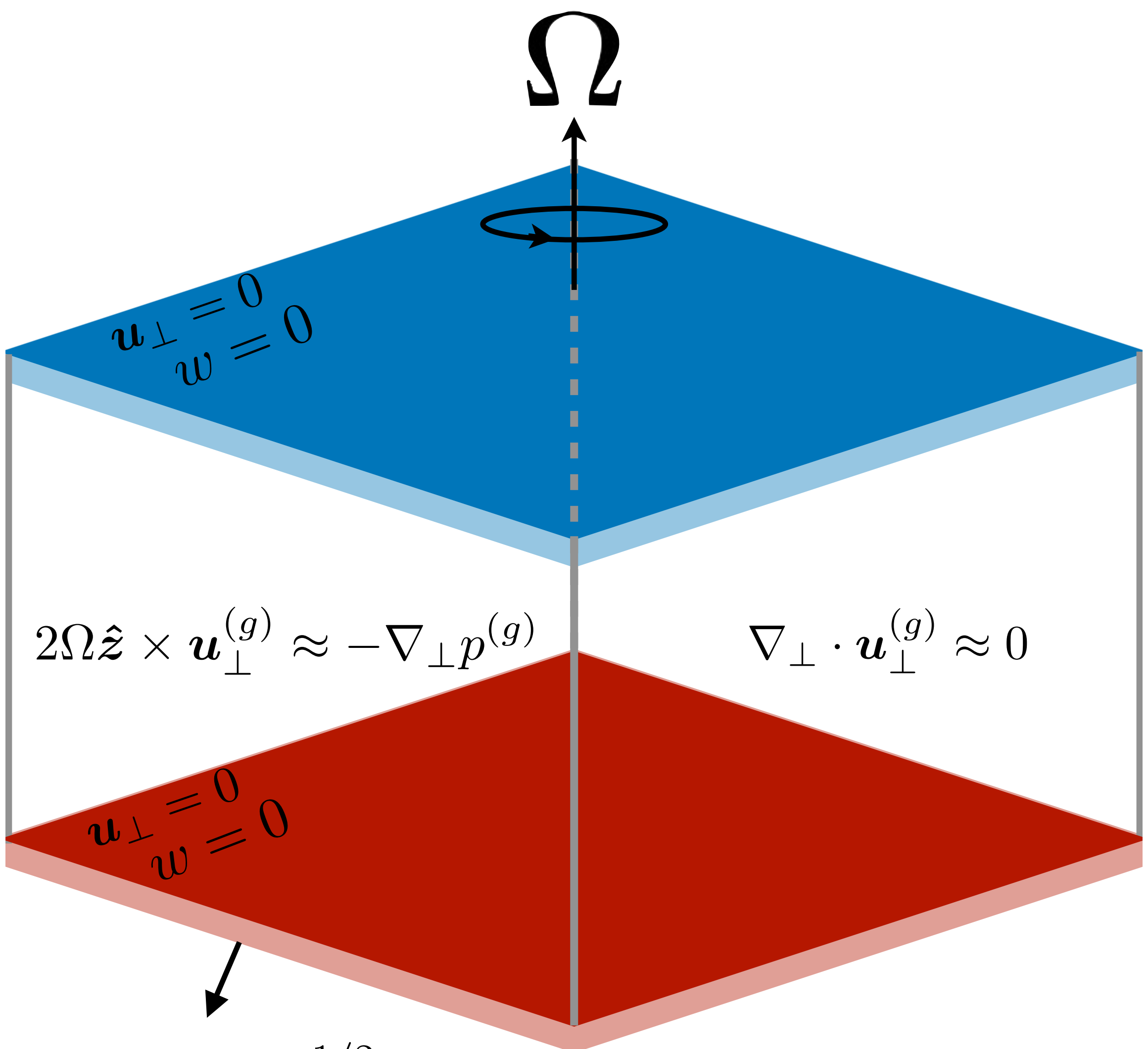
$$\boldsymbol{u}_\perp^{(c)} = \boldsymbol{u}_\perp^{(g)}(0) + \boldsymbol{u}_\perp^{(e)}(\mu)$$

*zero on bnd* *exponentially decaying into interior*

- Matches solution at bounding plate
- Coriolis viscous balance implies $\delta_e \sim (\nu/2\Omega)^{1/2}$

KJ et al JFM 2016

# Linear Stability Theory: Sensitivity btw NS & SF BC's




$\delta_e / H \sim E^{1/2}$

$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(e)} \approx \nu \partial_{\mu\mu} \boldsymbol{u}_\perp^{(e)}$$

- Geostrophic solution incompatible with no-slip boundaries.
- Evidence for (Ekman) boundary layer solution s.t.

$$\boldsymbol{u}_\perp^{(c)} = \boldsymbol{u}_\perp^{(g)}(0) + \boldsymbol{u}_\perp^{(e)}(\mu)$$

*zero on bnd*

*exponentially decaying into interior*

- Matches solution at bounding plate
- Coriolis viscous balance implies $\delta_e \sim (\nu/2\Omega)^{1/2}$

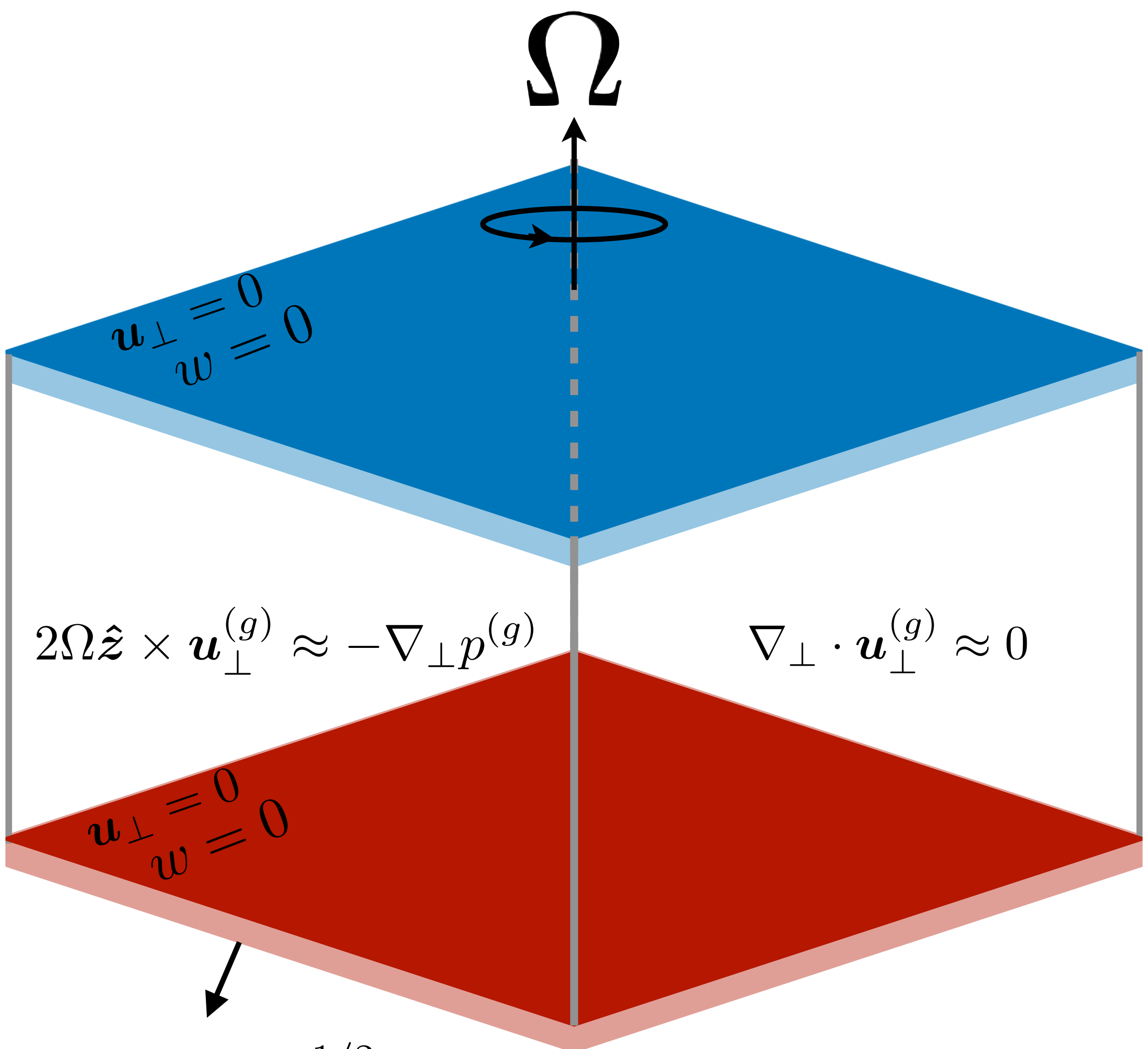
EBL solutions

$$u_0^{(e)}(\boldsymbol{x}_\perp, \mu, t) = -e^{-\frac{\mu}{\sqrt{2}}} \left( u_0^{(g)}(\boldsymbol{x}_\perp, 0, t) \cos \frac{\mu}{\sqrt{2}} - u_0^{(g)}(\boldsymbol{x}_\perp, 0, t) \sin \frac{\mu}{\sqrt{2}} \right)$$

$$v_0^{(e)}(\boldsymbol{x}_\perp, \mu, t) = -e^{-\frac{\mu}{\sqrt{2}}} \left( v_0^{(g)}(\boldsymbol{x}_\perp, 0, t) \cos \frac{\mu}{\sqrt{2}} + v_0^{(g)}(\boldsymbol{x}_\perp, 0, t) \sin \frac{\mu}{\sqrt{2}} \right)$$

*KJ et al JFM 2016*

# Linear Stability Theory: Sensitivity btw NS & SF BC's




$\delta_e/H \sim E^{1/2}$

$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(e)} \approx \nu\partial_{\mu\mu}\boldsymbol{u}_\perp^{(e)}$$

- Geostrophic solution incompatible with no-slip boundaries.
- Evidence for (Ekman) boundary layer solution s.t.

$$\boldsymbol{u}_\perp^{(c)} = \boldsymbol{u}_\perp^{(g)}(0) + \boldsymbol{u}_\perp^{(e)}(\mu)$$

*zero on bnd* *exponentially decaying into interior*

- Matches solution at bounding plate
- Coriolis viscous balance implies $\delta_e \sim (\nu/2\Omega)^{1/2}$

EBL solutions

$$u_0^{(e)}(\boldsymbol{x}_\perp,\mu,t) = -e^{-\frac{\mu}{\sqrt{2}}}\left(u_0^{(g)}(\boldsymbol{x}_\perp,0,t)\cos\frac{\mu}{\sqrt{2}} - u_0^{(g)}(\boldsymbol{x}_\perp,0,t)\sin\frac{\mu}{\sqrt{2}}\right)$$

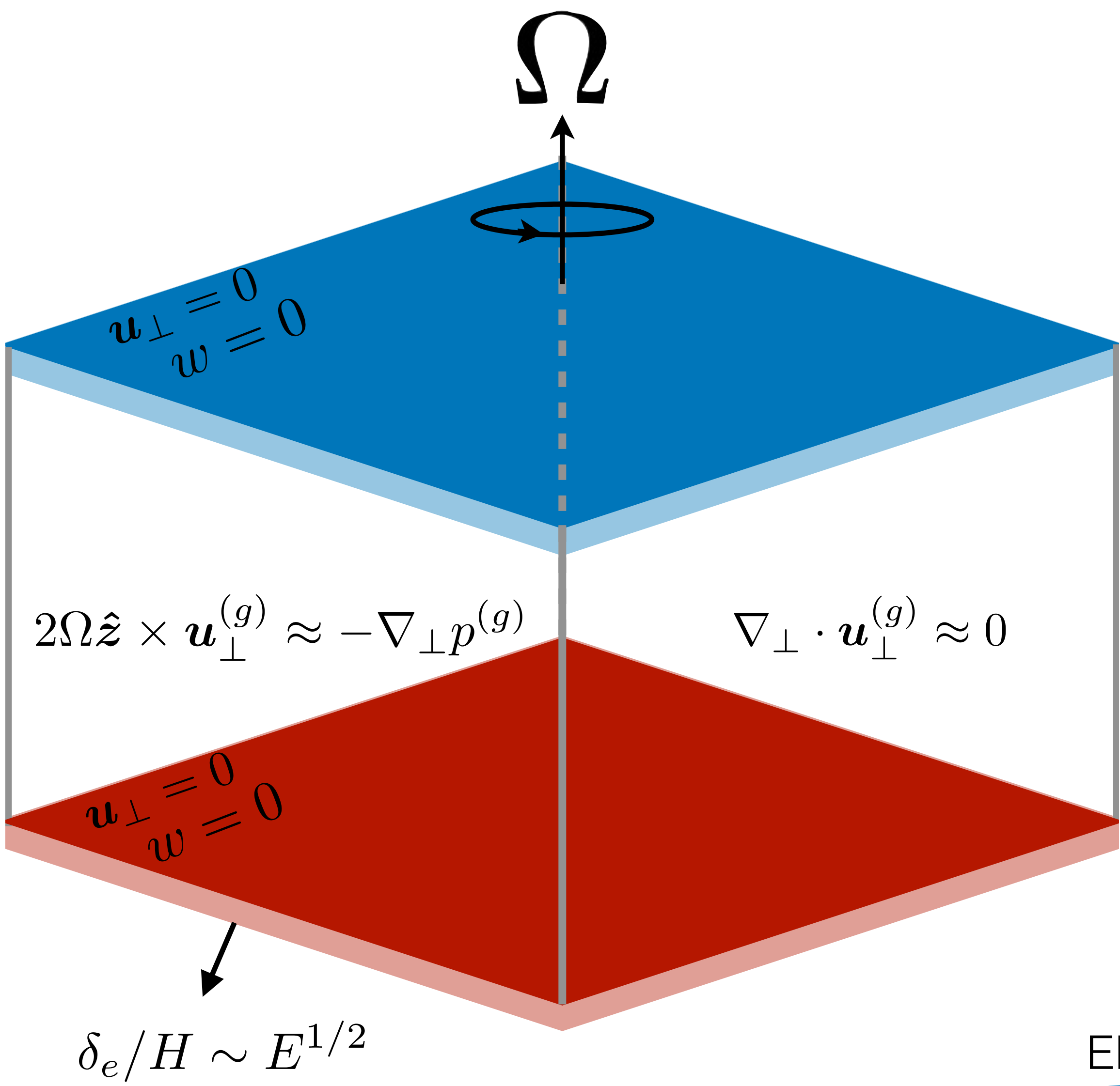
$$v_0^{(e)}(\boldsymbol{x}_\perp,\mu,t) = -e^{-\frac{\mu}{\sqrt{2}}}\left(v_0^{(g)}(\boldsymbol{x}_\perp,0,t)\cos\frac{\mu}{\sqrt{2}} + v_0^{(g)}(\boldsymbol{x}_\perp,0,t)\sin\frac{\mu}{\sqrt{2}}\right)$$

Ekman pumping and suction

$$w^{(e)}(\boldsymbol{x}_\perp,\mu,t) = \frac{E^{1/6}}{\sqrt{2}}\zeta_0^{(g)}(\boldsymbol{x}_\perp,0,t)\left(1 - e^{-\frac{\mu}{\sqrt{2}}}\left[\cos\frac{\mu}{\sqrt{2}} + \sin\frac{\mu}{\sqrt{2}}\right]\right)$$

