Lecture 3: Boulder Summer School

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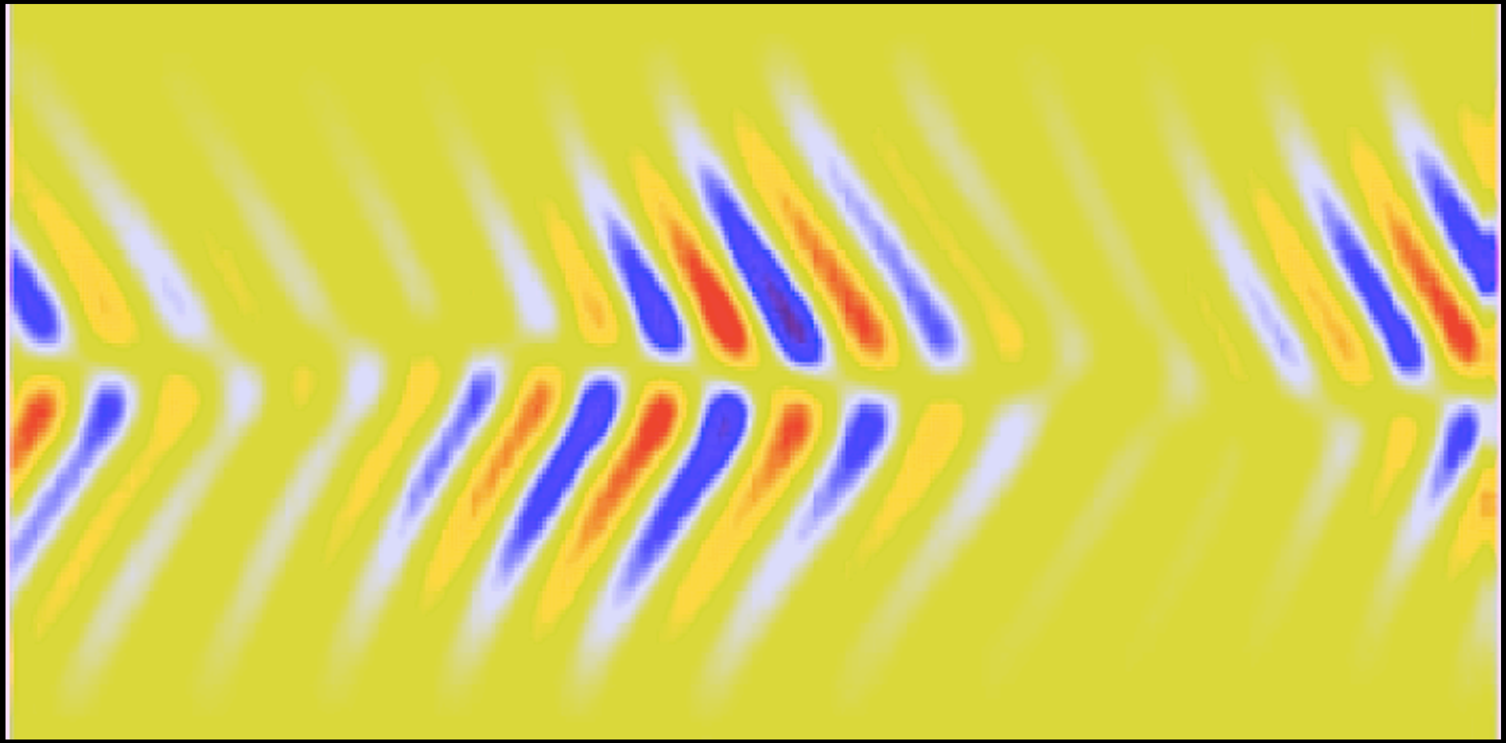
A mean field model of the Solar Dynamo



Dynamics vs Statistics

"The climate is what you expect; the weather is what you get."

Attributed to Mark Twain.

"More than any other theoretical procedure, numerical integration is also subject to the criticism that it yields little insight into the problem. The computed numbers are not only processed like data but they look like data, and a study of them may be no more enlightening than a study of real meteorological observations. An alternative procedure which does not suffer this disadvantage consists of deriving a new system of equations whose unknowns are the statistics themselves....

...This procedure can be very effective for problems where the original equations are linear, but, in the case of non-linear equations, the new system will inevitably contain more unknowns than equations, and can therefore not be solved, unless additional postulates are introduced."

Edward Lorenz, The Nature and Theory of the General Circulation of the Atmosphere (1967)

"Direct Statistical Simulation" (DSS)

Direct Statistical Simulation: Why *simulate* the statistics?

Low-order statistics: smoother in space than instantaneous flow.

Statistics evolve slowly in time, or not at all, and hence may be described by a fixed point, or at least a slow manifold.

Statistics usually describe mean/average behaviour and variations about that mean (e.g. 2pt correlations)

Statistics are less sensitive to changes in underlying parameters than detailed dynamics.

In geophysics/astrophysics correlations are *non-local* and highly anisotropic and inhomogeneous. Statistical formulations *must* respect this. They should also respect conservation laws

Solution of Statistical Equations is an old idea: Bousssinesq, Reynolds, Lorenz, Herring, Kraichnan, Frisch, Salmon, Monin & Yaglom, Majda

Direct Statistical Simulation: the options…

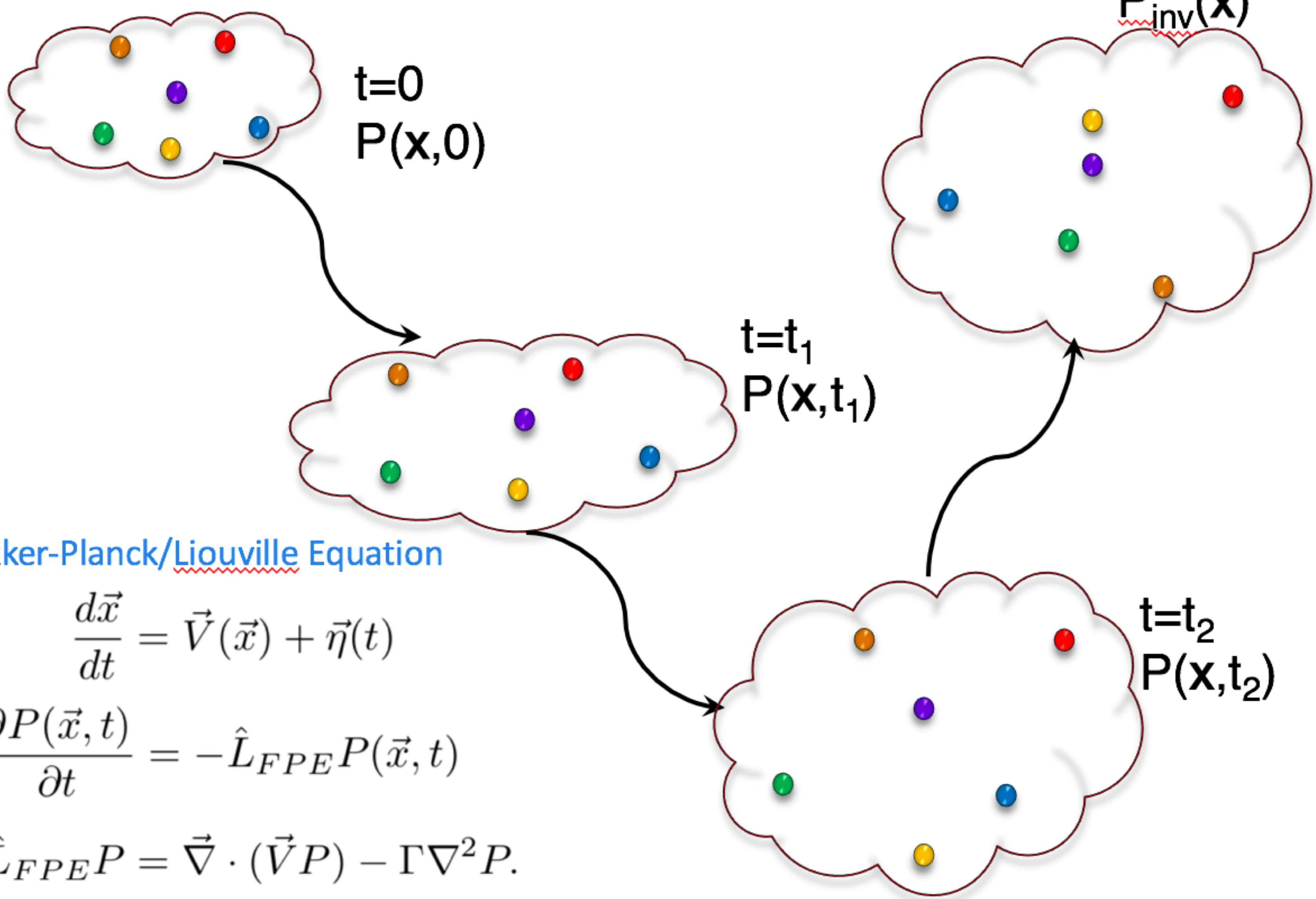
- For any dynamical system one could try to simulate pdfs
  - Perron-Frobenius Theory (see e.g. Beck & Schloegl 1993)
  - Fokker-Planck/Liouville equations (see e.g. Kadanoff 2000, Alwalla & Marston 2016, Cho et al 2016, Chen & Majda 2017)

- Kolmogorov/Kraichnan moment hierarchies
  - Usually assume isotropy/homogeneity of statistics

- Large Deviation Theory
  - Extreme/rare events or flipping (see e.g. Bouchet & Simonet 2009, Laurie & Bouchet 2015)

- Use cumulants
  - take into account inhomogeneity/anisotropy
  - Statistical closures
    - Relationship with deterministic approximations?

