Anton Andreev
Department of Physics, University of Washington, Seattle

**Field theoretic techniques in the theory of disordered systems**
**Boulder School on Condensed Matter Physics, 2013**

# 1 Preface

The purpose of these lectures is to give a pedagogical introduction into the diagrammatic and field theoretic techniques for disordered conductors. I will try to expose the main physical ideas behind the formalism and introduce a language that has been developed in the theory of disordered systems. Hopefully these lectures will serve as a basis for the more advanced courses in this school.

When studying theoretical techniques it is useful to have a clear qualitative picture of the physics they describe. I think this is especially important in the theory of disordered conductors, where a compact and intuitive language based on the diagrammatic technique has been developed. This language was developed in informal personal discussions between researchers advancing the field. It is ideally suited for personal communications but is difficult to present in a format of a monograph, where a certain degree of rigor is expected. Lectures at a summer school seem like an ideal setting to present this informal and intuitive language behind the formalism. [1] I would be happy even with a partial success in this regard.

# 2 Classical treatment of electron transport

In this section I will review the classical theory of electron transport.

## 2.1 Boltzmann kinetic equation

Let us begin with a brief survey of the Boltzmann kinetic equation. It will help us develop intuition and introduce terminology relevant for the subsequent discussion. More importantly it will help us place the quantum effects that will be the focus of this school in a broader context of general theory of transport phenomena in disordered conductors. I will assume familiarity with the Boltzmann equation and will be somewhat sloppy with notations hoping that some of them will be obvious. A more detailed exposition of the Boltzmann equation be found in many textbooks, for example Ref. [1]. The approach was introduced by Boltzmann, and was originally applied to classical gased. Peierls realized that it also applied to electrons [2] in metals as long their motion between collisions was classical. This condition is expressed by the inequality $k_F l \gg 1$, where $k_F$ is the Fermi momentum (in units of $\hbar$) and $l$ is the mean free path of electrons.

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[1] Anatoly Larkin liked to say that different branches of human knowledge can be classified into "oral" and "written" sciences. Written sciences are those that can be formalized and yield themselves to a rigorous exposition in a texbook (calculus is a good example). The oral sciences are best passed on in informal personal discussions. For example geometry in ancient Greece was taught in informal settings with pictures being drawn in the sand. This tradition is largely lost because of the difficulty of putting it into a written form.

[2] Strictly speaking it describes the motion of quasiparticles in the Landau theory of Fermi liquid, see e.g. Ref. [2]. They obey Fermi statistics and bare many similarities with non-interacting fermions.

The Boltzmann equation describes time evolution of the electron distribution function $f(\mathbf{r},\mathbf{p},t)$ in the presence of an external force $\mathbf{F}$, and is usually written in the form

$$(\partial_t + \mathbf{v}\cdot\nabla + \mathbf{F}\cdot\partial_{\mathbf{p}})\, f = \mathrm{St}[f]. \tag{1}$$

The left hand side describes the time evolution of $f$ due to the free motion of electrons, whereas the collision integral $\mathrm{St}[f]$ in the right hand side describes the evolution due to collisions of electrons with impurities, phonons, and other electrons. Collisions are treated as instantaneous and local (occurring at a single point in space). The collision integral should in general be expressed in terms of higher order correlation functions. Let us take electron-electron collisions as an example. In this case the collision integral is expressed in terms of the two-point distribution function for the colliding electrons. The two point function satisfies its own kinetic equation, where the collision integral is expressed in terms of higher order correlation functions. This procedure leads to a chain of coupled equation for all higher order correlation functions, known as the BBGKY (for Bogoliubov, Born, Green, Kirkwood, and Yvon) hierarchy.