*KJ et al JFM 2016*

# Linear Stability Theory: Sensitivity btw NS & SF BC's




- Geostrophic solution incompatible with no-slip boundaries.
- Evidence for (Ekman) boundary layer solution s.t.

$$\boldsymbol{u}_\perp^{(c)} = \boldsymbol{u}_\perp^{(g)}(0) + \boldsymbol{u}_\perp^{(e)}(\mu)$$

*zero on bnd* *exponentially decaying into interior*

- Matches solution at bounding plate
- Coriolis viscous balance implies $\delta_e \sim (\nu/2\Omega)^{1/2}$

$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(e)} \approx \nu\partial_{\mu\mu}\boldsymbol{u}_\perp^{(e)}$$

EBL solutions

$$u_0^{(e)}(\boldsymbol{x}_\perp,\mu,t) = -e^{-\frac{\mu}{\sqrt{2}}}\left(u_0^{(g)}(\boldsymbol{x}_\perp,0,t)\cos\frac{\mu}{\sqrt{2}} - u_0^{(g)}(\boldsymbol{x}_\perp,0,t)\sin\frac{\mu}{\sqrt{2}}\right)$$

$$v_0^{(e)}(\boldsymbol{x}_\perp,\mu,t) = -e^{-\frac{\mu}{\sqrt{2}}}\left(v_0^{(g)}(\boldsymbol{x}_\perp,0,t)\cos\frac{\mu}{\sqrt{2}} + v_0^{(g)}(\boldsymbol{x}_\perp,0,t)\sin\frac{\mu}{\sqrt{2}}\right)$$

As in atmos. and ocean applications filter Ekman layers by pumping/suction BC

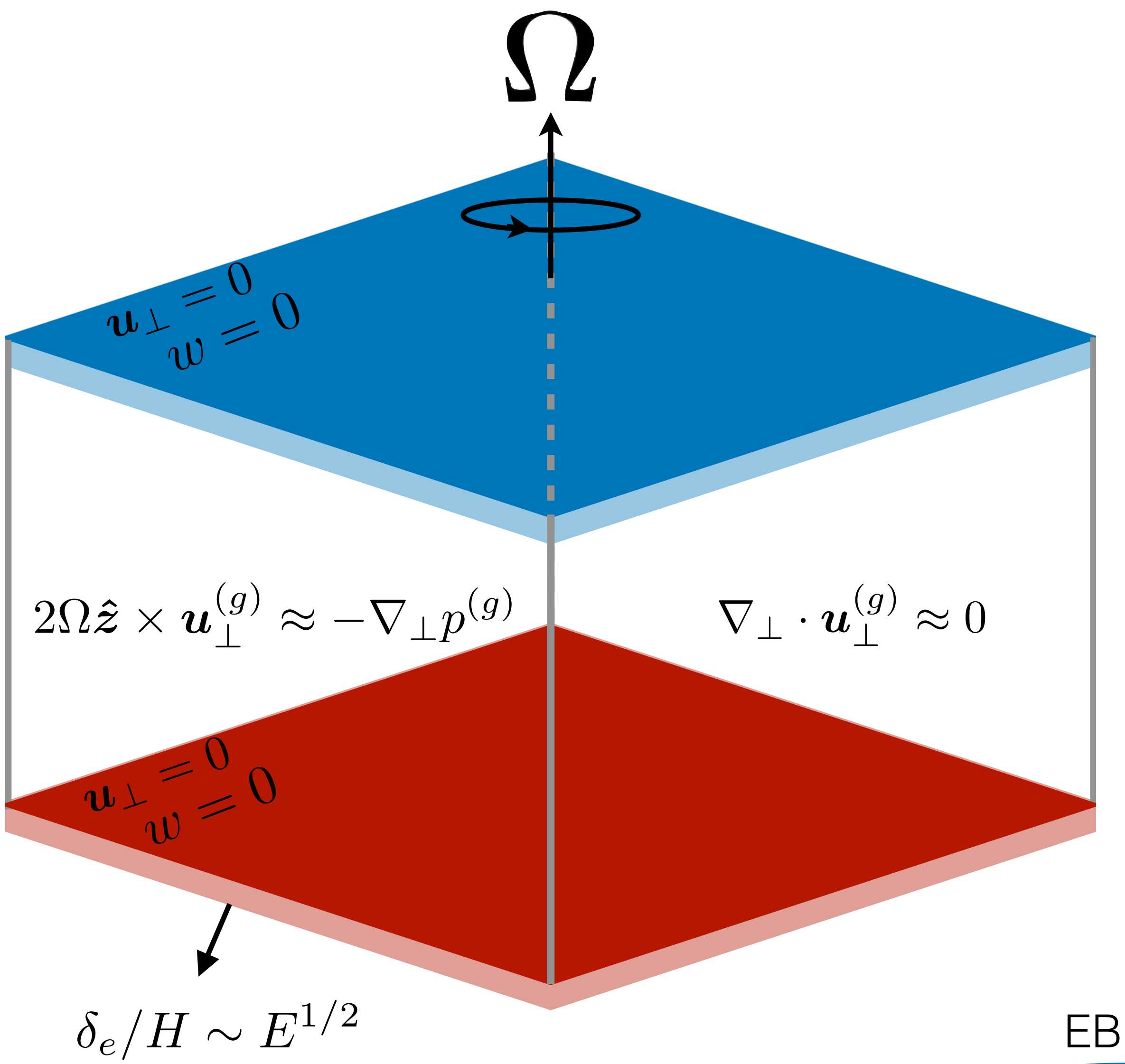
$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp\frac{E^{1//6}}{\sqrt{2}}\zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

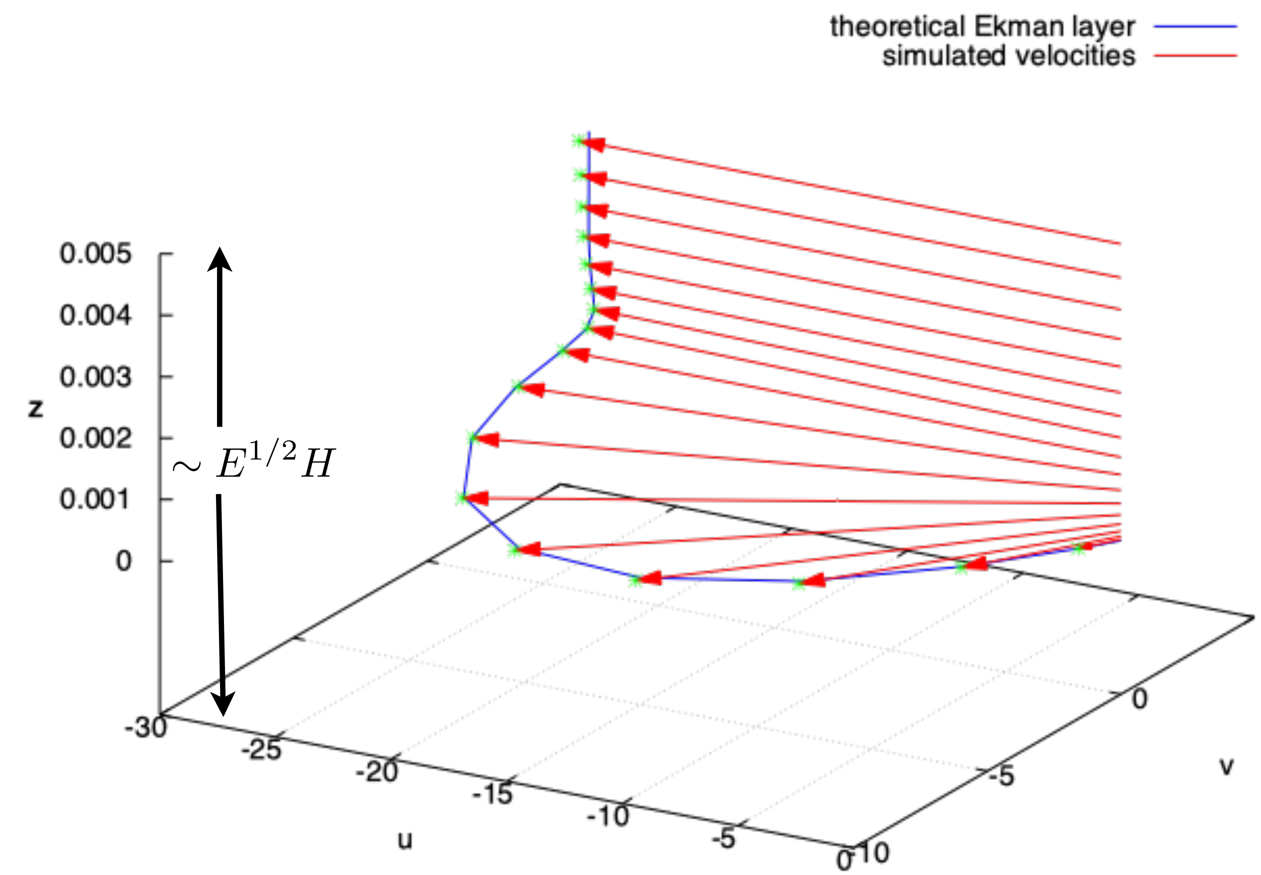
Ekman pumping and suction

$$w^{(e)}(\boldsymbol{x}_\perp,\mu,t) = \frac{E^{1/6}}{\sqrt{2}}\zeta_0^{(g)}(\boldsymbol{x}_\perp,0,t)\left(1 - e^{-\frac{\mu}{\sqrt{2}}}\left[\cos\frac{\mu}{\sqrt{2}} + \sin\frac{\mu}{\sqrt{2}}\right]\right)$$

*KJ et al JFM 2016*

# Linear Stability Theory: Sensitivity btw NS & SF BC's




$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(e)} \approx \nu\partial_{\mu\mu}\boldsymbol{u}_\perp^{(e)}$$

EBL solutions

$$u_0^{(e)}(\boldsymbol{x}_\perp,\mu,t) = -e^{-\frac{\mu}{\sqrt{2}}}\left(u_0^{(g)}(\boldsymbol{x}_\perp,0,t)\cos\frac{\mu}{\sqrt{2}} - u_0^{(g)}(\boldsymbol{x}_\perp,0,t)\sin\frac{\mu}{\sqrt{2}}\right)$$

$$v_0^{(e)}(\boldsymbol{x}_\perp,\mu,t) = -e^{-\frac{\mu}{\sqrt{2}}}\left(v_0^{(g)}(\boldsymbol{x}_\perp,0,t)\cos\frac{\mu}{\sqrt{2}} + v_0^{(g)}(\boldsymbol{x}_\perp,0,t)\sin\frac{\mu}{\sqrt{2}}\right)$$

As in atmos. and ocean applications filter Ekman layers by pumping/suction BC

$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp\frac{E^{1//6}}{\sqrt{2}}\zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