Direct Statistical Simulation: Simulating pdfs



Fokker-Planck/Liouville Equation

$$\frac{d\vec{x}}{dt} = \vec{V}(\vec{x}) + \vec{\eta}(t)$$

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\hat{L}_{FPE} P(\vec{x},t)$$

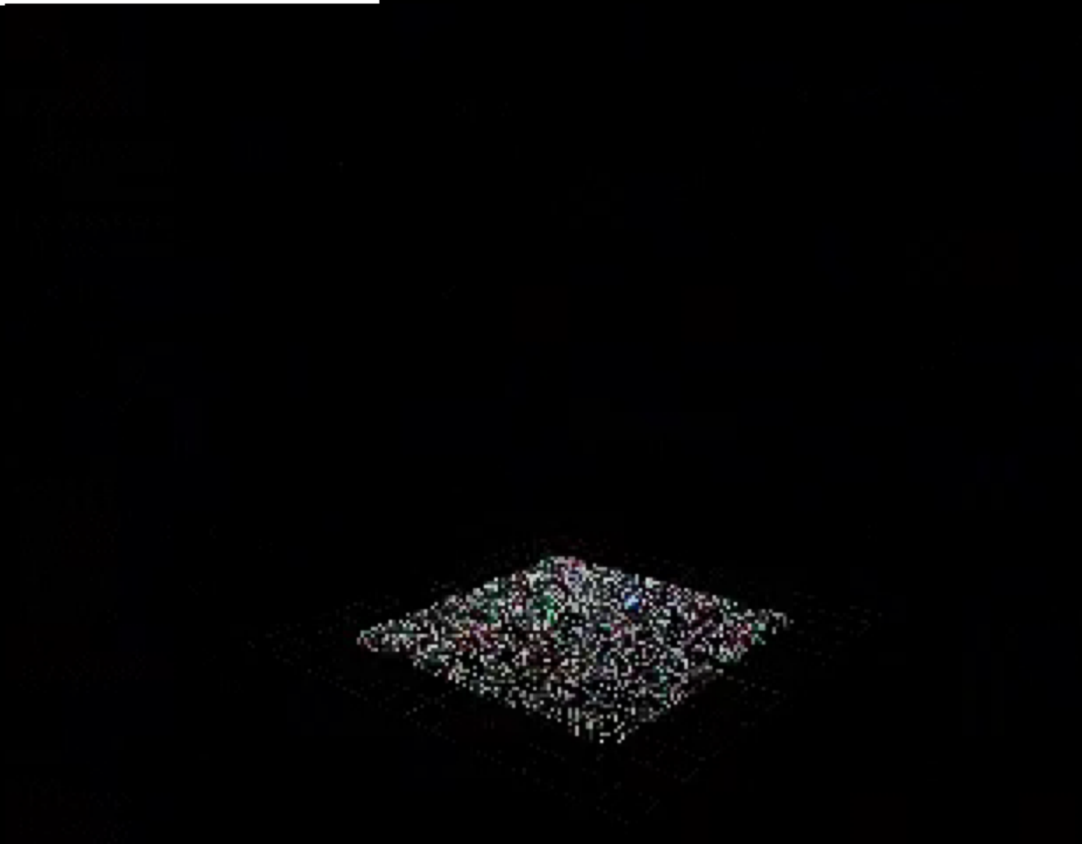
$$\hat{L}_{FPE} P = \vec{\nabla} \cdot (\vec{V} P) - \Gamma \nabla^2 P.$$

See e.g. Foias, Manley & Temam 2001

To Fix ideas: the (stochastically forced) Lorenz equations…

$$\dot{x} = \sigma(y - x) + \eta_1(t)$$
$$\dot{y} = x(\rho - z) - y + \eta_2(t)$$
$$\dot{z} = xy - \beta z + \eta_3(t)$$

- What does pdf look like?



https://hypertextbook.com/chaos/strange/

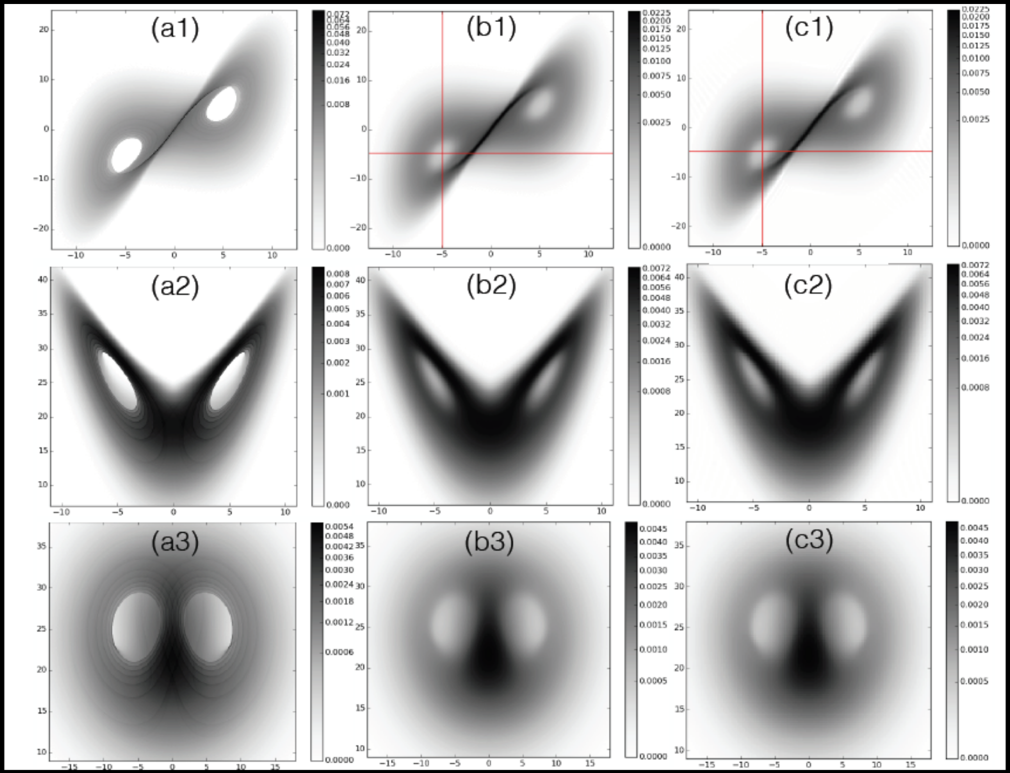
To Fix ideas: the (stochastically forced) Lorenz equations…

$$\dot{x} = \sigma(y - x) + \eta_1(t)$$
$$\dot{y} = x(\rho - z) - y + \eta_2(t)$$
$$\dot{z} = xy - \beta z + \eta_3(t)$$

- Simulation of pdfs: L/FPE

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\hat{L}_{FPE} P(\vec{x},t)$$

Alwalla & Marston (2016)



DNS **DNS+NOISE** **FPE**

$$\hat{L}_{FPE} P = \vec{\nabla} \cdot [(\sigma(y - x),\ x(\rho - z) - y,\ xy - \beta z)P] - \Gamma \nabla^2 P$$

Deterministic flow **noise**

To Fix ideas: the (stochastically forced) Lorenz equations…

- Use cumulant expansions

$$\dot{q}_i = F_i + L_{ij}\ q_j + Q_{ijk}\ q_j q_k + \eta_i .$$

ODE SYSTEM

$$q_i = \overline{q_i} + q_i'$$

REYNOLDS DECOMPOSITION

$$\begin{aligned} c_i &\equiv \overline{q_i} \\ c_{ij} &\equiv \overline{q_i' q_j'} \\ c_{ijk} &\equiv \overline{q_i' q_j' q_k'} \\ c_{ijk\ell} &\equiv \overline{q_i' q_j' q_k' q_\ell'} - c_{ij}\ c_{k\ell} - c_{ik}\ c_{j\ell} - c_{i\ell}\ c_{jk} \end{aligned}$$

DEFINE CUMULANTS

To Fix ideas: the (stochastically forced) Lorenz equations…

- Derive cumulant system using Hopf Functionals (or brute force)

$$\dot{q}_i = F_i + L_{ij}\ q_j + Q_{ijk}\ q_j q_k + \eta_i.$$

$$q_i = \overline{q_i} + q_i'$$

1st Cumulant Equation

$$\frac{dc_i}{dt} = \overline{\frac{dq_i}{dt}}$$

$$= F_i + L_{ij}\ c_j + Q_{ijk}\ (c_j c_k + c_{jk})$$

$$\frac{dc_{ij}}{dt} = 2\left\{\overline{\frac{dq_i'}{dt}\ q_j'}\right\}$$

$$= \{2L_{ik}\ c_{kj} + Q_{ik\ell}\ (4c_k c_{\ell j} + 2c_{k\ell j})\} + 2\Gamma_{ij}$$

2nd Cumulant Equation

Li et al (2022)

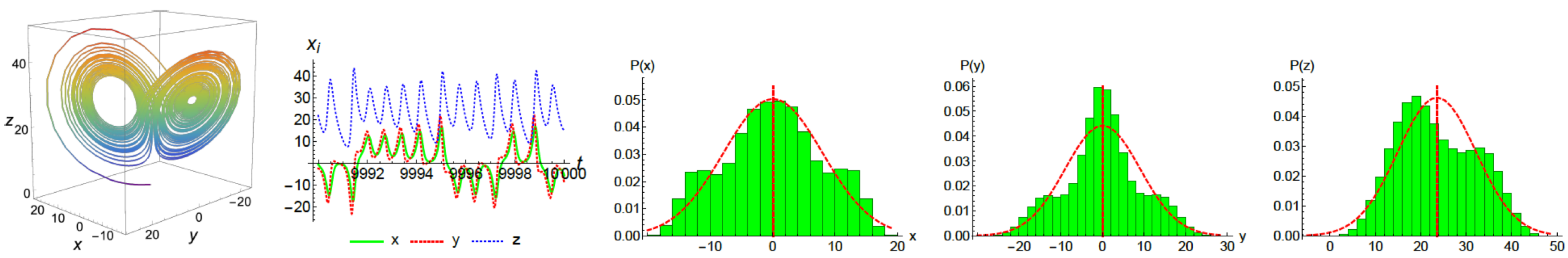
To Fix ideas: the (stochastically forced) Lorenz equations…

$$\dot{q}_i = F_i + L_{ij}\ q_j + Q_{ijk}\ q_j q_k + \eta_i .$$

$$\frac{dc_{ijk}}{dt} = 3\left\{ \overline{\frac{dq'_i}{dt} q'_j q'_k} \right\}$$

$$= \{3L_{im}\ c_{mjk} + 6Q_{imn}\ (c_m c_{njk} + c_{mj} c_{nk})\} - \frac{c_{ijk}}{\tau_d}$$

3rd Cumulant Equation



(a) (b) (c) (d) (e)

Cumulant	DNS	CE3 $(\tau_d = 0.1)$	CE3 $(\tau_d = 0.5)$
$\overline{x}$	$\mathcal{O}(10^{-4})$	0	0
$\overline{y}$	$\mathcal{O}(10^{-4})$	0	0
$\overline{z}$	24.796	25.188	25.000
$\overline{x'x'}$	3.966	4.030	4.000
$\overline{x'y'}$	3.966	4.030	4.000
$\overline{x'z'}$	$\mathcal{O}(10^{-5})$	0	0

Cumulant Expansion for PDEs

Take PDE or set of PDEs

e.g. Momentum, induction, energy equations

$$\partial_t q = \mathcal{L}[q] + \mathcal{N}[q\ q] + f(t)$$

Define Cumulants

1st cumulants: means

2nd cumulants: two-point correlation functions

cross correlations

Split into means and fluctuations

Reynolds averaging

$$q = \langle q \rangle + q'$$

$$c_1 = \langle q(\mathbf{x}) \rangle, \quad c_2 = \langle q'(\mathbf{x_1}) q'(\mathbf{x_2}) \rangle$$

$$c_3 = \langle q'(\mathbf{x_1}) q'(\mathbf{x_2}) q'(\mathbf{x_3}) \rangle$$

Derive Evolution Eqns for Cumulants

Use Hopf functional technique (Monin & Yaglom, Frisch) or brute force

$$\dot{c}_1 = \mathcal{L}[c_1] + \mathcal{N}[c_1 c_1 + c_2]$$

$$\dot{c}_2 = \mathcal{L}[c_2] + \mathcal{N}[c_1 c_2 + c_3] + \Gamma$$

$$\dot{c}_3 = \mathcal{L}[c_3] + \mathcal{N}[c_1 c_3 + c_2 c_2 + c_4] \ \ldots$$

Truncate cumulant hierarchy

Truncation at second order is a Quasilinear, realisable self-consistent mean-field theory (CE2)

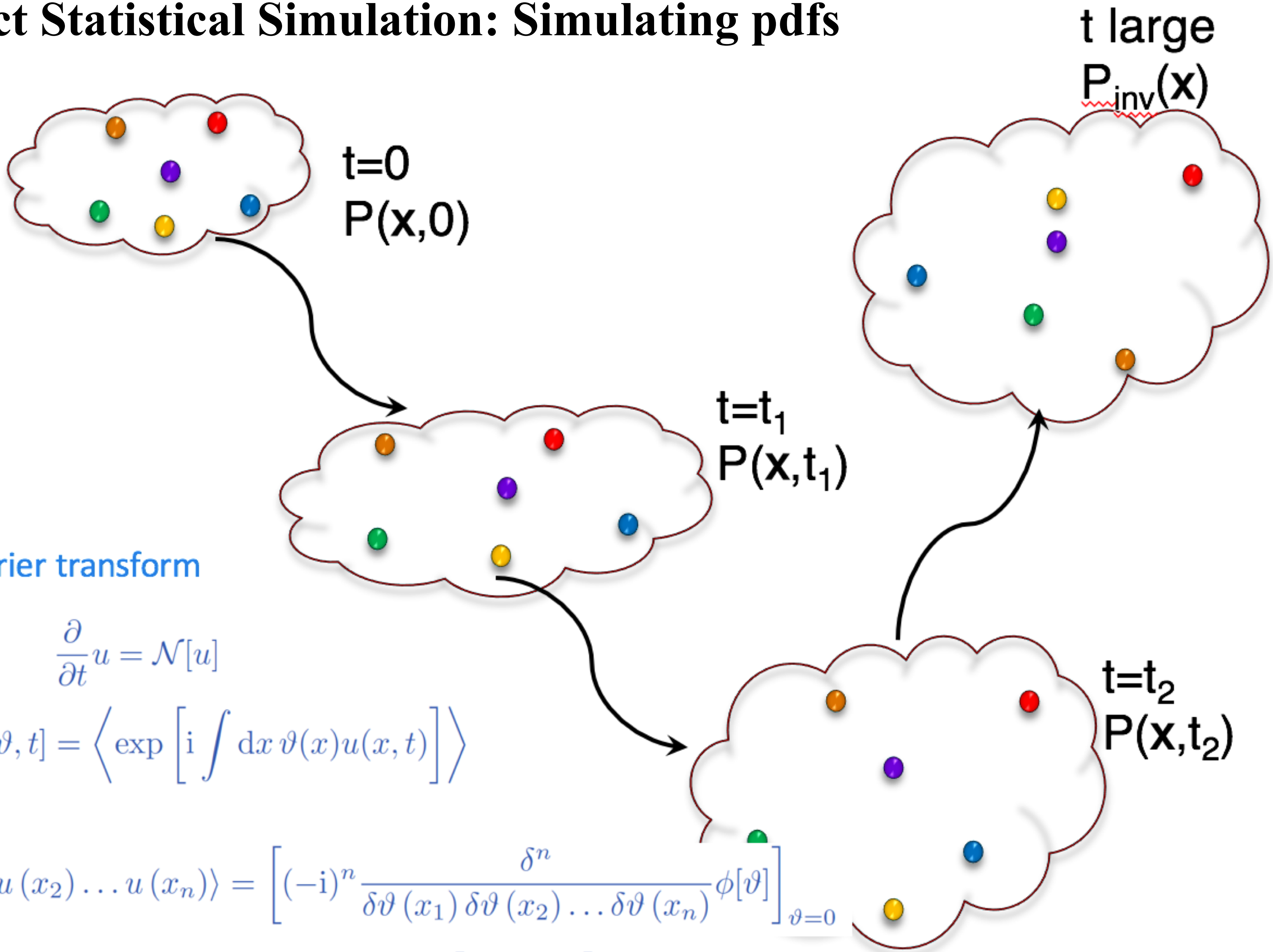
Formally analogous to Farrell & Ioannou S3T

Truncation at third order is an anisotropic, inhomogeneous EDQNM (CE3)

Conservation Laws

Realizability

Direct Statistical Simulation: Simulating pdfs



Fourier transform

$$\frac{\partial}{\partial t} u = \mathcal{N}[u]$$

$$\phi[\vartheta, t] = \left\langle \exp\left[\mathrm{i} \int \mathrm{d}x\, \vartheta(x) u(x,t)\right] \right\rangle$$

$$\langle u(x_1)\, u(x_2) \ldots u(x_n) \rangle = \left[(-\mathrm{i})^n \frac{\delta^n}{\delta\vartheta(x_1)\, \delta\vartheta(x_2) \ldots \delta\vartheta(x_n)} \phi[\vartheta] \right]_{\vartheta=0}$$

$$\frac{\partial}{\partial t} \phi[\vartheta, t] = \mathrm{i} \int \mathrm{d}x\, \vartheta(x) \mathcal{N}\left[-\mathrm{i} \frac{\delta}{\delta\vartheta(x)} \right] \phi[\vartheta, t]$$

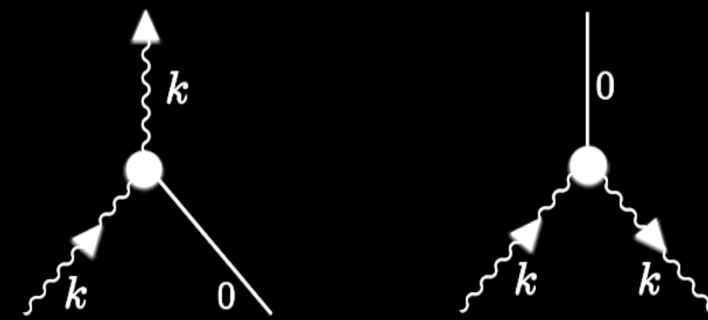
Hopf Equation

See e.g. Foias, Manley & Temam 2001

DSS: Cumulant Expansion to 2nd Order (CE2)

"The discarding of the fluctuating self-interaction then corresponds to closing the system of moment equations by discarding the third order cumulants" (Herring 1963).

$$\mathbf{c}(\mathbf{r}) \equiv \overline{\mathbf{u}(\mathbf{r})},$$
$$\mathbf{c}(\mathbf{r}_1, \mathbf{r}_2) \equiv \overline{\mathbf{u}'(\mathbf{r}_1) \otimes \mathbf{u}'(\mathbf{r}_2)}$$



Wave — Mean-Flow Interaction   Reynolds Forcing of Mean-Flow

$$\partial_t \mathbf{c}(\mathbf{r}_1) = \mathcal{L}[\mathbf{c}(\mathbf{r}_1)] + \mathcal{N}[\mathbf{c}(\mathbf{r}_1),\ \mathbf{c}(\mathbf{r}_1)] + \int \mathcal{N}[\mathbf{c}(\mathbf{r}_1, \mathbf{r}_2), \delta(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{I}] d\mathbf{r}_2$$

$$\partial_t \mathbf{c}(\mathbf{r}_1, \mathbf{r}_2) = \mathcal{L}_{\mathbf{c}(\mathbf{r}_1)}[\mathbf{c}(\mathbf{r}_1, \mathbf{r}_2)] + \mathcal{L}_{\mathbf{c}(\mathbf{r}_2)}[\mathbf{c}(\mathbf{r}_1, \mathbf{r}_2)]$$

Paradigm Problem:

Barotropic turbulence. Two-dimensional hydrodynamics on a sphere

$$\partial_t q = \mathcal{L}[q] + \mathcal{N}[q\ q] + f(t)$$

$$\partial_t \zeta = -\kappa\zeta - \nu_4 \nabla^4 \zeta - \mathbf{v}.\nabla(\zeta + 2\Omega \sin\theta) + \eta(t)$$

$$\zeta = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{r}}$$

$$\zeta(\theta, \phi) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \zeta_{\ell,m} Y_\ell^m(\theta, \phi)$$