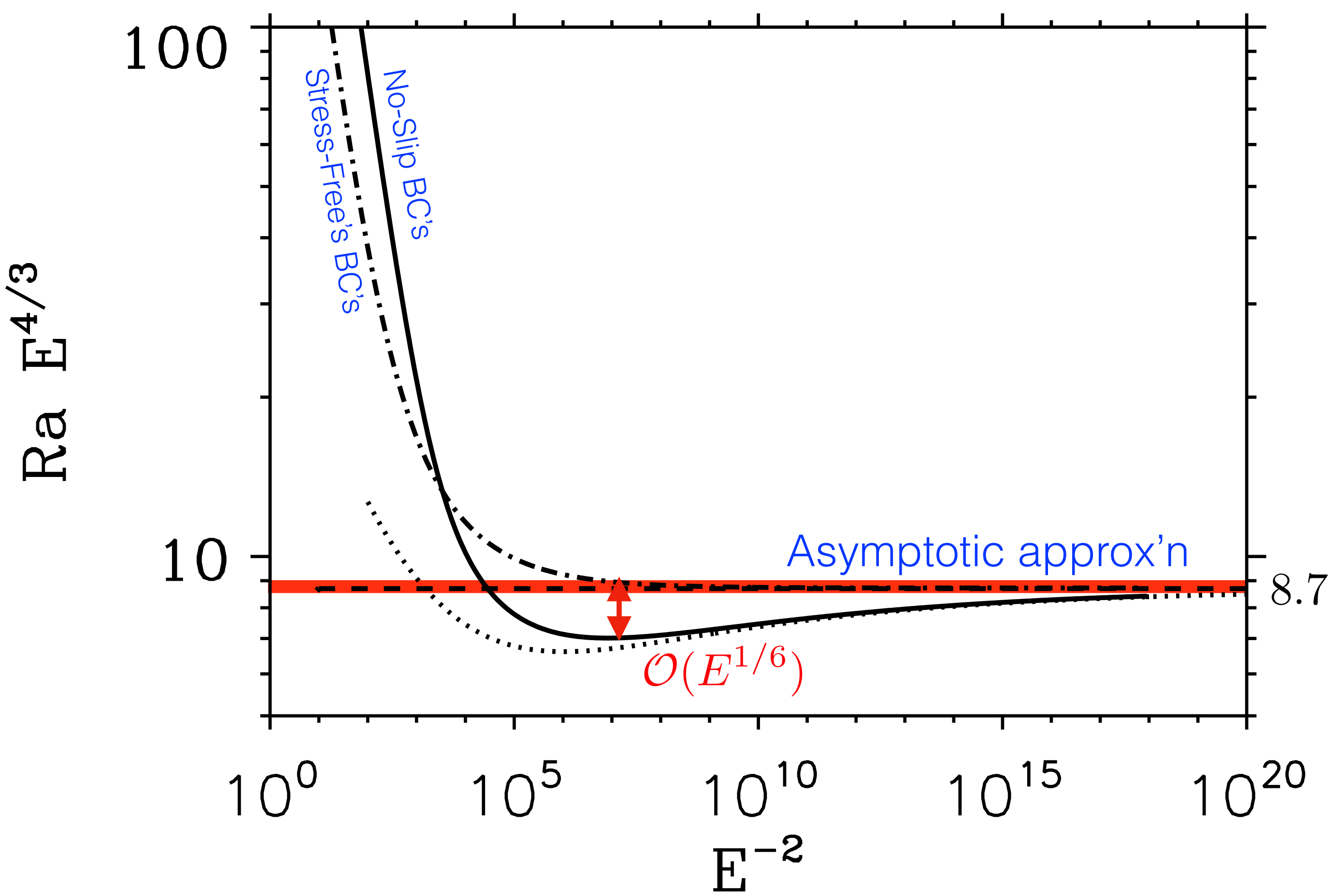
Ekman pumping and suction

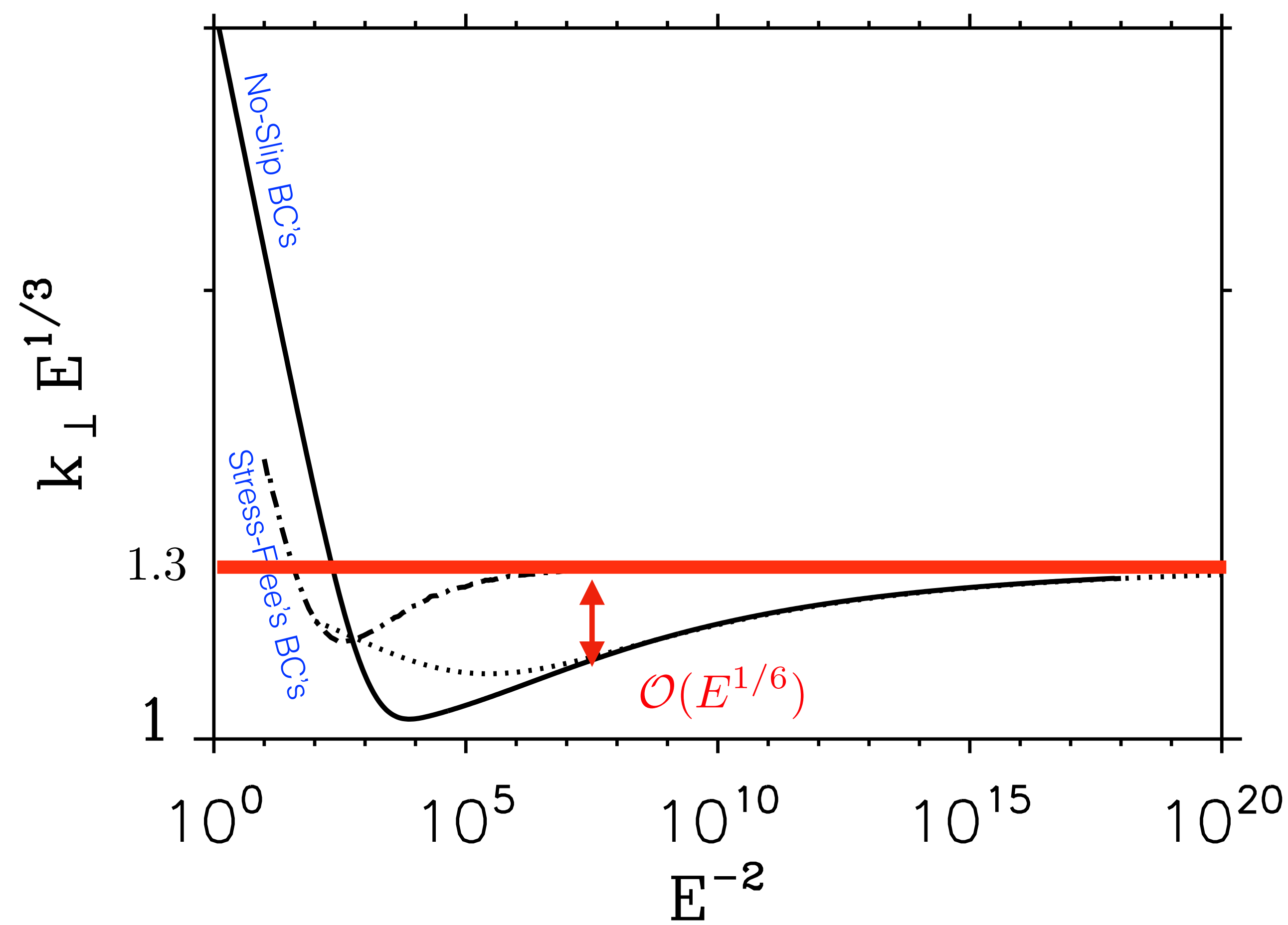
$$w^{(e)}(\boldsymbol{x}_\perp,\mu,t) = \frac{E^{1/6}}{\sqrt{2}}\zeta_0^{(g)}(\boldsymbol{x}_\perp,0,t)\left(1 - e^{-\frac{\mu}{\sqrt{2}}}\left[\cos\frac{\mu}{\sqrt{2}} + \sin\frac{\mu}{\sqrt{2}}\right]\right)$$

*KJ et al JFM 2016*

# Convergence to Asymptotic Limit

$$w^{(e)}(\boldsymbol{x}_\perp, Z_\pm, t) = \mp \frac{E^{1//6}}{\sqrt{2}} \zeta_0^{(g)}(\boldsymbol{x}_\perp, Z_\pm, t)$$

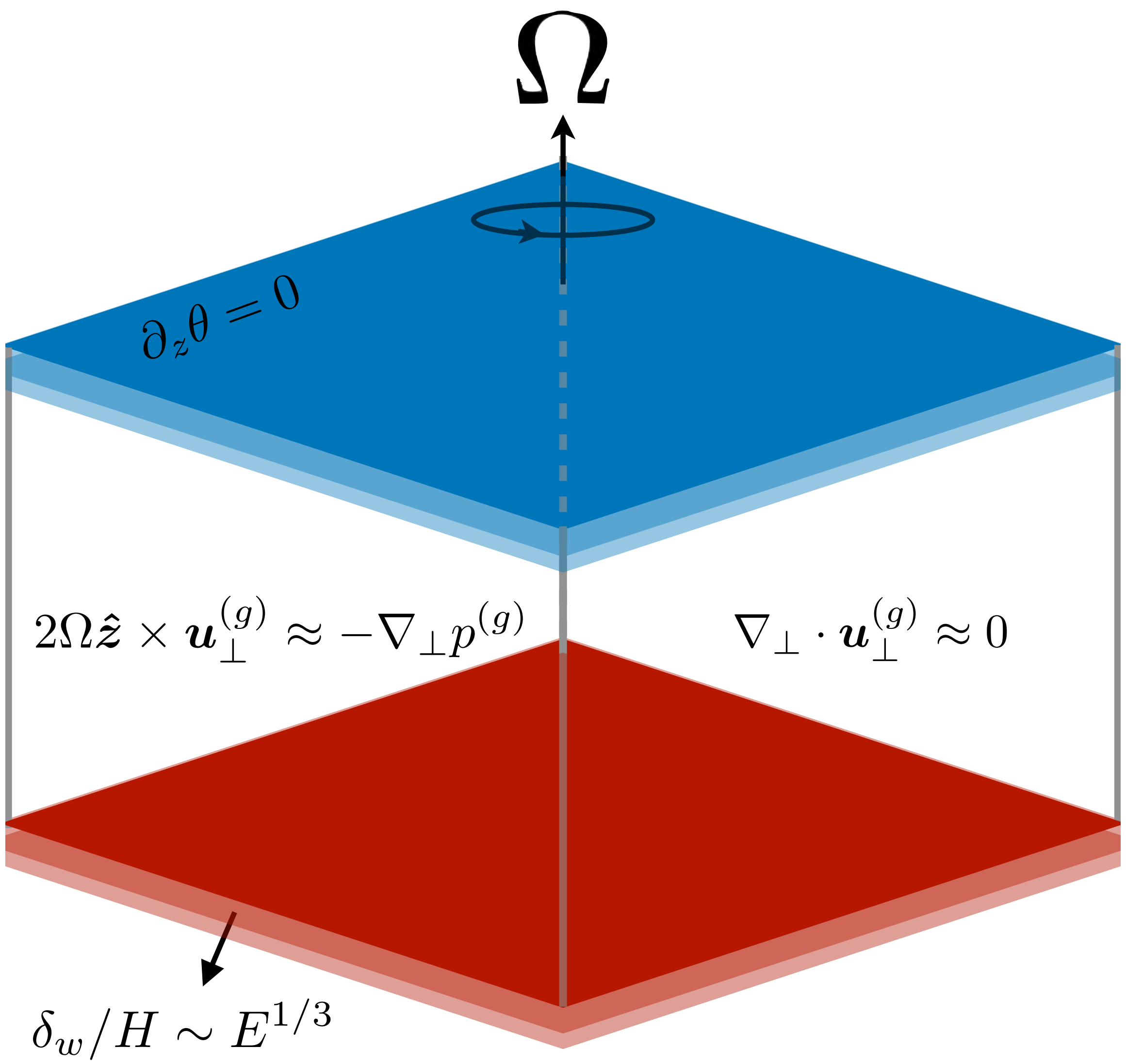



- Ekman pumping/suction accounts for the slow convergence
- Complete asymptotic picture for linear dynamics.

J. et al JFM '16

# Linear Stability Theory: Sensitivity btw FT-FF BC's




$$2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{(w)} \approx -\nabla p^{(w)} + g\alpha\theta^{(w)}\hat{\boldsymbol{z}}$$
$$\nabla_\perp \cdot \boldsymbol{u}_\perp^{w} \approx 0$$

- Evidence for thermal (wind) boundary layer solution s.t.

$$\partial_z\theta^{(c)} = \partial_Z\theta^{(g)}(0) + \partial_\eta\, \theta^{(w)}(\eta) \sim \mathcal{O}(E^{1/3}), \quad \theta^{(w)}(\eta) \sim E^{2/3}$$

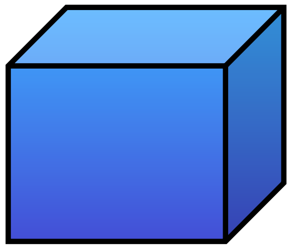
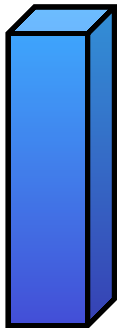
*zero on bnd* *exponentially decaying into interior*

- Correction drives thermal wind BL

# Suggestion of  a NL Asymptotic Theory

*QG Derivation - Unified View*

# QG System (*Ro << 1) Unified View*

| Hydrostatic-QG | Hydrostatic-QG | Non-Hydrostatic-QG |
|---|---|---|
|  |  |  |
| $A = \dfrac{\ell}{H} \gg 1$ | $\dfrac{\ell}{H} \sim 1$ | $\dfrac{\ell}{H} \ll 1$ |
| Charney (1948) | Embid & Majda (1998) | Julien et al (2006) |

All cases must satisfy a leading order geostrophic balance

$$2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -\nabla p \quad \Rightarrow \quad \frac{1}{Ro} \hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu \nabla p \quad \Rightarrow \quad Eu = \frac{1}{Ro}$$

$$\Downarrow$$

$$p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$$

# Governing Equations (dimensional)

## Incompressible Navier-Stokes Equations (iNSE): buoyantly driven

$$\left(D_t^{\perp} + w\partial_z\right)' \boldsymbol{u}_\perp + 2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{\rho_0}\nabla_\perp p' + \nu\left(\nabla_\perp^2 + \partial_{zz}\right)\boldsymbol{u}_\perp \quad \text{h. mtm}$$

$$\left(D_t^{\perp} + w\partial_z\right)' w = -\frac{1}{\rho_0}\partial_z p' + g\alpha\vartheta'\hat{\boldsymbol{z}} + \nu\left(\nabla_\perp^2 + \partial_{zz}\right) w \quad \text{v. mtm}$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + \partial_z w = 0 \quad \text{mass cons.}$$

$$\left(D_t^{\perp} + w\partial_z\right)' \vartheta' + w\partial_z\overline{T} = \kappa\left(\nabla_\perp^2 + \partial_{zz}\right)\vartheta' \quad \text{fluct. temp}$$

$$\partial_t\overline{T} + \partial_z\left(\overline{w\vartheta'}\right) = \kappa\partial_{zz}\overline{T} \quad \text{mean temp}$$

Horizontal material advection: $D_t^{\perp} = \partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp$

Assume anisotropy btw horizontal & vertical

$$\boldsymbol{u}_\perp \sim [U], \quad \boldsymbol{x}_\perp \sim [\ell]$$
$$w \sim [W], \quad z \sim [H]$$

Characteristic Thermodynamic scales

$$p \sim [P], \quad \overline{T} \sim [\Delta T], \quad \vartheta' \sim [\vartheta^*]$$

# Governing Equations (non-dimensional)

## Non-dimensional iNSE:

$$\left(D_t^\perp + A\delta_w w\partial_z\right)' \boldsymbol{u}_\perp + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{Ro}\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\boldsymbol{u}_\perp$$ h. mtm

$$\delta_w\left(D_t^\perp + A\delta_w w\partial_z\right)' w = -EuA\partial_z p' + \Gamma\delta_\vartheta\vartheta'\hat{\boldsymbol{z}} + \frac{1}{Re}\left(\nabla_\perp^2 + A^2\partial_{zz}\right) w$$ v. mtm

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + A\delta_w\partial_z w = 0$$ mass cons.

$$\delta_\vartheta\left(D_t^\perp + A\delta_w w\partial_z\right)' \vartheta' + w\partial_z\overline{T} = \frac{\delta_\vartheta}{Pe}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\vartheta'$$ fluct. temp

$$\partial_t\overline{T} + A\delta_\vartheta\delta_w\partial_z\left(\overline{w\vartheta'}\right) = \frac{A^2}{Pe}\partial_{zz}\overline{T}$$ mean temp

## Non-dimensional parameters:

$$Ro = \frac{U}{2\Omega\ell}, \quad Eu = \frac{P}{\rho_0 U^2}, \quad \Gamma = \frac{g\alpha\Delta T\ell}{U^2}, \quad Re = \frac{U\ell}{\nu}, \quad Pe = \frac{U\ell}{\kappa}$$

Rossby | Euler | Buoyancy | Reynolds | Peclet

## Anisotropy parameters:

$$\delta_w = \frac{W}{U}, \quad \delta_\vartheta = \frac{\vartheta^*}{\Delta T}, \quad A = \frac{\ell}{H}$$

# Governing Equations (non-dimensional)

Non-dimensional iNSE:

$$\left(D_t^\perp + A\delta_w w\partial_z\right)' \boldsymbol{u}_\perp + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{Ro}\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\boldsymbol{u}_\perp$$

h. mtm

$$\delta_w\left(D_t^\perp + A\delta_w w\partial_z\right)' w = -EuA\partial_z p' + \Gamma\delta_\vartheta\vartheta'\hat{\boldsymbol{z}} + \frac{1}{Re}\left(\nabla_\perp^2 + A^2\partial_{zz}\right) w$$

v. mtm

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + A\delta_w\partial_z w = 0$$

mass cons.

$$\delta_\vartheta\left(D_t^\perp + A\delta_w w\partial_z\right)'\vartheta' + w\partial_z\overline{T} = \frac{\delta_\vartheta}{Pe}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\vartheta'$$

fluct. temp

$$\partial_t\overline{T} + A\delta_\vartheta\delta_w\partial_z\left(\overline{w\vartheta'}\right) = \frac{A^2}{Pe}\partial_{zz}\overline{T}$$

mean temp

Non-dimensional parameters:

$$Ro = \frac{U}{2\Omega\ell}, \quad Eu = \frac{P}{\rho_0 U^2}, \quad \Gamma = \frac{g\alpha\Delta T\ell}{U^2}, \quad Re = \frac{U\ell}{\nu}, \quad Pe = \frac{U\ell}{\kappa}$$

Rossby | Euler | Buoyancy | Reynolds | Peclet

Anisotropy parameters:

$$\delta_w = \frac{W}{U}, \quad \delta_\vartheta = \frac{\vartheta^*}{\Delta T}, \quad A = \frac{\ell}{H}$$

# Governing Equations (Energetic Scales)

Non-dimensional iNSE:

$$\left(D_t^{\perp} + A\delta_w w \partial_z\right)' \boldsymbol{u}_{\perp} + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp} = -\frac{1}{Ro}\nabla_{\perp} p' + \frac{1}{Re}\left(\nabla_{\perp}^2 + A^2 \partial_{zz}\right)\boldsymbol{u}_{\perp}$$ h. mtm