Radial component of the vorticity

Other problems at CE2:

2D turbulence on beta-plane (Farrell & Ioannou 2007, Tobias & Marston 2013, Bakas & Ioannou 2014)

Joint instabilities in the Tachocline (Plummer et al 2017)

MRI in shearing box (Squire & Bhattacherjee 2014)

Convection in a Busse Annulus

Saturation of Inertial (symmetric) instabilities

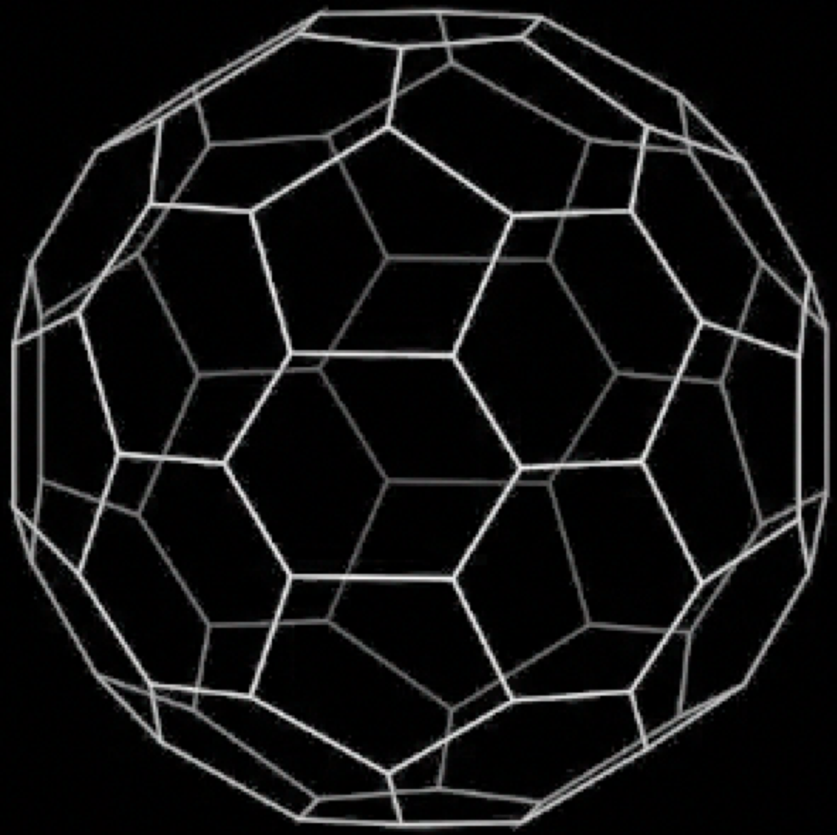
Comparison

- Compare two techniques:
- DNS using spherical geodesic grid
  - Second-order leapfrog
  - Robert filter
  - Multigrid algorithm
- Fully spectral Direct Statistical Simulation
  - Semi-implicit Krylov method
  - Approach to a simple attractor (fixed point, limit cycle?)

What do we expect?

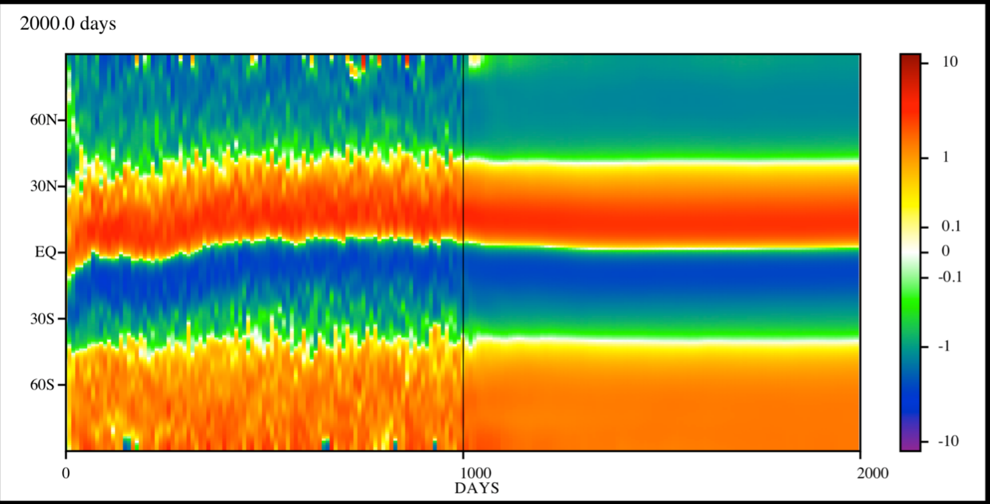
- Small-scale turbulence will drive jets via PV homogenisation (Rhines 1975, McIntrye 2003, Dritschel & McIntyre 2008, Vallis & Maltrud 1993, Manfroi & Young 1998, Sukoriansky & Galperin, 2008)
  - Correlations of nonlinear Rossby waves drives zonal flow via Reynolds Stresses
  - Non-trivial interactions lead to generations of mean flows and angular momentum transport

DNS Movie of hydro jet formation

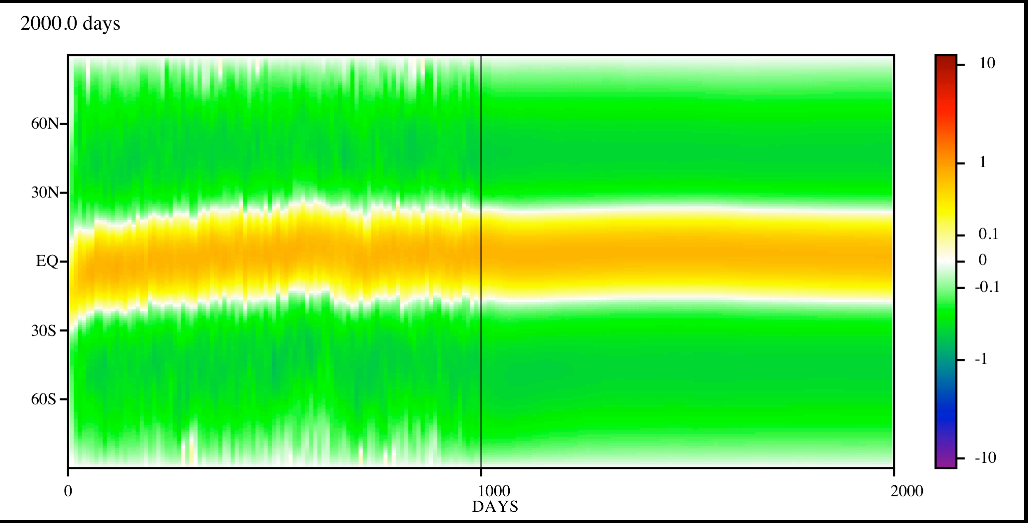


Relative Vorticity

Hydrodynamics: DNS

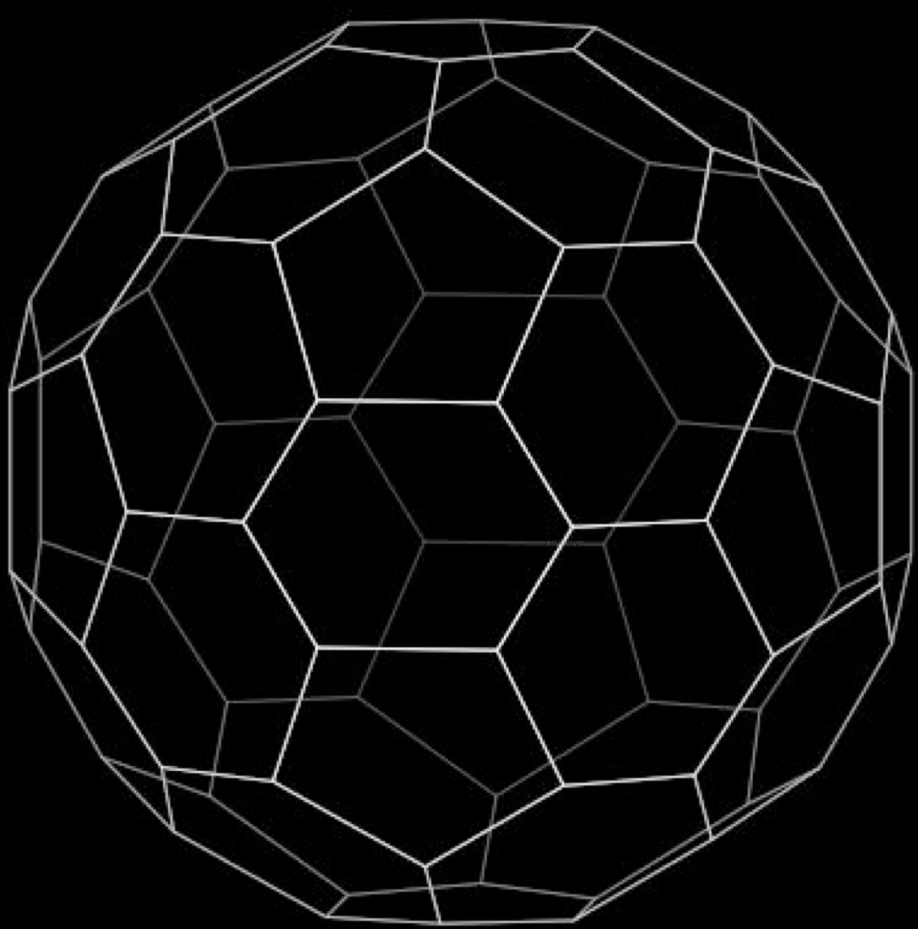


- Zonally averaged zonal radial vorticity



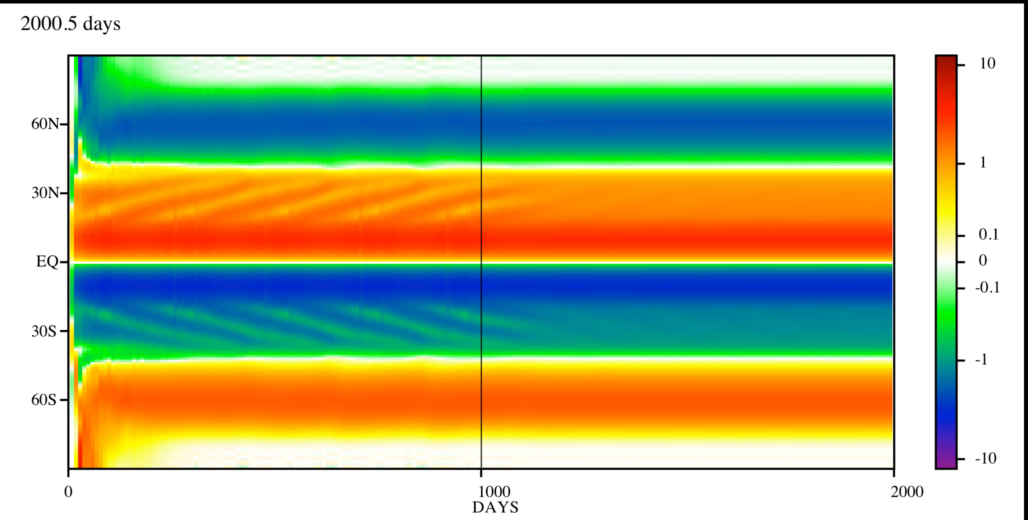
- Zonally averaged zonal azimuthal velocity