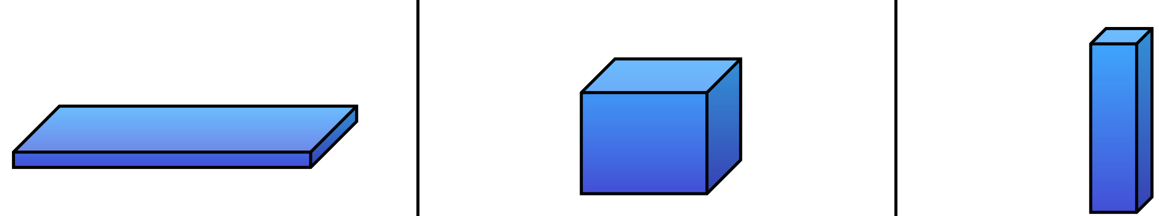
$$\delta_w \left(D_t^{\perp} + A\delta_w w \partial_z\right)' w = -\frac{A}{Ro}\partial_z p' + \Gamma \delta_\vartheta \vartheta' \hat{\boldsymbol{z}} + \frac{1}{Re}\left(\nabla_{\perp}^2 + A^2 \partial_{zz}\right) w$$ v. mtm

$$\nabla_{\perp} \cdot \boldsymbol{u}_{\perp} + A\delta_w \partial_z w = 0$$ mass cons.

$$\delta_\vartheta \left(D_t^{\perp} + A\delta_w w \partial_z\right)' \vartheta' + A\delta_w w \partial_z \overline{T} = \frac{\delta_\vartheta}{Pe}\left(\nabla_{\perp}^2 + A^2 \partial_{zz}\right)\vartheta'$$ fluct. temp

$$\partial_t \overline{T} + A\delta_\vartheta \delta_w \partial_z \left(\overline{w\vartheta'}\right) = \frac{A^2}{Pe}\partial_{zz}\overline{T}$$ mean temp

Aspect ratio of energetic scales:



$A =$ $\frac{\ell}{H} \gg 1$ | $\frac{\ell}{H} \sim 1$ | $\frac{\ell}{H} \ll 1$

# Governing Equations (Energetic Scales)

Non-dimensional iNSE:

$$\left(D_t^{\perp} + A\delta_w w \partial_z\right)' \boldsymbol{u}_\perp + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{Ro}\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + A^2 \partial_{zz}\right)\boldsymbol{u}_\perp \qquad \text{h. mtm}$$

$$\delta_w \left(D_t^{\perp} + A\delta_w w \partial_z\right)' w = -\frac{A}{Ro}\partial_z p' + \Gamma \delta_\vartheta \vartheta' \hat{\boldsymbol{z}} + \frac{1}{Re}\left(\nabla_\perp^2 + A^2 \partial_{zz}\right) w \qquad \text{v. mtm}$$

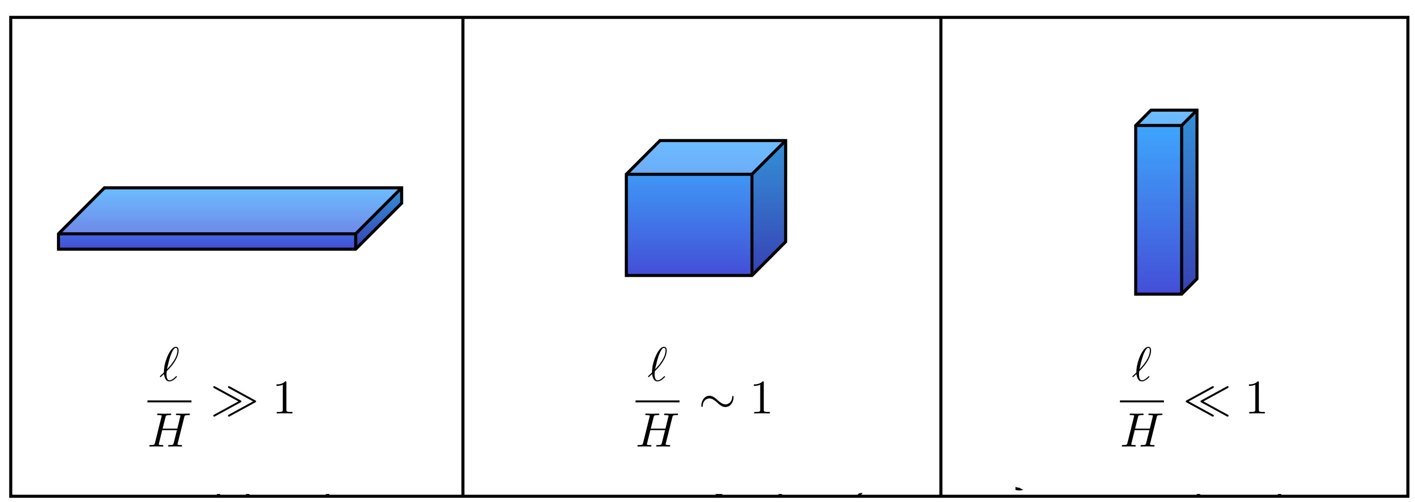
$$\nabla_\perp \cdot \boldsymbol{u}_\perp + A\delta_w \partial_z w = 0 \qquad \text{mass cons.}$$

$$\delta_\vartheta \left(D_t^{\perp} + A\delta_w w \partial_z\right)' \vartheta' + A\delta_w w \partial_z \overline{T} = \frac{\delta_\vartheta}{Pe}\left(\nabla_\perp^2 + A^2 \partial_{zz}\right)\vartheta' \qquad \text{fluct. temp}$$

$$\partial_t \overline{T} + A\delta_\vartheta \delta_w \partial_z \left(\overline{w\vartheta'}\right) = \frac{A^2}{Pe}\partial_{zz}\overline{T} \qquad \text{mean temp}$$

Aspect ratio of energetic scales:

$A =$




?? For arbitrary *A* determine the criteria for which the dynamics remains rotationally constrained!!!

# Governing Equations (Geostrophy)

Non-dimensional iNSE:

$$\left(D_t^{\perp} + A\delta_w w\partial_z\right)' \boldsymbol{u}_\perp + \boxed{\frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{Ro}\nabla_\perp p'} + \frac{1}{Re}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\boldsymbol{u}_\perp \qquad \text{h. mtm}$$

$$\delta_w\left(D_t^{\perp} + A\delta_w w\partial_z\right)' w = -\frac{A}{Ro}\partial_z p' + \Gamma\delta_\vartheta\vartheta'\hat{\boldsymbol{z}} + \frac{1}{Re}\left(\nabla_\perp^2 + A^2\partial_{zz}\right) w \qquad \text{v. mtm}$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + A\delta_w\partial_z w = 0 \qquad \text{mass cons.}$$

$$\delta_\vartheta\left(D_t^{\perp} + A\delta_w w\partial_z\right)' \vartheta' + A\delta_w w\partial_z\overline{T} = \frac{\delta_\vartheta}{Pe}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\vartheta' \qquad \text{fluct. temp}$$

$$\partial_t\overline{T} + A\delta_\vartheta\delta_w\partial_z\left(\overline{w\vartheta'}\right) = \frac{A^2}{Pe}\partial_{zz}\overline{T} \qquad \text{mean temp}$$

Geostrophy enforced as the dominant balance?

?? For arbitrary *A* determine the criteria for Quasi-Geostrophic dynamics!!!

Find the distinguished limits - (relationship to *Ro*)

Assess the hallmark characteristics of QG dynamics

# Return to Governing Equations (dimensional)

Incompressible Navier-Stokes Equations (iNSE): buoyantly driven

$$\left(D_t^{\perp} + w\partial_z\right)' \boldsymbol{u}_\perp + 2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{\rho_0}\nabla_\perp p' + \nu\left(\nabla_\perp^2 + \partial_{zz}\right)\boldsymbol{u}_\perp$$ h. mtm

$$\left(D_t^{\perp} + w\partial_z\right)' w = -\frac{1}{\rho_0}\partial_z p' + g\alpha\vartheta'\hat{\boldsymbol{z}} + \nu\left(\nabla_\perp^2 + \partial_{zz}\right) w$$ v. mtm

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + \partial_z w = 0$$ mass cons.

$$\left(D_t^{\perp} + w\partial_z\right)' \vartheta' + w\partial_z \overline{T} = \kappa\left(\nabla_\perp^2 + \partial_{zz}\right)\vartheta'$$ fluct. temp

$$\partial_t \overline{T} + \partial_z\left(\overline{w\vartheta'}\right) = \kappa\partial_{zz}\overline{T}$$ mean temp

Horizontal material advection: $D_t^{\perp} = \partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp$

Assess the hallmark characteristics of QG dynamics

# Hallmarks of QG Theory

Coriolis-Inertia-Archemedean (CIA) balance

*Aurnou, Julien & Horn PRR 2020*

# Hallmarks of QG Theory (*Ro<<1*)

Coriolis-Inertia-Archemedean (CIA) balance

a) C~I: Axial planetary vortex stretching balanced by vortical advection

$$\nabla \times \left(D_t^{\perp} \boldsymbol{u}_{\perp}\right) \sim \nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp})$$

$$\frac{U^2}{\ell^2} \sim \frac{2\Omega W}{H} \quad \Longrightarrow \quad \delta_w \sim \frac{Ro}{A}$$

*Aurnou, Julien & Horn PRR 2020*

# Hallmarks of QG Theory

Coriolis-Inertia-Archemedean (CIA) balance

a) C~I: Axial planetary vortex stretching balanced by vortical advection

$$\nabla \times \left(D_t^{\perp} \boldsymbol{u}_{\perp}\right) \sim \nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp})$$

$$\frac{U^2}{\ell^2} \sim \frac{2\Omega W}{H} \implies \delta_w \sim \frac{Ro}{A}$$

b) C~A: Axial planetary vortex stretching balances buoyant forcing

$$\nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp}) \sim \nabla \times (g\alpha\vartheta \hat{\boldsymbol{z}})$$

$$\frac{2\Omega U}{H} \sim \frac{g\alpha\vartheta^*}{\ell} \implies \Gamma\delta_\vartheta \sim \frac{A}{Ro}$$

*Aurnou, Julien & Horn PRR 2020*

# Hallmarks of QG Theory

Coriolis-Inertia-Archemedean (CIA) balance

a) C~I: Axial planetary vortex stretching balanced by vortical advection

$$\nabla \times \left(D_t^{\perp} \boldsymbol{u}_{\perp}\right) \sim \nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp})$$
$$\frac{U^2}{\ell^2} \sim \frac{2\Omega W}{H} \quad \Longrightarrow \quad \delta_w \sim \frac{Ro}{A}$$

b) C~A: Axial planetary vortex stretching balances buoyant forcing

$$\nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp}) \sim \nabla \times (g\alpha\vartheta \hat{\boldsymbol{z}})$$
$$\frac{2\Omega U}{H} \sim \frac{g\alpha\vartheta^*}{\ell} \quad \Longrightarrow \quad \Gamma\delta_{\vartheta} \sim \frac{A}{Ro}$$

c) buoyant tendencies driven by advection of mean temperature

$$D_t^{\perp}\vartheta \sim w\partial_z \overline{T}$$
$$\frac{U\vartheta^*}{\ell} \sim \frac{W\Delta T}{H} \quad \Longrightarrow \quad \delta_{\vartheta} \sim \delta_w A \sim Ro$$

*Aurnou, Julien & Horn PRR 2020*

# Hallmarks of QG Theory

Coriolis-Inertia-Archemedean (CIA) balance

a) C~I: Axial planetary vortex stretching balanced by vortical advection

$$\nabla \times \left(D_t^{\perp} \boldsymbol{u}_{\perp}\right) \sim \nabla \times \left(2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp}\right)$$
$$\frac{U^2}{\ell^2} \sim \frac{2\Omega W}{H} \quad \Longrightarrow \quad \delta_w \sim \frac{Ro}{A}$$

b) C~A: Axial planetary vortex stretching balances buoyant forcing

$$\nabla \times \left(2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp}\right) \sim \nabla \times \left(g\alpha\vartheta \hat{\boldsymbol{z}}\right)$$
$$\frac{2\Omega U}{H} \sim \frac{g\alpha\vartheta^*}{\ell} \quad \Longrightarrow \quad \Gamma\delta_\vartheta \sim \frac{A}{Ro}$$

c) buoyant tendencies driven by advection of mean temperature

$$D_t^{\perp}\vartheta \sim w\partial_z \overline{T}$$
$$\frac{U\vartheta^*}{\ell} \sim \frac{W\Delta T}{H} \quad \Longrightarrow \quad \delta_\vartheta \sim \delta_w A \sim Ro$$

d) ** buoyancy flux feedback to mean temperature **

$$\partial_z \left(w\vartheta'\right) \sim \kappa \partial_{zz} \overline{T}$$
$$\frac{w\vartheta^*}{H} \sim \frac{\kappa \Delta T}{H^2} \quad \Longrightarrow \quad \delta_\vartheta \delta_w \sim \frac{A}{Pe}$$

*Aurnou, Julien & Horn PRR 2020*

# Hallmarks of QG Theory (Distinguish limits determined)

## Coriolis-Inertia-Archemedean (CIA) balance

a) C~I: Axial planetary vortex stretching balanced by vortical advection

$$\nabla \times \left(D_t^{\perp} \boldsymbol{u}_{\perp}\right) \sim \nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp})$$
$$\frac{U^2}{\ell^2} \sim \frac{2\Omega W}{H} \implies \delta_w \sim \frac{Ro}{A}$$

b) C~A: Axial planetary vortex stretching balances buoyant forcing

$$\nabla \times (2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp}) \sim \nabla \times (g\alpha\vartheta \hat{\boldsymbol{z}})$$
$$\frac{2\Omega U}{H} \sim \frac{g\alpha\vartheta^*}{\ell} \implies \Gamma \sim \frac{A}{Ro^2}$$

c) buoyant tendencies driven by advection of mean temperature

$$D_t^{\perp}\vartheta \sim w\partial_z \overline{T}$$
$$\frac{U\vartheta^*}{\ell} \sim \frac{W\Delta T}{H} \implies \delta_\vartheta \sim Ro$$

d) ** buoyancy flux feedback to mean temperature **

$$\partial_z \left(w\vartheta'\right) \sim \kappa \partial_{zz} \overline{T}$$
$$\frac{w\vartheta^*}{H} \sim \frac{\kappa \Delta T}{H^2} \implies Pe, Re \sim \frac{A^2}{Ro^2}$$

*Aurnou, Julien & Horn PRR 2020*

# QG Observations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi,\ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}},\ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$\left(D_t^\perp + Ro\, w\partial_z\right)' \boldsymbol{u}_\perp + \boxed{\frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\frac{1}{Ro}\nabla_\perp p'} + \frac{1}{\widetilde{Re}}\frac{Ro^2}{A^2}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\boldsymbol{u}_\perp$$ h. mtm

$$\frac{Ro^2}{A^2}\left(D_t^\perp + Ro\, w\partial_z\right)' w = -\partial_z p' + \widetilde{\Gamma}\vartheta' + \frac{1}{\widetilde{Re}}\frac{Ro^4}{A^4}\left(\nabla_\perp^2 + A^2\partial_{zz}\right) w$$ v. mtm

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + Ro\,\partial_z w = 0$$ mass cons.

$$\left(D_t^\perp + Ro\, w\partial_z\right)' \vartheta' + w\partial_z \overline{T} = \frac{1}{\widetilde{Pe}}\frac{Ro^2}{A^2}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\vartheta'$$ fluct. temp

$$\frac{1}{Ro^2}\partial_t \overline{T} + \partial_z\left(\overline{w\vartheta'}\right) = \frac{1}{\widetilde{Pe}}\partial_{zz}\overline{T}$$ mean temp

Vertical advection subdominant

# QG Observations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \;\; \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \;\; \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \boldsymbol{u}_\perp + \boxed{\frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp \approx -\frac{1}{Ro}\nabla_\perp p'} + \frac{1}{\widetilde{Re}}\frac{Ro^2}{A^2}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\boldsymbol{u}_\perp \qquad \text{h. mtm}$$

$$\frac{Ro^2}{A^2}D_t^\perp w \approx -\partial_z p' + \widetilde{\Gamma}\vartheta' + \frac{1}{\widetilde{Re}}\frac{Ro^4}{A^4}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)w \qquad \text{v. mtm}$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + Ro\,\partial_z w = 0 \qquad \text{mass cons.}$$

$$D_t^\perp \vartheta' + w\partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}}\frac{Ro^2}{A^2}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\vartheta' \qquad \text{fluct. temp}$$

$$\boxed{\frac{1}{Ro^2}\partial_t \overline{T}} + \partial_z\left(\overline{w\vartheta'}\right) = \frac{1}{\widetilde{Pe}}\partial_{zz}\overline{T} \qquad \text{mean temp}$$

Vertical advection subdominant

Mean dynamics evolves on slow timescale

# QG Observations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \boldsymbol{u}_\perp + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp \approx -\frac{1}{Ro}\nabla_\perp p' + \frac{1}{\widetilde{Re}}\frac{Ro^2}{A^2}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\boldsymbol{u}_\perp$$ h. mtm

$$\frac{Ro^2}{A^2}D_t^\perp w \approx -\partial_z p' + \widetilde{\Gamma}\vartheta' + \frac{1}{\widetilde{Re}}\frac{Ro^4}{A^4}\left(\nabla_\perp^2 + A^2\partial_{zz}\right) w$$ v. mtm

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + Ro\,\partial_z w = 0$$ mass cons.

$$D_t^\perp \vartheta' + w\partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}}\frac{Ro^2}{A^2}\left(\nabla_\perp^2 + A^2\partial_{zz}\right)\vartheta'$$ fluct. temp

$$\frac{1}{Ro^2}\partial_t \overline{T} + \partial_z\left(\overline{w\vartheta'}\right) = \frac{1}{\widetilde{Pe}}\partial_{zz}\overline{T}$$ mean temp

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn

# QG Equations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \zeta - \partial_z w \approx \frac{1}{\widetilde{Re}} \frac{Ro^2}{A^2} \left( \nabla_\perp^2 + A^2 \partial_{zz} \right) \zeta$$ z. vort

$$\frac{Ro^2}{A^2} D_t^\perp w \approx -\partial_z \psi + \widetilde{\Gamma} \vartheta' + \frac{1}{\widetilde{Re}} \frac{Ro^4}{A^4} \left( \nabla_\perp^2 + A^2 \partial_{zz} \right) w$$ z.mtm

$$D_t^\perp \vartheta' + w \partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}} \frac{Ro^2}{A^2} \left( \nabla_\perp^2 + A^2 \partial_{zz} \right) \vartheta'$$ fluct. temp

$$\frac{1}{Ro^2} \partial_t \overline{T} + \partial_z \left( \overline{w \vartheta'} \right) = \frac{1}{\widetilde{Pe}} \partial_{zz} \overline{T}$$ mean temp

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn

# QG Equations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \zeta - \partial_z w \approx \frac{1}{\widetilde{Re}} \frac{Ro^2}{A^2} \left(\nabla_\perp^2 + A^2 \partial_{zz}\right) \zeta$$ z. vort

$$\frac{Ro^2}{A^2} D_t^\perp w \approx -\partial_z \psi + \widetilde{\Gamma} \vartheta' + \frac{1}{\widetilde{Re}} \frac{Ro^4}{A^4} \left(\nabla_\perp^2 + A^2 \partial_{zz}\right) w$$ z.mtm

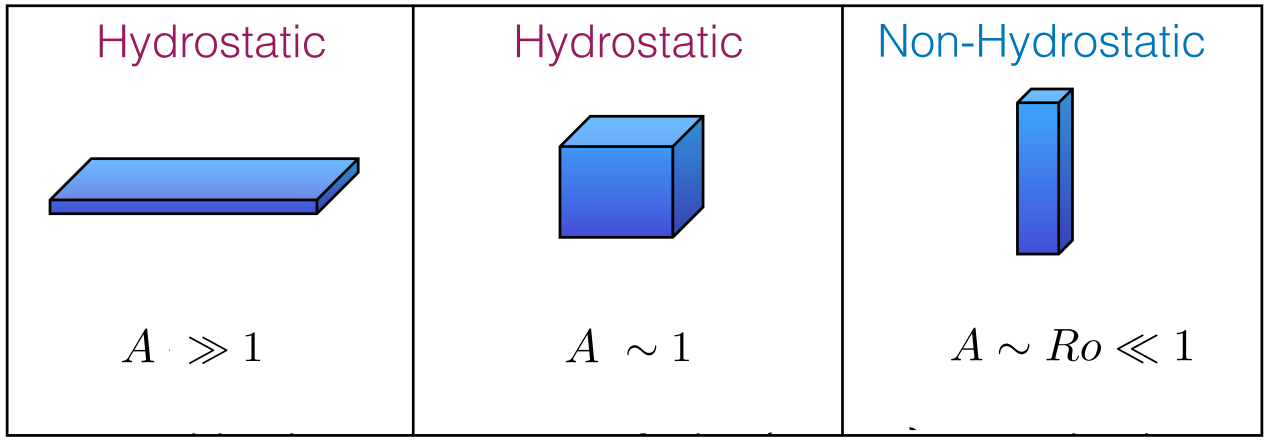
$$D_t^\perp \vartheta' + w \partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}} \frac{Ro^2}{A^2} \left(\nabla_\perp^2 + A^2 \partial_{zz}\right) \vartheta'$$ fluct. temp

$$\frac{1}{Ro^2} \partial_t \overline{T} + \partial_z \left(\overline{w \vartheta'}\right) = \frac{1}{\widetilde{Pe}} \partial_{zz} \overline{T}$$ mean temp

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn



# QG Equations for arbitrary $A$ and $Ro \ll 1$

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \zeta - \partial_z w \approx \frac{1}{\widetilde{Re}} \frac{Ro^2}{A^2} \left(\nabla_\perp^2 + A^2 \partial_{zz}\right) \zeta \qquad \text{z. vort}$$

$$\frac{Ro^2}{A^2} D_t^\perp w \approx \boxed{-\partial_z \psi + \widetilde{\Gamma} \vartheta'} + \frac{1}{\widetilde{Re}} \frac{Ro^4}{A^4} \left(\nabla_\perp^2 + A^2 \partial_{zz}\right) w \qquad \text{z.mtm}$$

$$D_t^\perp \vartheta' + w \partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}} \frac{Ro^2}{A^2} \left(\nabla_\perp^2 + A^2 \partial_{zz}\right) \vartheta' \qquad \text{fluct. temp}$$

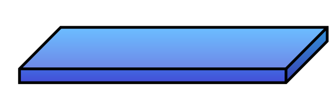
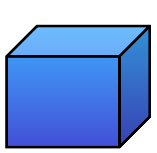
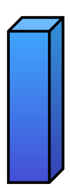
$$\frac{1}{Ro^2} \partial_t \overline{T} + \partial_z \left(\overline{w \vartheta'}\right) = \frac{1}{\widetilde{Pe}} \partial_{zz} \overline{T} \qquad \text{mean temp}$$

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn

Dynamics of energetic scales:

| Hydrostatic | Hydrostatic | Non-Hydrostatic |
|---|---|---|
|  |  |  |
| $A \gg 1$ | $A \sim 1$ | $A \sim Ro \ll 1$ |

# QG Equations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \zeta - \partial_z w \approx 0$$ z. vort

$$0 \approx -\partial_z \psi + \widetilde{\Gamma} \vartheta'$$ z.mtm

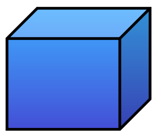
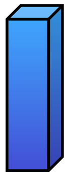
$$D_t^\perp \vartheta' + w \partial_z \overline{T} \approx 0$$ fluct. temp

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn

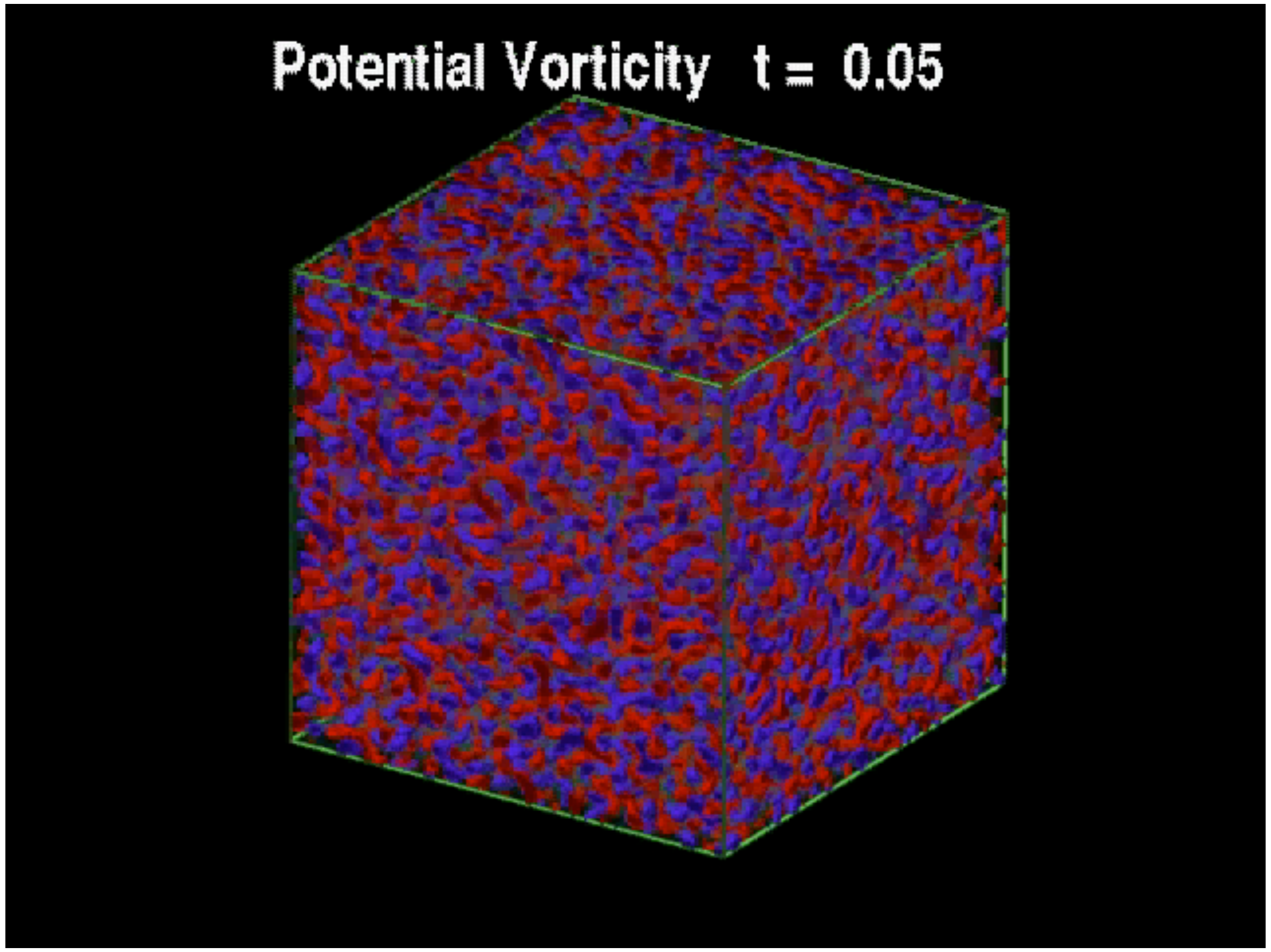
Dynamics of energetic scales:

| Hydrostatic | Hydrostatic | Non-Hydrostatic |
|---|---|---|
|  |  |  |
| $A \gg 1$ | $A \sim 1$ | $A \sim Ro \ll 1$ |

# QG Equations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:




$$D_t Q = 0, \quad Q = \left( \nabla_\perp^2 \psi + \partial_z \left( \frac{\partial_z \psi}{\partial_z \overline{\overline{T}}} \right) \right)$$

*Weiss & McWilliams 1994*

# QG Observations for arbitrary *A* and *Ro* << *1*

Geostrophy: $p = \psi, \ \boldsymbol{u}_\perp = -\nabla \times \psi \hat{\boldsymbol{z}}, \ \zeta = \nabla_\perp^2 \psi$

Non-dimensional iNSE:

$$D_t^\perp \boldsymbol{u}_\perp + \boxed{\frac{1}{Ro} \hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp \approx -\frac{1}{Ro} \nabla_\perp p'} + \frac{1}{\widetilde{Re}} \frac{Ro^2}{A^2} \left( \nabla_\perp^2 + A^2 \partial_{zz} \right) \boldsymbol{u}_\perp \qquad \text{h. mtm}$$

$$\frac{Ro^2}{A^2} D_t^\perp w \approx -\partial_z p' + \widetilde{\Gamma} \vartheta' + \frac{1}{\widetilde{Re}} \frac{Ro^4}{A^4} \left( \nabla_\perp^2 + A^2 \partial_{zz} \right) w \qquad \text{v. mtm}$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp + Ro \, \partial_z w = 0 \qquad \text{mass cons.}$$

$$D_t^\perp \vartheta' + w \partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}} \frac{Ro^2}{A^2} \left( \nabla_\perp^2 + A^2 \partial_{zz} \right) \vartheta' \qquad \text{fluct. temp}$$

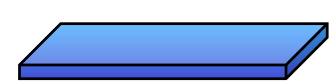
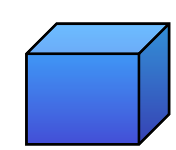
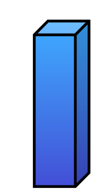
$$\boxed{\frac{1}{Ro^2} \partial_t \overline{T}} + \partial_z \left( \overline{w \vartheta'} \right) = \frac{1}{\widetilde{Pe}} \partial_{zz} \overline{T} \qquad \text{mean temp}$$

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn

Dynamics of energetic scales:

| Hydrostatic | Hydrostatic | Non-Hydrostatic |
| --- | --- | --- |
|  |  |  |
| $A \gg 1$ | $A \sim 1$ | $A \sim Ro \ll 1$ |

# QG Equations for arbitrary *A* and *Ro* << 1

Geostrophy: $p = \psi, \quad \boldsymbol{u}_\perp \approx -\nabla \times \psi \widehat{\boldsymbol{z}}$