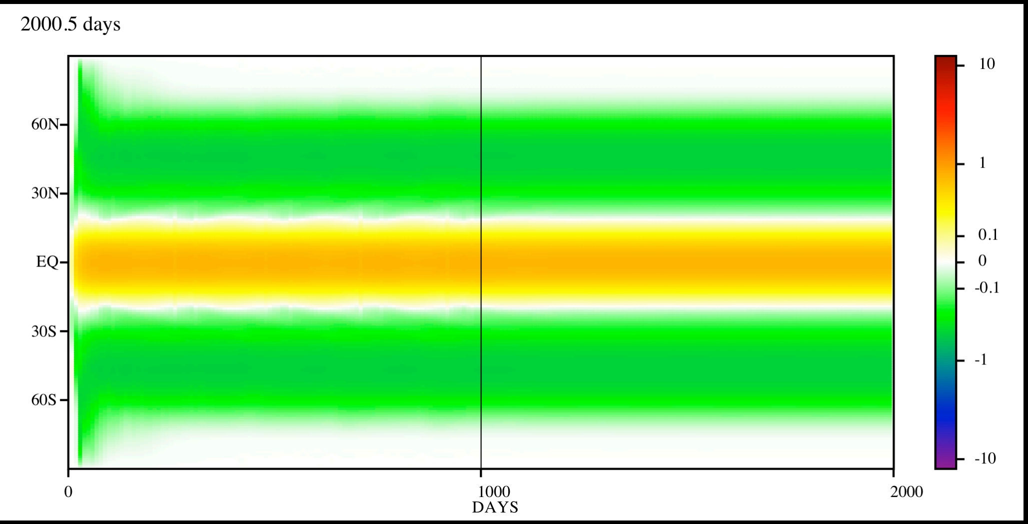
DSS Movie of hydro jet formation



Relative Vorticity

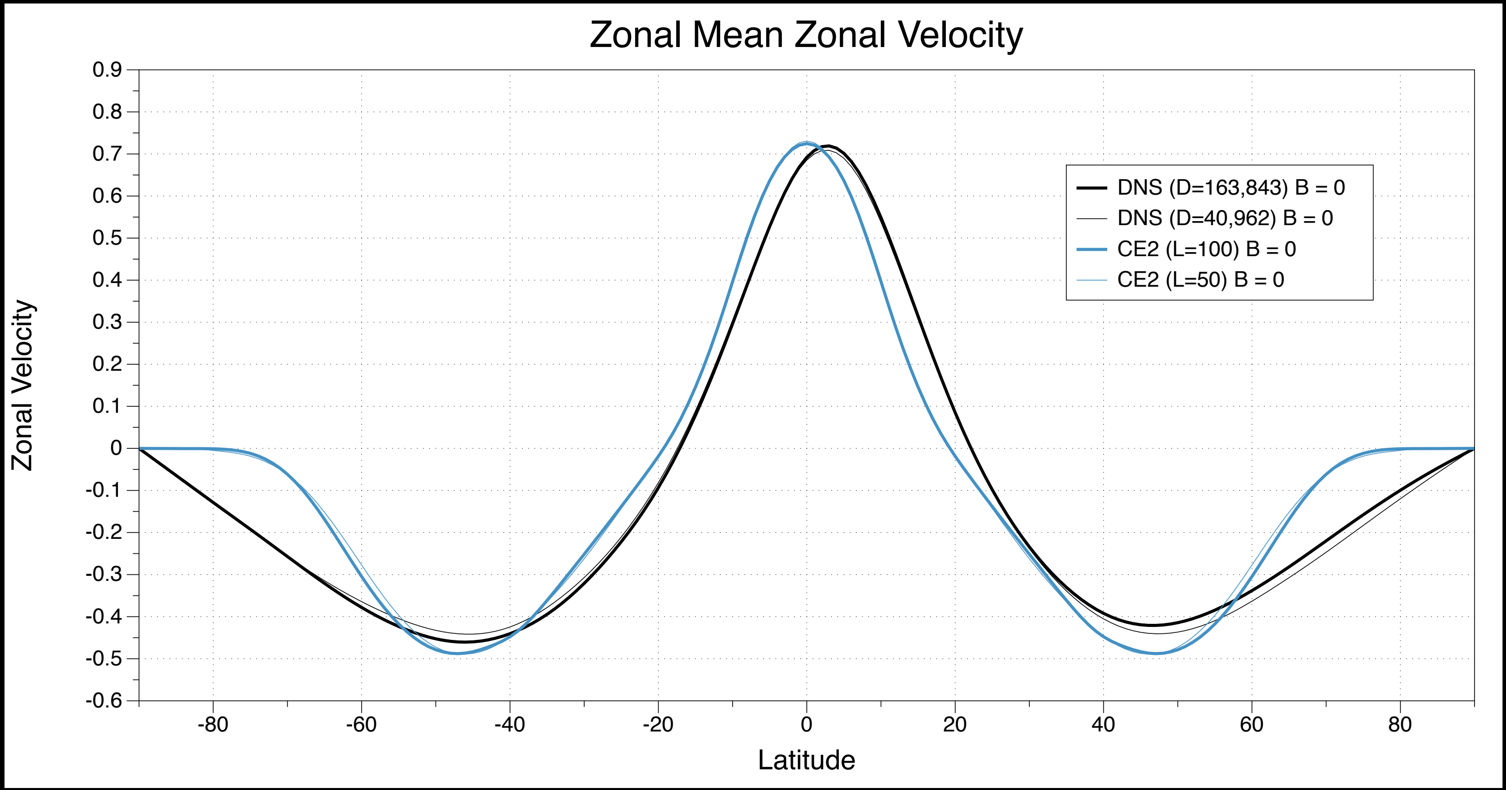
Hydrodynamic: DSS





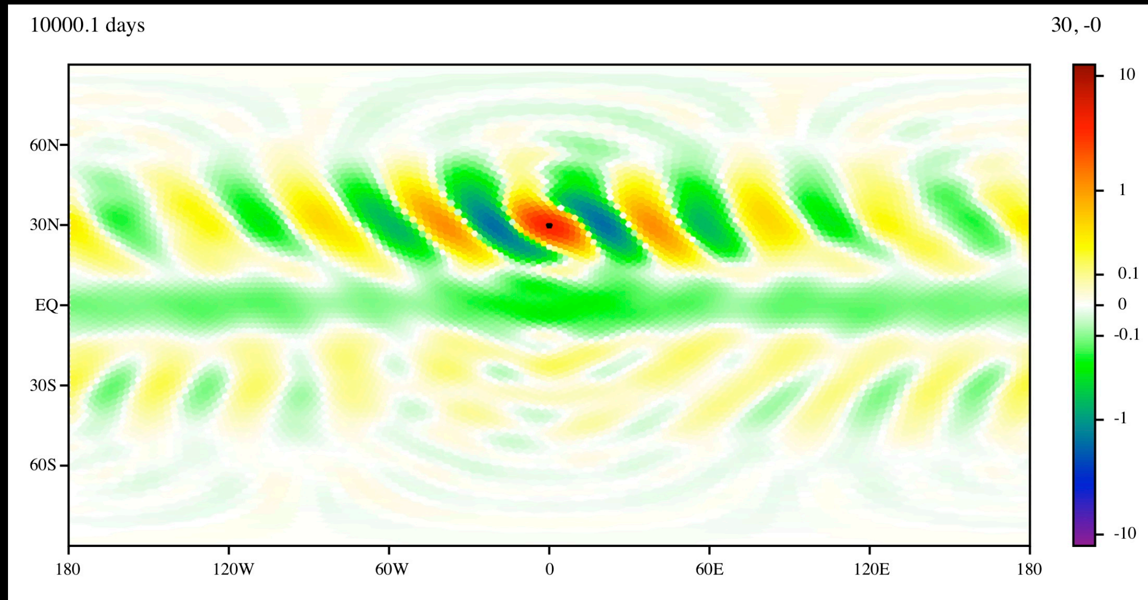
- DSS reproduces driving of jets.
- This is possible even for a cumulant expansion truncated at second order.
- These jets are therefore driven directly via Reynolds stresses.
- Recall: No inverse (or forward) cascade in this model.