Non-dimensional iNSE:

$$D_t^\perp \zeta - \partial_z w \approx \frac{1}{\widetilde{Re}} \nabla_\perp^2 \zeta$$ z. vort

$$D_t^\perp w \approx -\partial_z \psi + \widetilde{\Gamma} \vartheta' + \frac{1}{\widetilde{Re}} \nabla_\perp^2 w$$ z.mtm

$$D_t^\perp \vartheta' + w \partial_z \overline{T} \approx \frac{1}{\widetilde{Pe}} \nabla_\perp^2 \vartheta'$$ fluct. temp

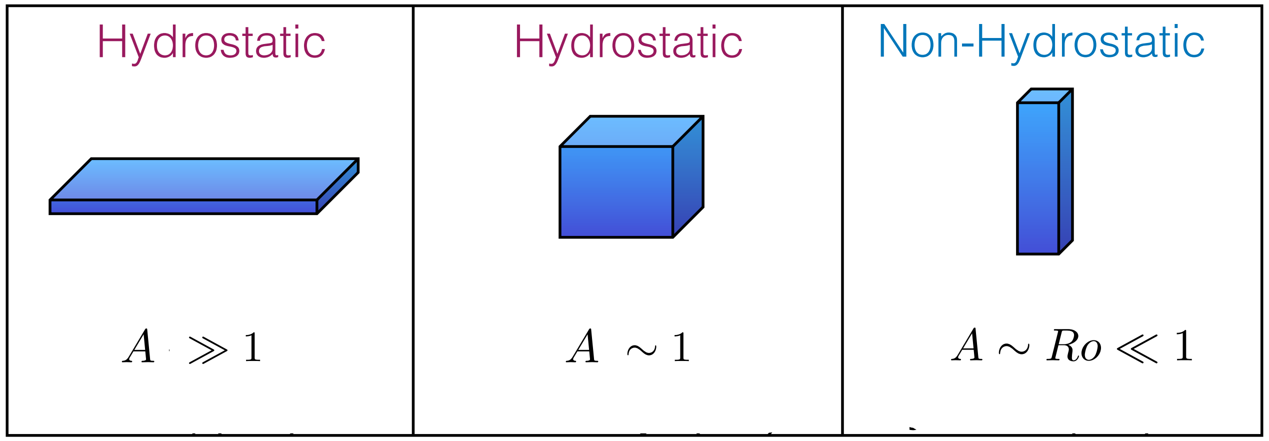
$$\frac{1}{Ro^2} \partial_t \overline{T} + \partial_z \left( \overline{w \vartheta'} \right) = \frac{1}{\widetilde{Pe}} \partial_{zz} \overline{T}$$ mean temp

Vertical advection subdominant

Mean dynamics evolves on slow timescale

Project onto geostrophic manifold - curl mtm eqn

Dynamics of energetic scales:



# NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

**Balance:** $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3} \vartheta.$

**Vert. Vorticity**

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\zeta - \partial_Z w = \nabla_\perp^2 \zeta$$

**Vert. Velocity**

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp) w + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\,\theta$$

**Temp. Fluct.**

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\theta + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

**Mean. Temp.**

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

**Two control parameters**

$$\widetilde{Ra} = Ra E^{4/3}, \; Pr$$

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3} \vartheta.$

Vert. Vorticity

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\zeta - \partial_Z w = \nabla_\perp^2 \zeta$$

Vert. Velocity

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp) w + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\,\theta$$

Temp. Fluct.

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\theta + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

Mean. Temp.

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Two control parameters

$$\widetilde{Ra} = Ra E^{4/3}, \; Pr$$

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0 \quad \text{limit}$

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3}\vartheta.$

Vert. Vorticity
$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

Vert. Velocity
$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\, \theta$$

Temp. Fluct.
$$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

Mean. Temp.
$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Two control parameters

$$\widetilde{Ra} = Ra E^{4/3}, \;\; Pr$$

- Isotropic velocity magnitudes (non-hydrostatic dynamics)

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0 \quad \text{limit}$

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3} \vartheta.$

Vert. Vorticity $$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

Vert. Velocity $$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr} \theta$$

Temp. Fluct. $$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

Mean. Temp. $$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Two control parameters

$$\widetilde{Ra} = Ra E^{4/3}, \ Pr$$

- Isotropic velocity magnitudes (non-hydrostatic dynamics)
- No vertical inertial advection (hallmark of QG theory)

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

Balance: $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3} \vartheta.$

Vert. Vorticity

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

Vert. Velocity

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr} \theta$$

Temp. Fluct.

$$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

Mean. Temp.

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Two control parameters

$$\widetilde{Ra} = Ra E^{4/3}, \ Pr$$

Reduced BCs.

$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

- Isotropic velocity magnitudes (non-hydrostatic dynamics)
- No vertical inertial advection (hallmark of QG theory)
- Horizontal dissipation only (no vertical dissipation - filtered mtm. Ekman bl's)

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

**Balance:** $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3} \vartheta.$

**Vert. Vorticity**

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

**Vert. Velocity**

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr} \theta$$

**Temp. Fluct.**

$$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

**Mean. Temp.**

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

**Two control parameters**

$$\widetilde{Ra} = Ra E^{4/3}, \ Pr$$

**Reduced BCs.**

$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

- Isotropic velocity magnitudes (non-hydrostatic dynamics)
- No vertical inertial advection (hallmark of QG theory)
- Horizontal dissipation only (no vertical dissipation - filtered mtm. Ekman bl's)
- Vertical diffusion for mean temperature only (TBL can develop)

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0 \quad \text{limit}$

**Balance:** $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3}\vartheta.$

**Vert. Vorticity**

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

**Vert. Velocity**

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\,\theta$$

**Temp. Fluct.**

$$\partial_t \theta + J[\psi, \theta] + w\partial_Z \overline{T} = \frac{1}{Pr}\nabla_\perp^2 \theta$$

**Mean. Temp.**

$$\partial_Z(\overline{w\theta}) = \frac{1}{Pr}\partial_{ZZ}\overline{T}$$

**Two control parameters**

$$\widetilde{Ra} = RaE^{4/3}, \ Pr$$

**Reduced BCs.**

$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

- Isotropic velocity magnitudes (non-hydrostatic dynamics)
- No vertical inertial advection (hallmark of QG theory)
- Horizontal dissipation only (no vertical dissipation - filtered mtm. Ekman bl's)
- Vertical diffusion for mean temperature only (TBL can develop)
- Slow inertial waves only. No fast (unbalanced) waves

$$\omega_{wave}^2 = \frac{k_Z^2}{k_\perp^2} \in (c, \infty)$$

# NH-Quasi-Geostrophic (RRBC) $Ro \to 0$ limit

**Balance:** $p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + E^{1/3}\vartheta.$

**Vert. Vorticity**

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

**Vert. Velocity**

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr}\,\theta$$

**Temp. Fluct.**

$$\partial_t \theta + J[\psi, \theta] + w\partial_Z \overline{T} = \frac{1}{Pr}\nabla_\perp^2 \theta$$

**Mean. Temp.**

$$\partial_Z(\overline{w\theta}) = \frac{1}{Pr}\partial_{ZZ}\overline{T}$$

**Reduced BCs.**

$$w = 0, \quad T(0) = 1, \quad T(1) = 0$$

Conserved quantities: Energy (volume averaged) and Potential Vorticity (pointwise)

$$E = \frac{1}{2}\left\langle |\nabla^\perp \psi|^2 + w^2 \right\rangle_V, \qquad Q_{PV} = \left(\zeta + \frac{\widetilde{Ra}}{Pr}\partial_Z\left(\frac{\theta}{\partial_Z \overline{T}}\right)\right) + J[w, \theta]$$

Enstrophy not conserved. Forward cascade?

# Quasi-Geostrophic RBC

$Ro \to 0 \quad \text{limit}$

$$p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + \epsilon \vartheta.$$

$$\partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta$$

$$\partial_t w + J[\psi, w] + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr} \theta$$

$$\partial_t \theta + J[\psi, \theta] + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Quasilinear PDE

Single-Mode solutions:

Pose: $w = \hat{W}(Z,t) h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$ $\qquad J[h,h] = 0$

# Quasi-Geostrophic RBC

$Ro \to 0 \quad \text{limit}$

$$p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + \epsilon \vartheta.$$

$$\partial_t \zeta - \partial_Z w = \nabla_\perp^2 \zeta$$

$$\partial_t w + \partial_Z \psi = \nabla_\perp^2 w + \frac{\widetilde{Ra}}{Pr} \theta$$

Quasilinear PDE

$$\partial_t \theta + w \partial_Z \overline{T} = \frac{1}{Pr} \nabla_\perp^2 \theta$$

$$\partial_Z (\overline{w\theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Single-Mode solutions:

Pose: $w = \hat{W}(Z,t) h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$ $\qquad J[h,h] = 0$

# Quasi-Geostrophic RBC

$Ro \to 0 \quad \text{limit}$

$p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + \epsilon \vartheta.$

$$k_\perp^2 \partial_t \hat{\Psi} + \partial_Z \hat{W} = -k_\perp^4 \hat{\Psi}$$

$$\partial_t \hat{W} + \partial_Z \hat{\Psi} = -k_\perp^2 \hat{W} + \frac{\widetilde{Ra}}{Pr} \hat{\Theta}$$

Quasilinear PDE

$$\partial_t \hat{\Theta} + w \partial_Z \overline{T} = -\frac{1}{Pr} k_\perp^2 \hat{\Theta}$$

$$\partial_Z (\hat{W} \hat{\Theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Single-Mode solutions:

Pose: $w = \hat{W}(Z,t) h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$ $\qquad J[h,h] = 0$

# Quasi-Geostrophic RBC

$Ro \to 0 \quad \text{limit}$

$$p' = \Psi, \quad \boldsymbol{u} = (-\partial_y \Psi, \partial_x \Psi, w), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \nabla_\perp^2 \Psi, \quad T = \overline{T}(Z) + \epsilon\vartheta.$$

$$\cancel{k_\perp^2 \partial_t \hat{\Psi}} + \partial_Z \hat{W} = -k_\perp^4 \hat{\Psi}$$

$$\cancel{\partial_t \hat{W}} + \partial_Z \hat{\Psi} = -k_\perp^2 \hat{W} + \frac{\widetilde{Ra}}{Pr} \hat{\Theta}$$

Steady convection

$$\cancel{\partial_t \hat{\Theta}} + w \partial_Z \overline{T} = -\frac{1}{Pr} k_\perp^2 \hat{\Theta}$$

$$\partial_Z (\hat{W}\hat{\Theta}) = \frac{1}{Pr} \partial_{ZZ} \overline{T}$$

Single-Mode solutions:

Pose: $$w = \hat{W}(Z,t) h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$$

$$J[h,h] = 0$$

# Nonlinear Solutions - Exact Coherent Structures (ECS)

Pose:

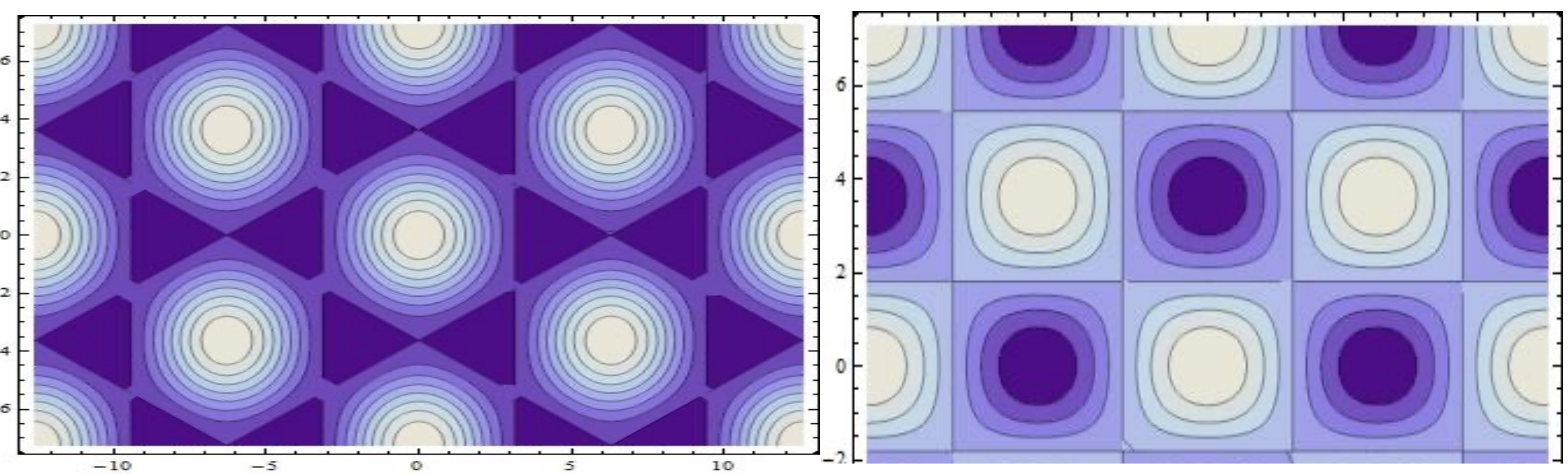
$$w = \hat{W}(Z)h(x,y), \quad \nabla_\perp^2 h + k_\perp^2 h = 0$$

Find:

$$\partial_{ZZ}\hat{W} - k_\perp^2 \left[\widetilde{Ra}\partial_Z \overline{T} + k_\perp^4\right]\hat{W} = 0,$$

$$\partial_Z \overline{T} = -\left(\frac{k_\perp^2}{k_\perp^2 + Pr^2\hat{W}^2}\right) Nu \qquad Nu = \left[\int_0^1 \left(\frac{k_\perp^2}{k_\perp^2 + Pr^2\hat{W}^2}\right) dZ\right]^{-1}$$

Spatially Extended (SE)



$h(x,y)$ top view

consequence of no vertical advection!

$$\boldsymbol{u}_\perp \cdot \nabla_\perp h = 0$$

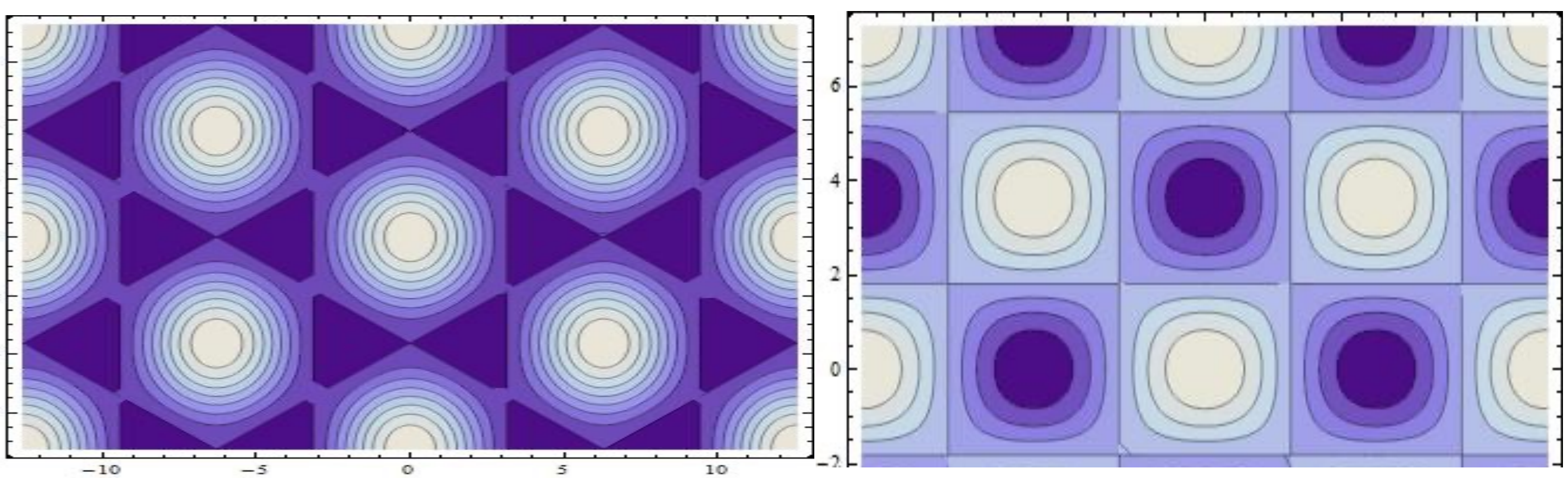
# Nonlinear Solutions - Exact Coherent Structures (ECS)

Pose: $$w = \hat{W}(Z)h(x,y)e^{i\omega t} + c.c., \quad \nabla_\perp^2 h + k_\perp^2 h = 0$$

Find: $$\partial_{ZZ}\hat{W} - k_\perp^2\left(\frac{i\omega}{Pr} + k_\perp^2\right)\left[\left(\frac{i\omega}{Pr} + k_\perp^2\right) + \frac{(k_\perp^2 - i\omega)}{(\omega^2 + k_\perp^4)}\widetilde{Ra}\partial_Z\overline{T}\right]\hat{W} = 0$$

$$\partial_Z\overline{T} = -\left(\frac{\omega^2 + k_\perp^4}{\omega^2 + k_\perp^4 + Pr^2 k_\perp^6|\hat{W}|^2}\right)Nu$$

Spatially Extended (SE)

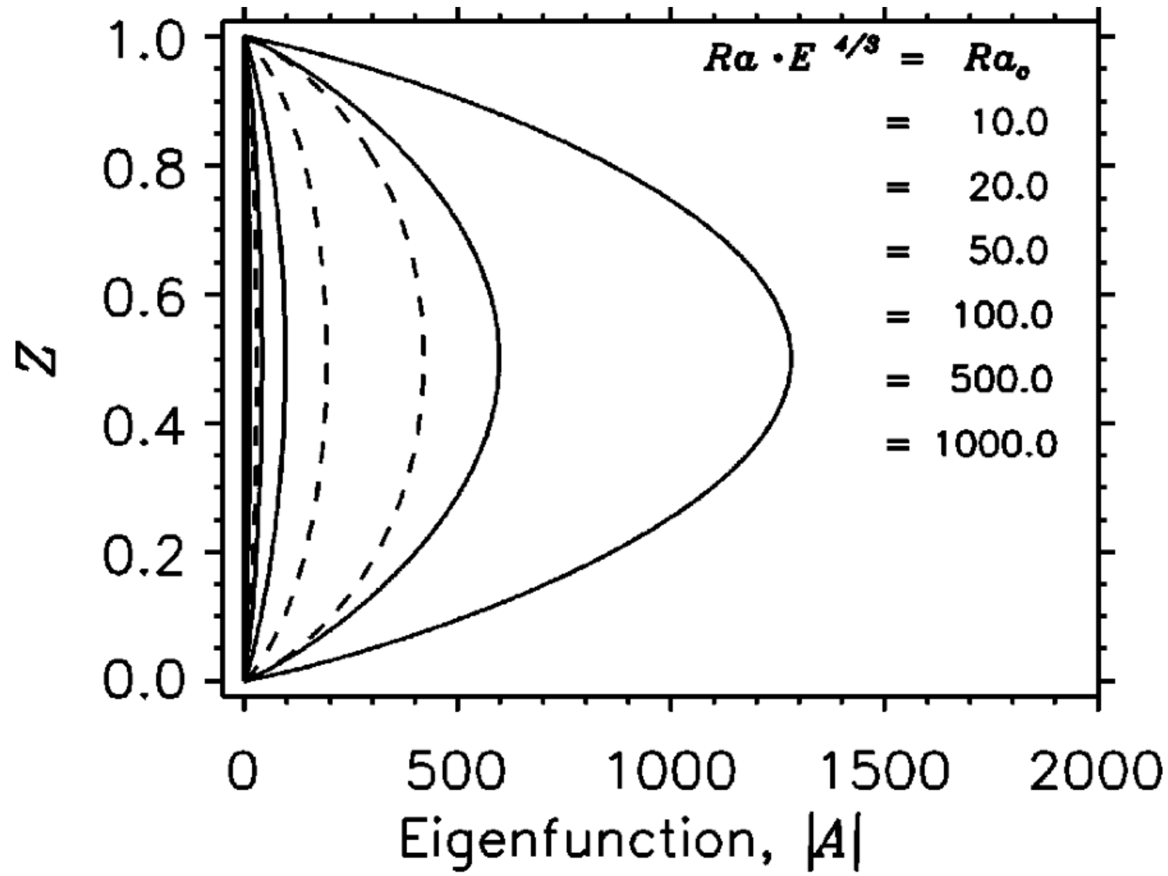



$h(x,y)$ top view

consequence of no vertical advection!

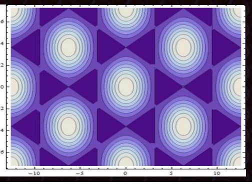
$$\boldsymbol{u}_\perp \cdot \nabla_\perp h = 0$$

# Nonlinear Solutions - Exact Coherent Structures (ECS)

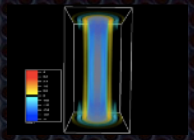


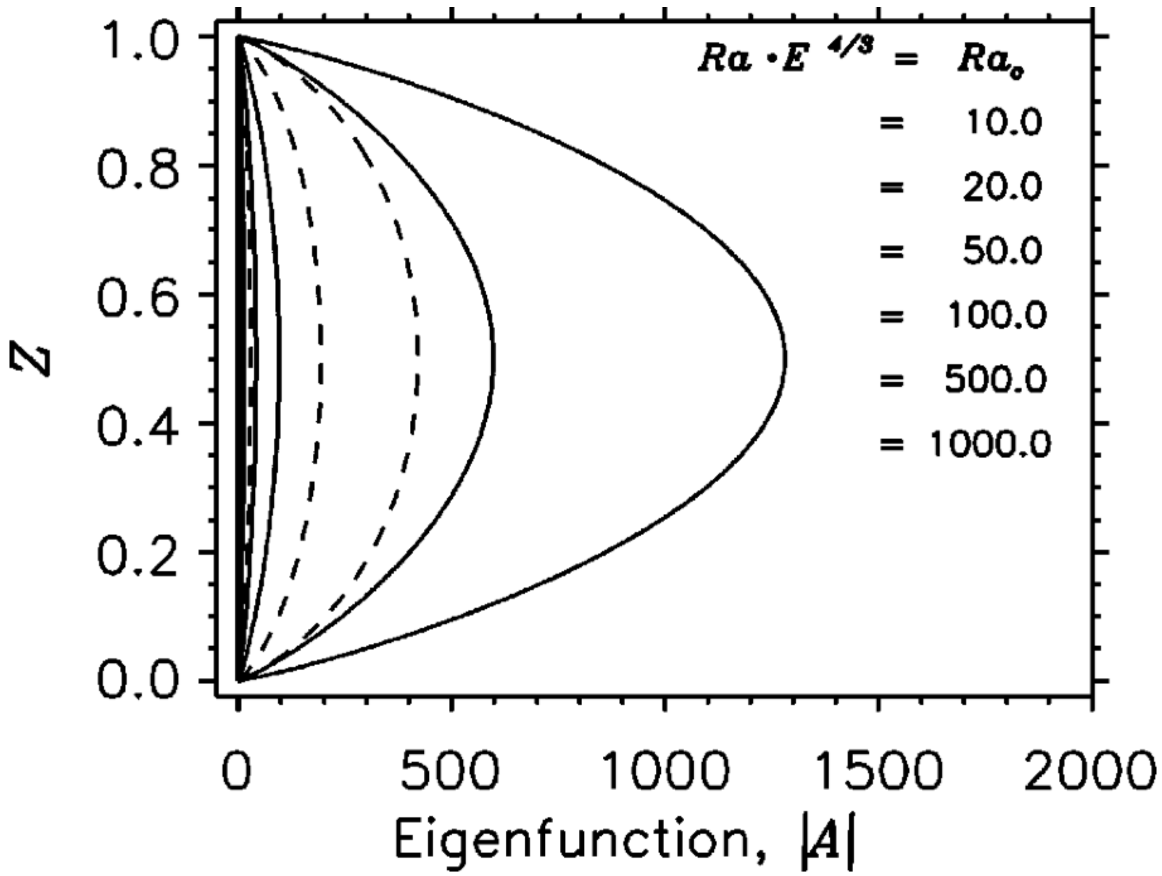

Sinusoidal vertical structure

For both steady and oscillatory convection



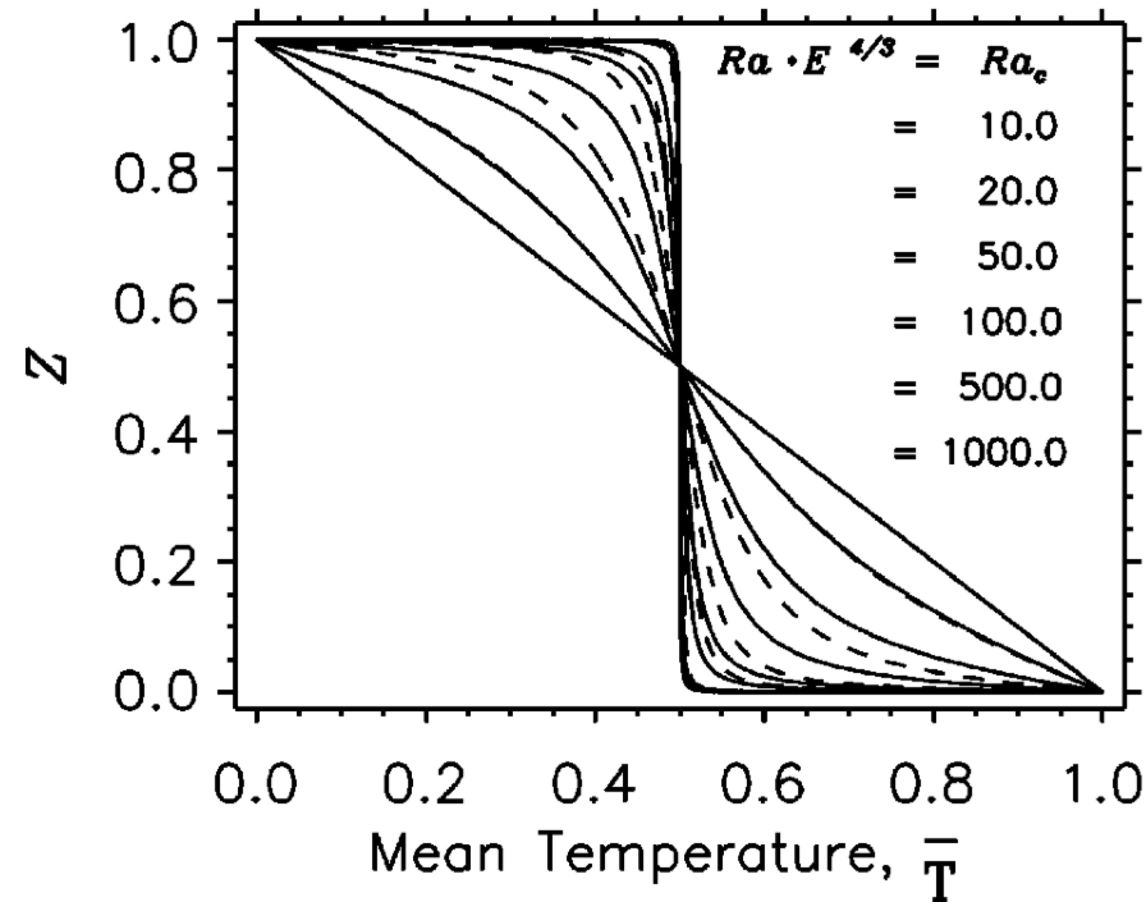
# Nonlinear Solutions - Exact Coherent Structures (ECS)



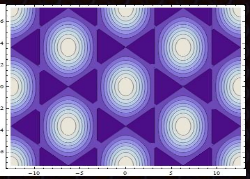



Sinusoidal vertical structure

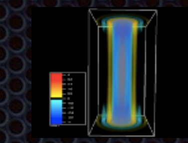
For both steady and oscillatory convection