Hydrodynamic: DSS

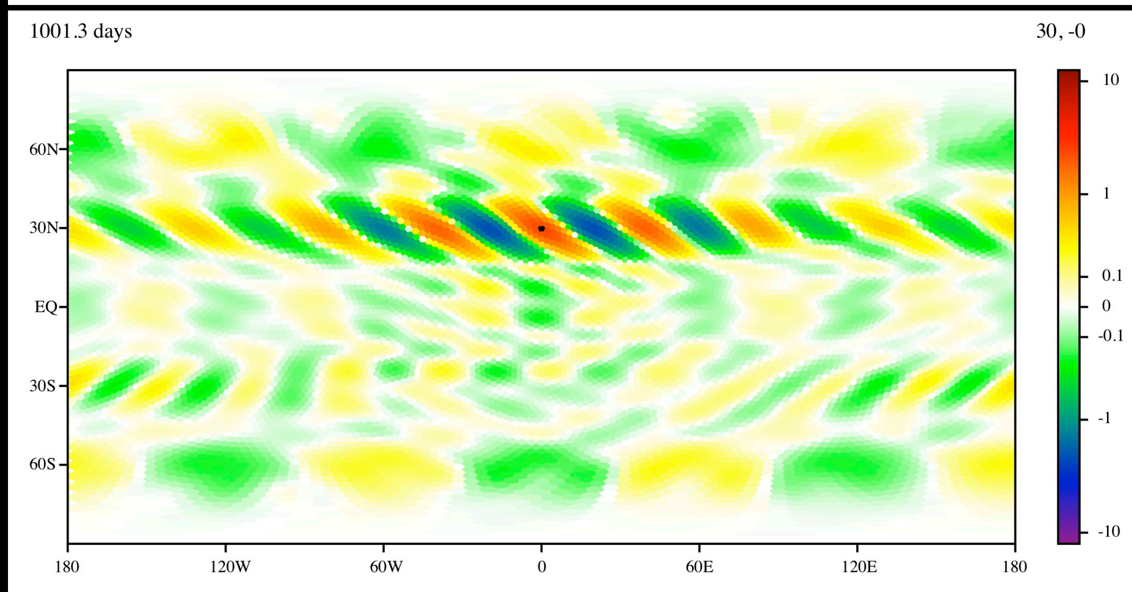


- Detailed comparison: DSS reproduces shape of jet very well (except at poles where mean is small)
- No eddy-eddy scattering

Covariance/2pt correlation?



DNS



CE2

Rank Instability of CE2

Nivarti, Kerswell, Marston & Tobias
arxiv:2202.04127

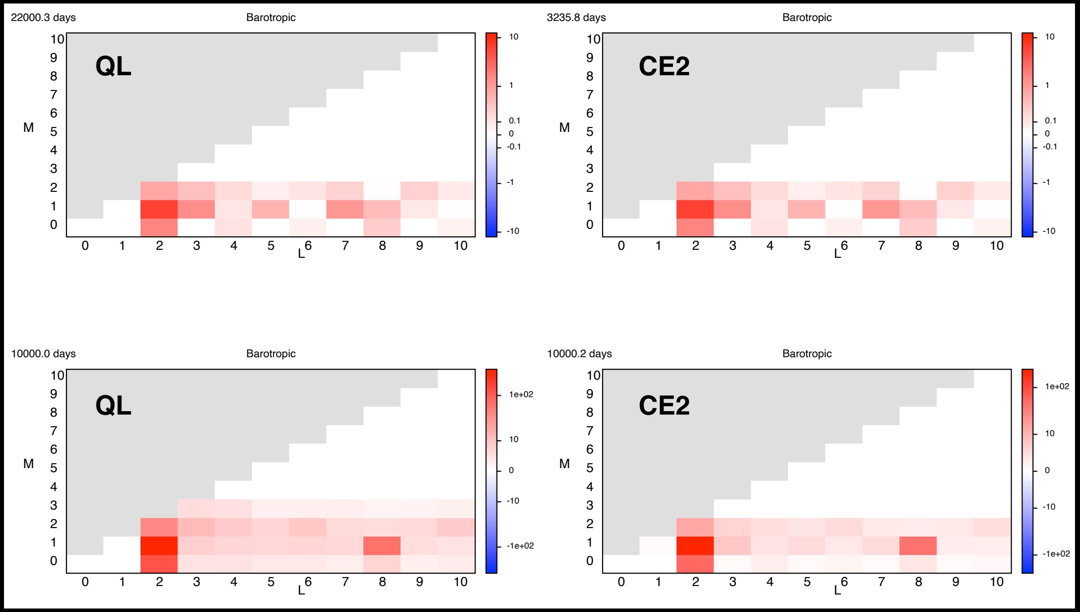
$$\dot{x} = 2x$$

$$\dot{y} = -y$$

$$\frac{d}{dt}\,\overline{xx} = 4\,\overline{xx}$$

$$\frac{d}{dt}\,\overline{xy} = \overline{xy}$$

$$\frac{d}{dt}\,\overline{xx} = -2\,\overline{yy}$$



How to improve on QL/CE2/SSST?

- Generalise the quasilinear approximation
  - Derive statistical theory (GCE2) corresponding to GQL approximation [Nivarti, Marston & Tobias in preparation]

- Include eddy/eddy → eddy interactions
  - CE2.5/CE3 [Marston et al (2015) arXiv:1412.0381]

- Change the definition of averaging to ensemble averaging [Allawala et al 2020]

How to improve on QL/CE2/SSST?

$$\frac{1}{\tau}\,\mathbf{c}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \{\mathcal{N}[\mathbf{c}(\mathbf{r}_1,\mathbf{r}_2),\mathbf{c}(\mathbf{r}_1,\mathbf{r}_3)]\}$$

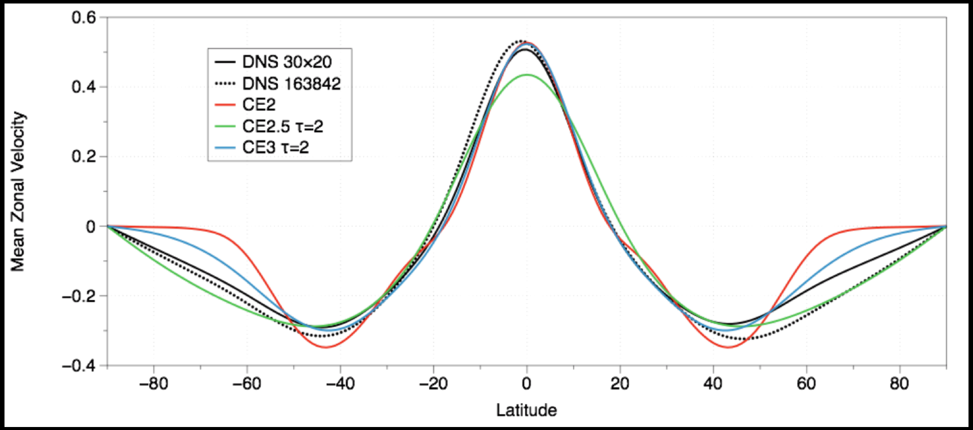
- CE2.5 is the generalization of the eddy-damped quasi-normal Markovian approximation (EDQNM) to anisotropic & inhomogeneous turbulent flows. Appears to be realizable.

- [Marston et al (2015) arXiv:1412.0381]

Improvements on CE2/ Beyond Quasilinearity

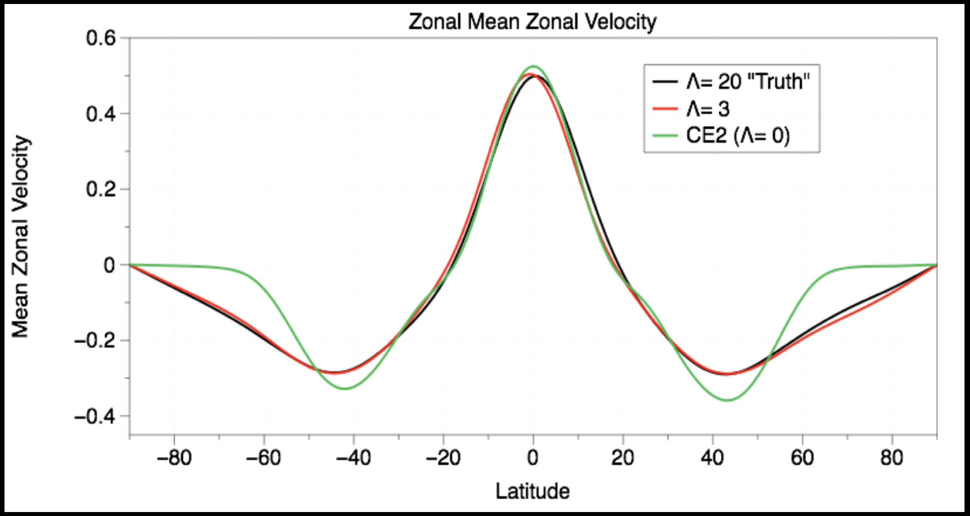
Include eddy/eddy → eddy interactions (CE2.5/CE3)

$$\dot{c}_1 = \mathcal{L}[c_1] + \mathcal{N}[c_1 c_1 + c_2]$$
$$\dot{c}_2 = \mathcal{L}[c_2] + \mathcal{N}[c_1 c_2 + c_3] + \Gamma$$
$$\dot{c}_3 = \mathcal{L}[c_3] + \mathcal{N}[c_1 c_3 + c_2 c_2 + c_4]$$