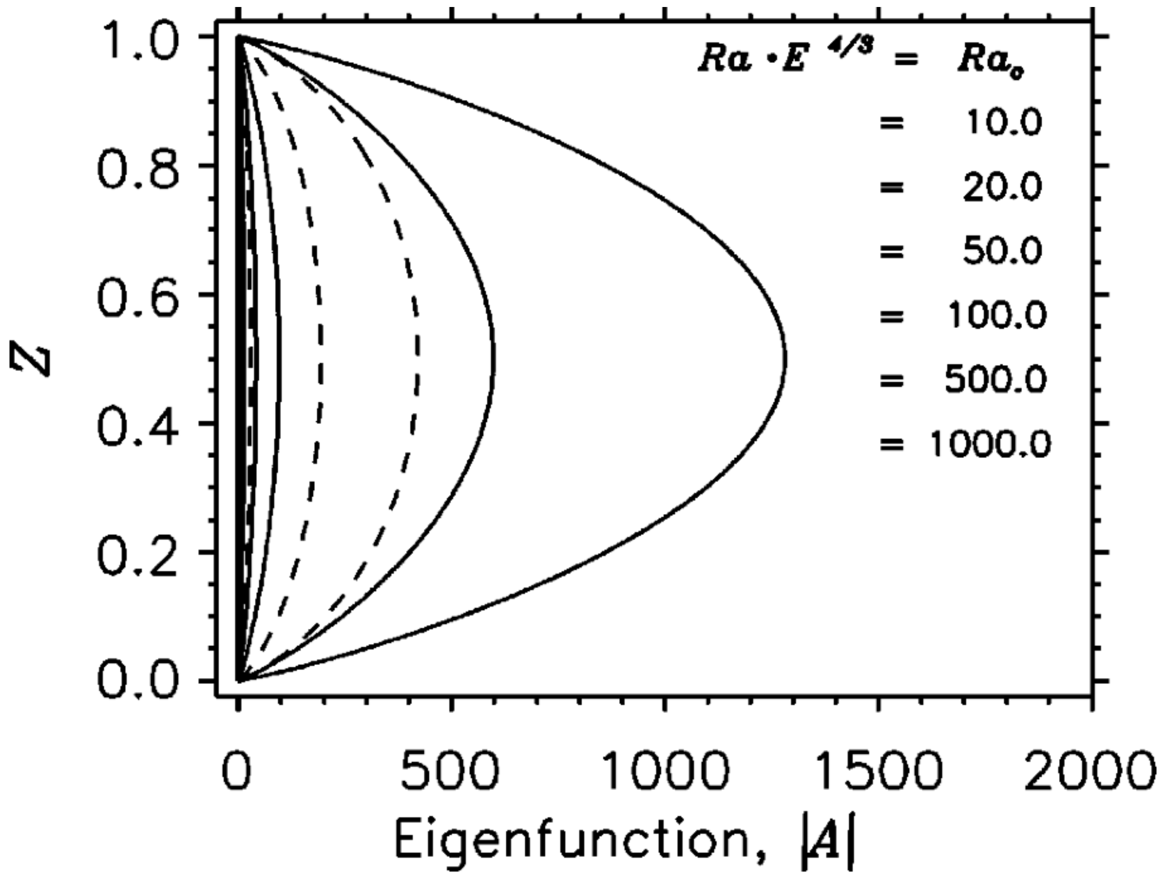


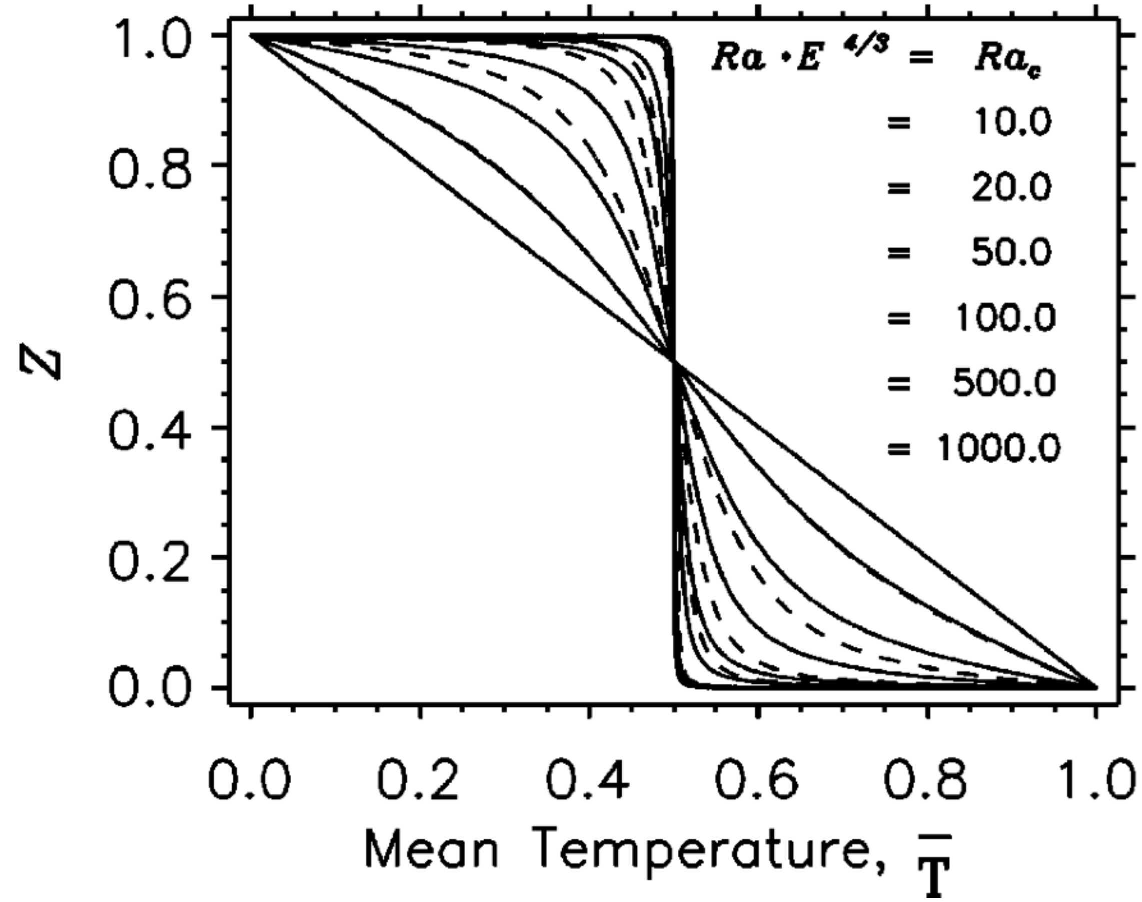

Development of TBL +Isothermal Interior

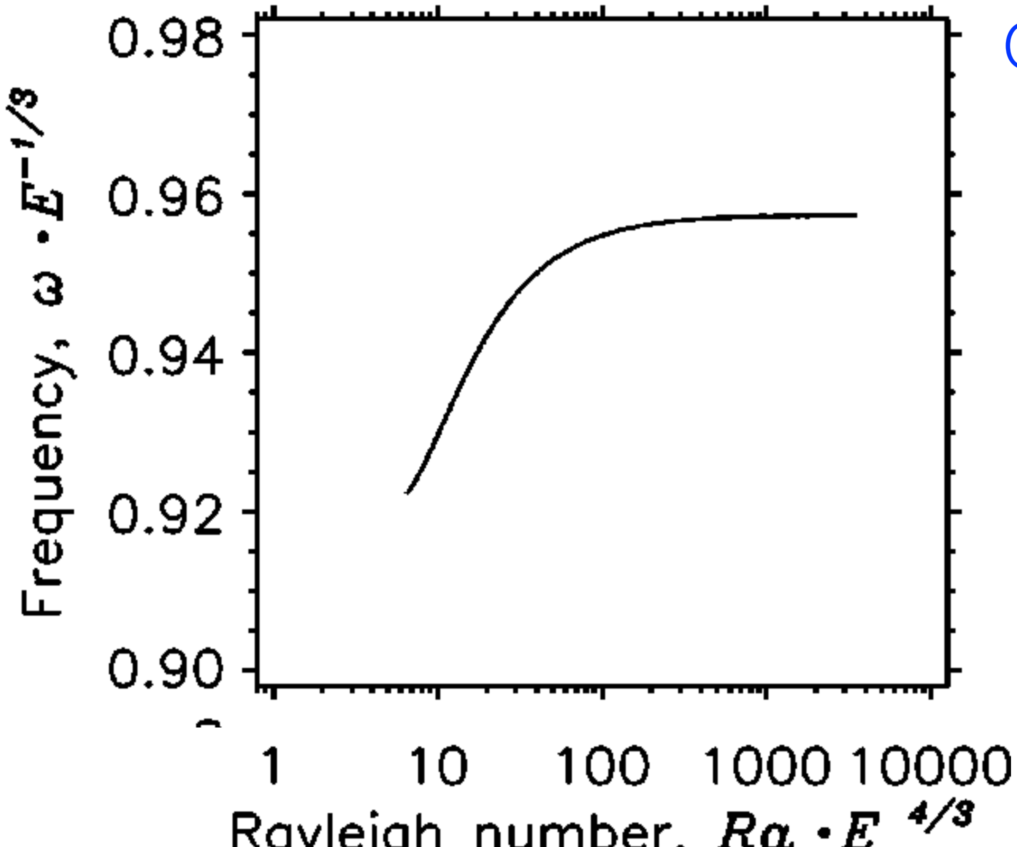


# Nonlinear Solutions - Exact Coherent Structures (ECS)

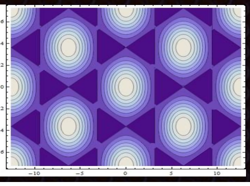




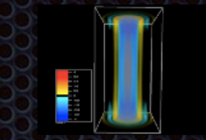



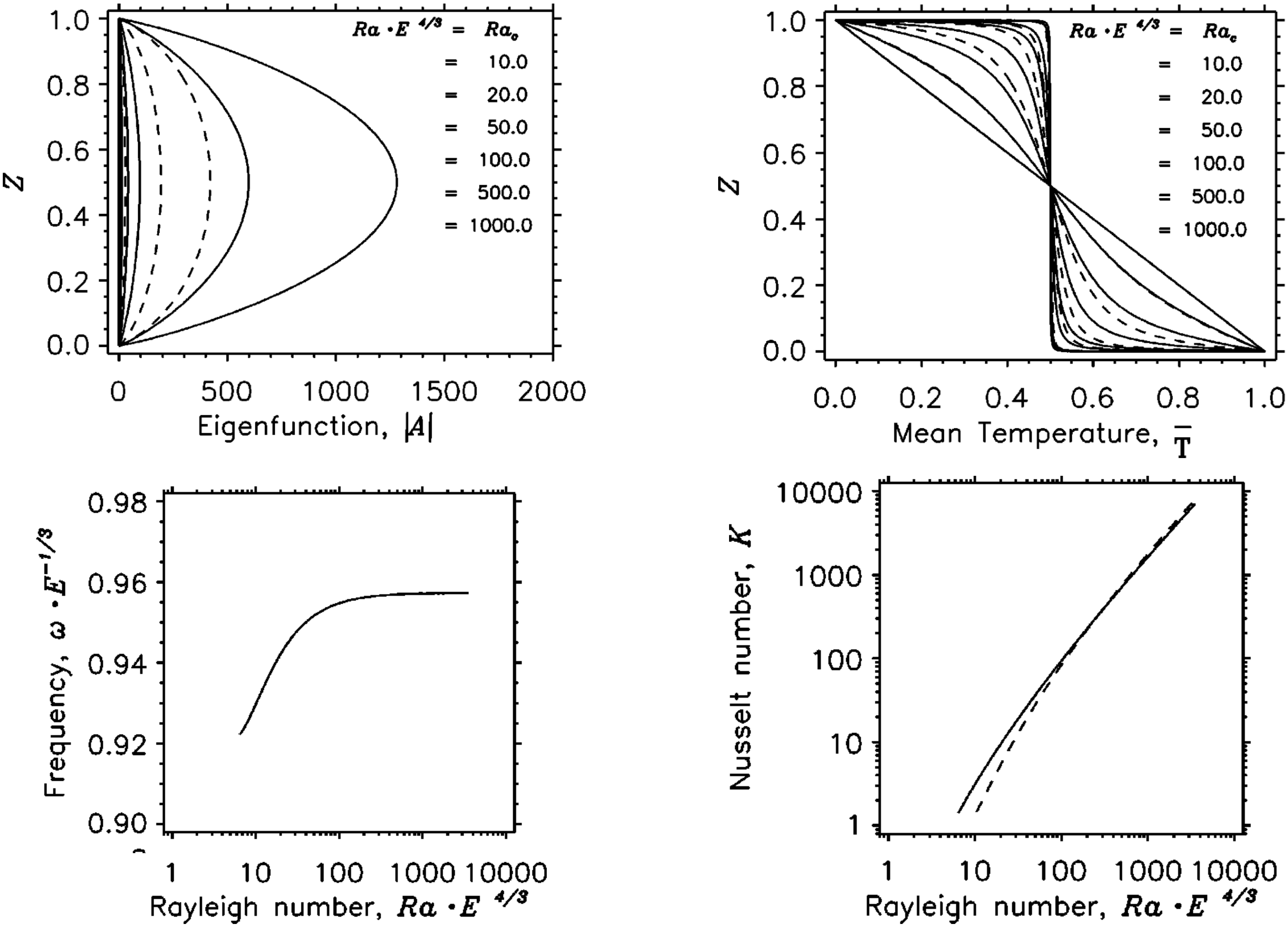




Oscillations saturate

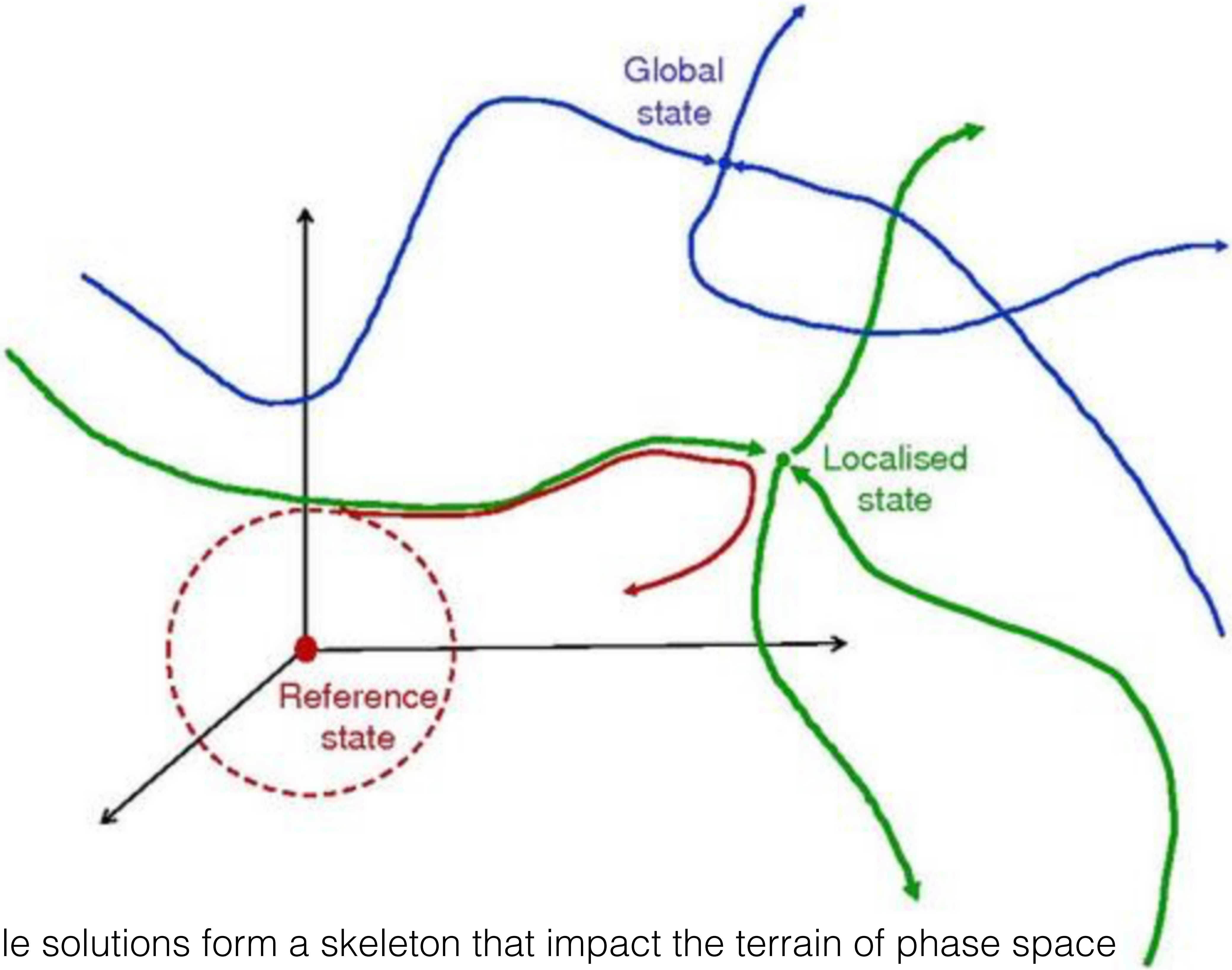


# Nonlinear Solutions - Exact Coherent Structures (ECS)





# Dynamics - Exact Coherent Structures (ECS)




Unstable solutions form a skeleton that impact the terrain of phase space

(open questions) potential to use as basis functions for further reduction