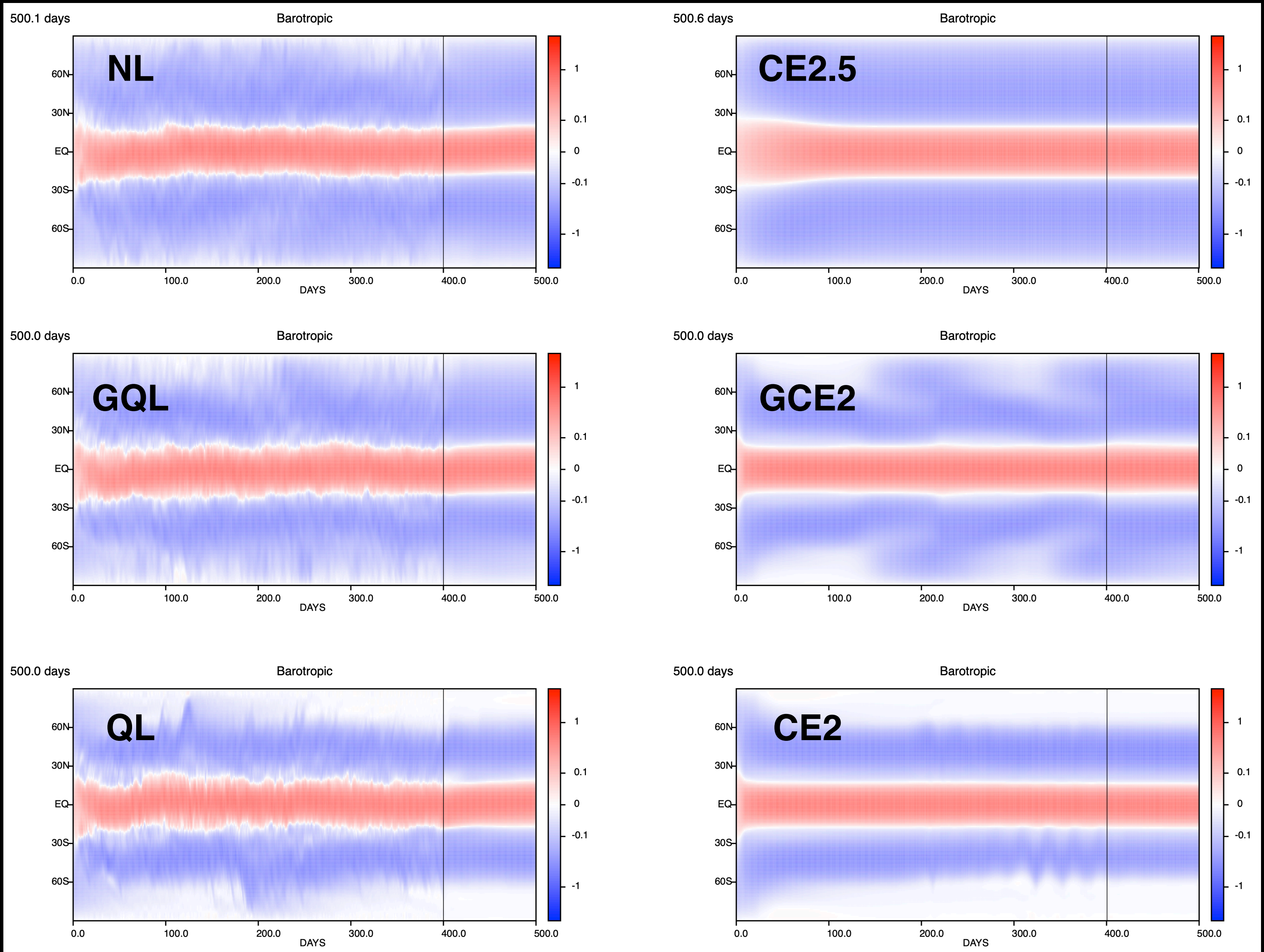
Generalise the definition of mean to large scales (low modes) (GCE2)

$$\partial_t q = \mathcal{L}[q] + \mathcal{N}[q\ q] \quad q = \ell + h$$
$$\partial_t l = \mathcal{L}[l] + \mathcal{N}[l\ l] + \mathcal{N}[h\ h]$$
$$\partial_t h = \mathcal{L}[h] + \mathcal{N}[l\ h]$$
$$\partial_t (h, h) = \mathcal{L}[(h, h)] + 2\mathcal{N}[l\ (h], h)$$



Statistical implementation of GQL

Summary



500.1 days
Barotropic
NL
500.6 days
Barotropic
CE2.5
500.0 days
Barotropic
GQL
500.0 days
Barotropic
GCE2
500.0 days
Barotropic
QL
500.0 days
Barotropic
CE2
60N
30N
EQ
30S
60S
1
0.1
0
-0.1
-1
0.0
100.0
200.0
300.0
400.0
500.0
DAYS

Model Reduction via POD

- Can we do the calculation keeping only the information that is needed? i.e. reduce the basis…

$$c_2 = \langle q'(\mathbf{x_1}) q'(\mathbf{x_2}) \rangle$$

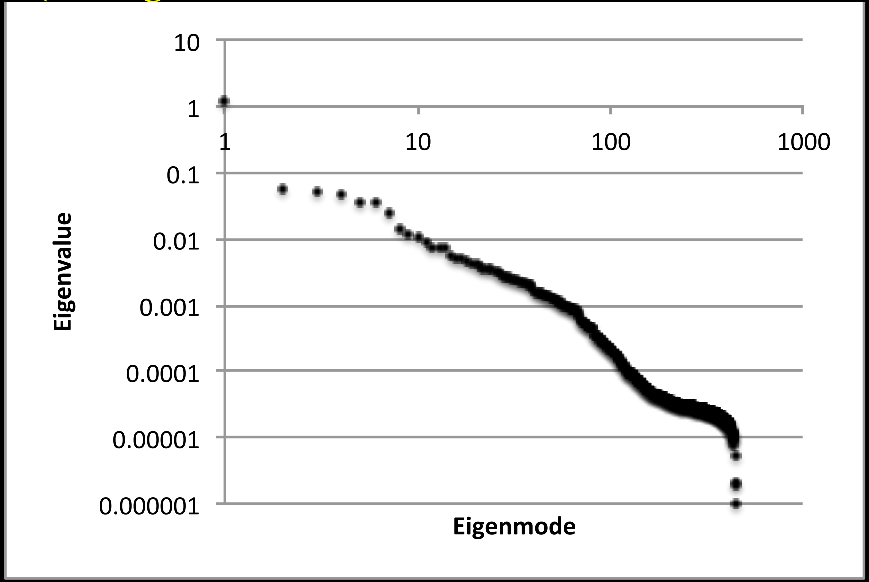
$$c_2(\mathbf{x}_1, \mathbf{x}_2) = \sum_N \lambda_i \phi_i(\mathbf{x}_1) \phi_i(\mathbf{x_2})$$

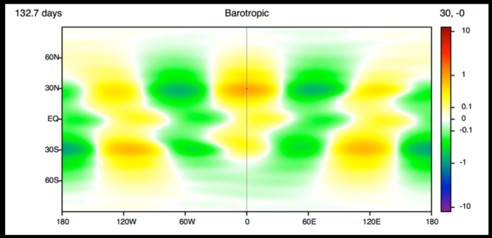
$$\approx \sum_{i=1} \lambda_i \phi_i(\mathbf{x}_1) \phi_i(\mathbf{x_2})$$

Schmidt Decomposition

Proper Orthogonal Decomposition Principal Component Analysis

(see e.g. Holmes et al *Turbulence, Coherent Structures, Dynamical Systems and Symmetry, 2012*)





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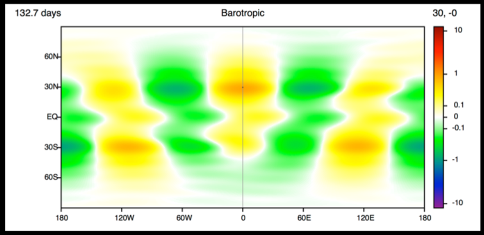
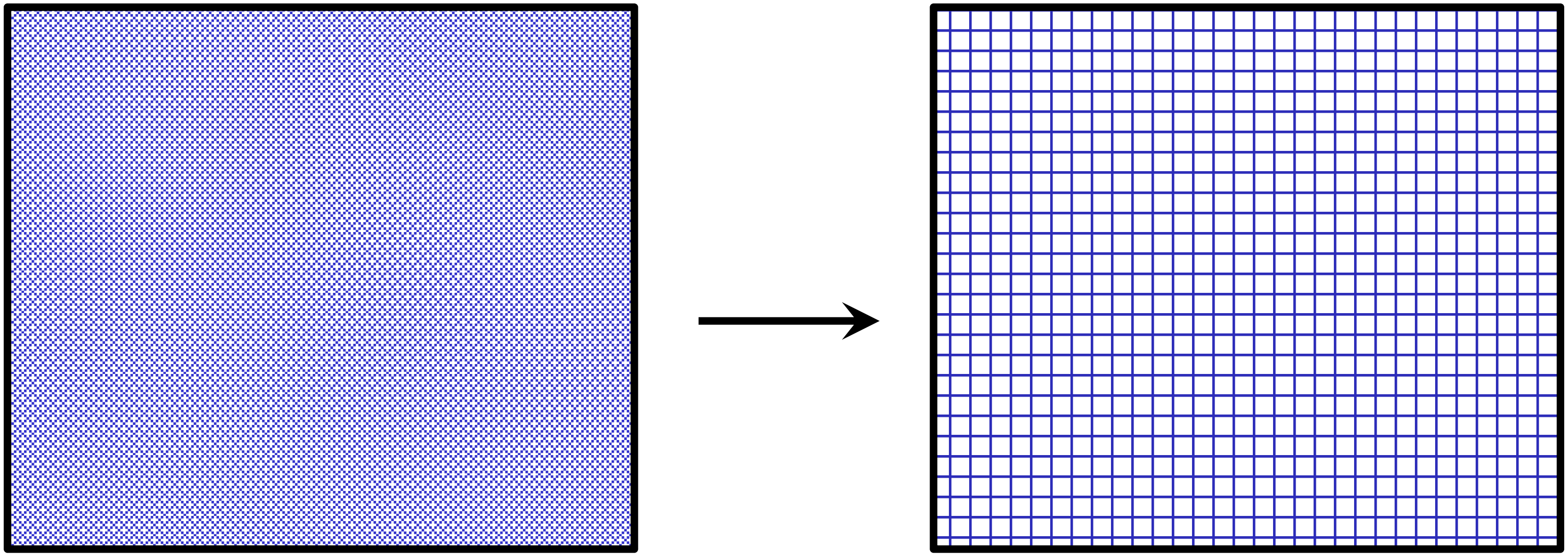


3

DSS as a subgrid model
(Kuan Li)



DNS on a very fine grid

DNS on a coarse grid + DSS of unresolved scales

- Can DSS be used dynamically?
- Think of the first cumulant as the DNS solution on a coarse grid
- Second cumulant is correlation of fluctuations from all the unresolved scales, projected onto the coarse grid
- Very expensive??
    - Just keep first PoD mode of second cumulant (how accurate?)
    - Matrix-free method (Li)

DSS as a subgrid model (Kuan Li)

$$\frac{\partial}{\partial t}\mathbf{f} = \mathcal{N}(\mathbf{f},\mathbf{f}) + \mathcal{L}(\mathbf{f})$$

$$\mathbf{f} = \zeta \oplus n, \qquad \mathcal{N}(\mathbf{f},\mathbf{f}) = \mathcal{J}(\psi,\zeta) \oplus \mathcal{J}(\psi,n)$$

$$C_{\mathbf{ff}} = \sum_i \mathbf{s}_i \otimes \mathbf{s}_i,$$

$$\begin{aligned}\frac{\partial}{\partial t}C_{\mathbf{f}} &= \mathcal{N}(C_{\mathbf{f}},C_{\mathbf{f}}) + \mathcal{L}(C_{\mathbf{f}}) + \mathbf{h},\\ \frac{\partial}{\partial t}C_{\mathbf{ff}} &= \sum_i \mathbf{s}_i \otimes \mathbf{p}_i + \mathbf{p}_i \otimes \mathbf{s}_i.\end{aligned}$$

$$\mathbf{h} = \sum_i \mathcal{N}(\mathbf{s}_i,\mathbf{s}_i)$$

$$\mathbf{p}_i = \left[\mathcal{N}(C_{\mathbf{f}},\mathbf{s}_i) + \mathcal{N}(\mathbf{s}_i,C_{\mathbf{f}}) + \mathcal{L}(\mathbf{s}_i)\right] + 2\tau_d\left[\mathcal{N}(\mathbf{h},\mathbf{s}_i) + \mathcal{N}(\mathbf{s}_i,\mathbf{h})\right].$$

Quasilinear terms

EDQNM type terms

DSS as a subgrid model: An example (Kuan Li)

Hasegawa-Wakatani

$$\frac{\partial \zeta}{\partial t} + \{\phi, \zeta\} = \alpha(\phi - n) - \mu\nabla^4\zeta$$
$$\frac{\partial n}{\partial t} + \{\phi, n\} = \alpha(\phi - n) - \kappa\frac{\partial \phi}{\partial y} - \mu\nabla^4 n$$

$$\kappa = -\partial/\partial x \ln n_0$$

$$\nabla_\perp^2 \phi = \zeta.$$

$$\text{zonal} : \langle f \rangle \equiv \frac{1}{L_y}\int f\, dy, \quad \text{nonzonal} : \tilde{f} \equiv f - \langle f \rangle$$

(Modified) Hasegawa-Wakatani

$$\frac{\partial \zeta}{\partial t} + \{\phi, \zeta\} = \alpha(\tilde{\phi} - \tilde{n}) - \mu\nabla^4\zeta$$
$$\frac{\partial n}{\partial t} + \{\phi, n\} = \alpha(\tilde{\phi} - \tilde{n}) - \kappa\frac{\partial \phi}{\partial y} - \mu\nabla^4 n$$

M. Wakatani and A. Hasegawa, “A collisional drift wave description of plasma edge turbulence”, Physics of Fluids, 27 (3), 1984.

A Hasegawa and M Wakatani, “Self-Organization of Electrostatic Turbulence in a Cylindrical Plasma”, Physical Review Letters, 59 (14), 1987.

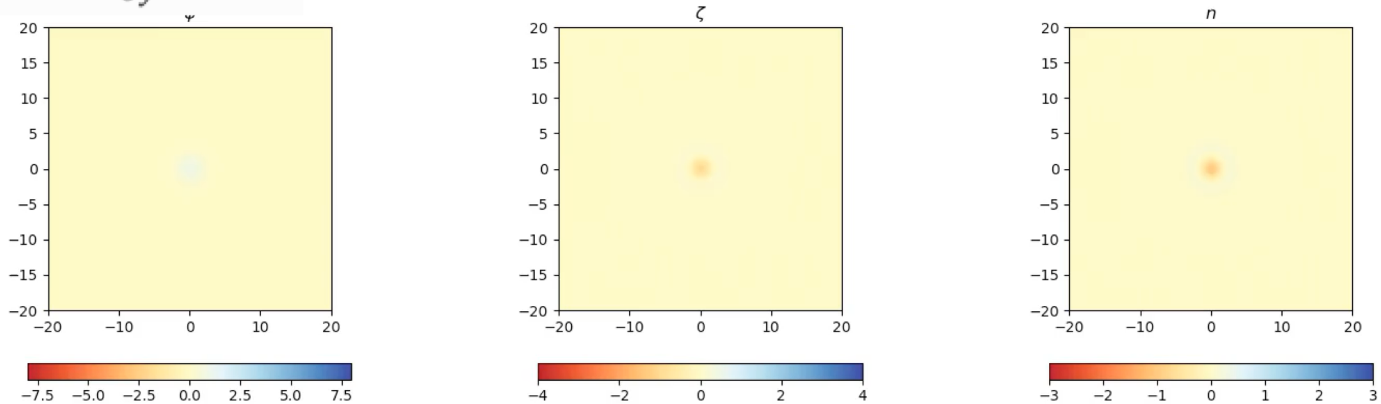
Numata, R., Ball, R., & Dewar, R. L, “Bifurcation in electrostatic resistive drift wave turbulence”. Physics of Plasmas, 14 (10), 102312, 2007

DSS as a subgrid model: An example (Kuan Li)

$$\frac{\partial \zeta}{\partial t} + \{\phi, \zeta\} = \alpha(\tilde{\phi} - \tilde{n}) - \mu \nabla^4 \zeta$$

$$\frac{\partial n}{\partial t} + \{\phi, n\} = \alpha(\tilde{\phi} - \tilde{n}) - \kappa \frac{\partial \phi}{\partial y} - \mu \nabla^4 n$$

DNS



$\kappa=1$, $\mu=10^{-4}$, $\alpha = 1$

Code verified against simulations with Gkeyll

http://ammar-hakim.org/sj/je/je17/je17-hasegawa-wakatani.html

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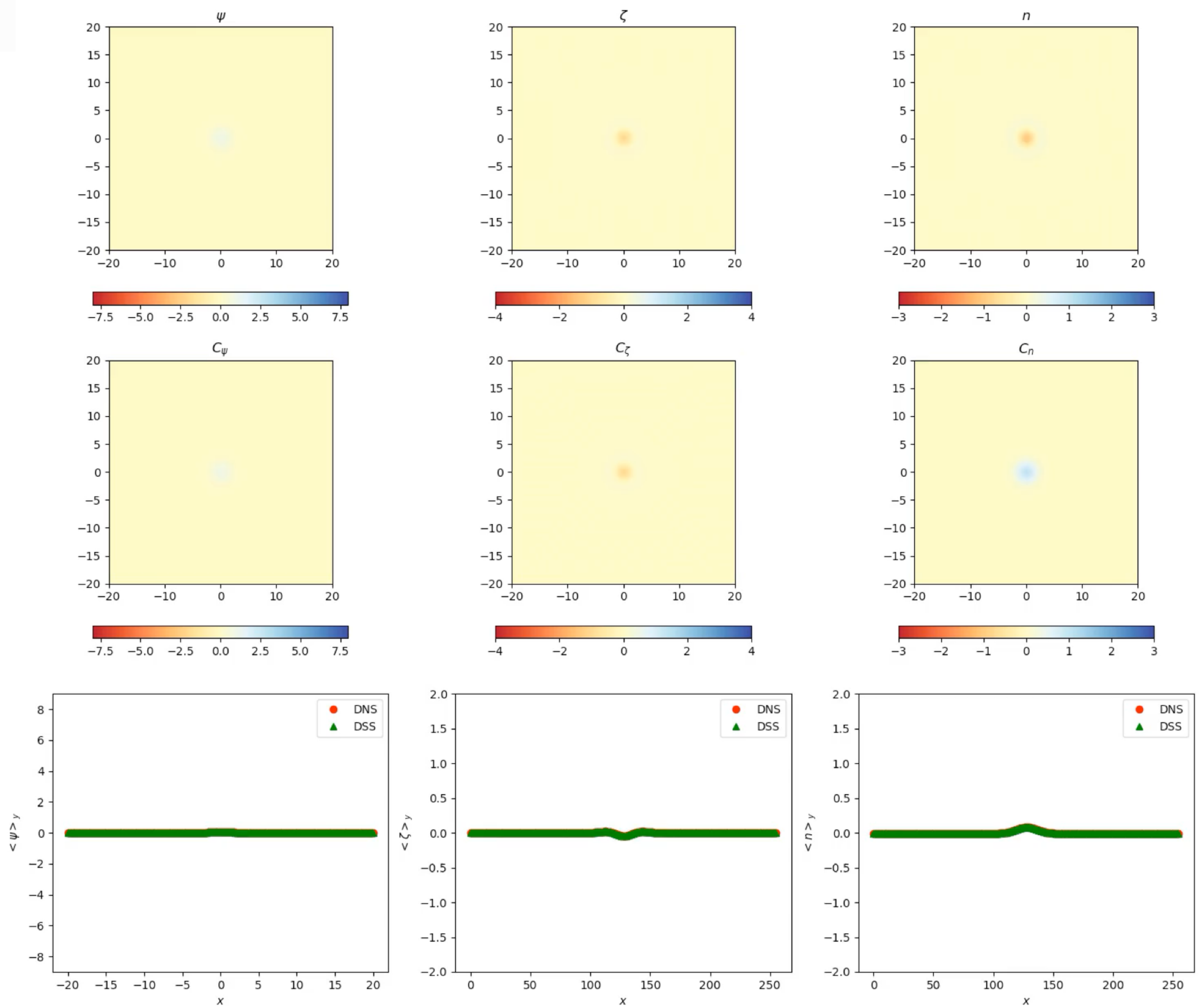
DNS (120^2)

DSS 30^2

COMPARISON



Conclusions

- Methods from non-equilibrium statistical mechanics *may* be useful in determining statistics of geophysical/astrophysical systems that include interactions of mean flows and fields with turbulence. *Direct Statistical Simulation.*
- One can move beyond quasilinearity by
  - Going to higher order in cumulant expansion
  - Generalising the quasilinear approximation (GQL)
- Significant speed-up can be achieved by using a reduced basis.
- This technique may form the basis for conservative statistical sub-grid models.
- For astrophysical fluids/dynamos this may offer a fast, efficient, conservative, self-consistent mean-field theory