A crucial assumption underlying the Boltzmann description is that of relaxation of correlations. The basic idea is that many-body correlations, which are necessarily induced by collisions between particles, decay during the long stretches of subsequent free motion. Therefore the distribution functions of the colliding electrons may be assumed uncorrelated. This enables us to express the higher order distribution functions appearing in the collision integral in terms of the one-point function $f$ itself. This procedure decouples the different levels in the hierarchy and makes Eq. (1) a closed equation for the one particle distribution function $f$. According to the principle of relaxation of correlations collisions of different types (electron-electron, electron-phonon, electron-impurity) are also treated as uncorrelated and are described by independent collision integrals. An important outcome of this treatment is that the resistivity turns out to be proportional to the sum of momentum relaxation rates — the result known as the Matthiessen's rule.

In semiconductors and many metals at low temperatures the umklapp processes are frozen out. Therefore electron-electron collisions conserve (quasi-)momentum, and the resistivity is controlled by electron-impurity and electron-phonon collisions. The latter are also suppressed at low temperatures, and the low temperature (so called residual) resistivity is determined by electron-impurity scattering. The corresponding collision integral has the form

$$\mathrm{St}[f] = \int (dp')W(\mathbf{p},\mathbf{p}')[f(\mathbf{r},\mathbf{p}',t) - f(\mathbf{r},\mathbf{p},t)]. \tag{2}$$

Here $W(\mathbf{p},\mathbf{p}')$ denotes the scattering probability per unit time form state $\mathbf{p}'$ to state $\mathbf{p}$, $(dp') = \frac{d^3p'}{(2\pi\hbar)^3}$, and we used the Stückelberg transformation of the collision integral.[3]

In linear response we typically consider monochromatic electric fields $\mathbf{E} = -\mathbf{F}/e = \mathbf{E}(\mathbf{r})e^{-i\omega t}$ ($e$ is the absolute value of the electron charge) and write the distribution function as $f = f_0(\epsilon) + \delta f$, where $f_0(\epsilon)$ is the equilibrium distribution function which depends only on the energy $\epsilon$. Recalling that the equilibrium distribution nullifies the collision integral

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[3] This transformation relies on the identity $\int(dp')W(\mathbf{p}',\mathbf{p}) = \int(dp')W(\mathbf{p},\mathbf{p}')$, which follows from unitarity of scattering, see §§ 2, 3 of Ref. [1]

and passing to the Fourier representation we obtain to linear order in $\mathbf{E}$,

$$(-i\omega + \mathbf{v}\cdot\nabla)\,\delta f - \mathrm{St}[\delta f] = e\mathbf{E}(\mathbf{r})\cdot\mathbf{v}\frac{df_0(\epsilon)}{d\epsilon}. \tag{3}$$

Let us assume the simplest spectrum $\epsilon = p^2/2m$ and write the nonequilibrium part of the electron distribution in the form $\delta f(\epsilon, \mathbf{r}, \mathbf{n})$ where $\mathbf{n}$ is the momentum direction. Keeping in mind that the electron energy is conserved in collisions with impurities we observe the distribution functions at each different energies obey independent equations,

$$\mathcal{B}_\epsilon \delta f(\epsilon, \mathbf{r}, \mathbf{n}) = ev\mathbf{E}(\mathbf{r})\cdot\mathbf{n}\,\frac{df_0(\epsilon)}{d\epsilon}, \tag{4}$$

where $\mathcal{B}_\epsilon$ denotes the Boltzmann operator defined by

$$\mathcal{B}_\epsilon \delta f(\epsilon, \mathbf{r}, \mathbf{n}) = \left(-i\omega + v\mathbf{n}\cdot\nabla + \frac{1}{\tau}\right)\delta f(\epsilon, \mathbf{r}, \mathbf{n}) - n_i v \int d\mathbf{n}'\sigma_\epsilon(\mathbf{n}, \mathbf{n}')\delta f(\epsilon, \mathbf{r}, \mathbf{n}'). \tag{5}$$

Here we expressed the scattering probability in terms of the electron velocity $v = p/m$, impurity concentration $n_i$ and the (energy dependent) differential scattering cross-section $\sigma_\epsilon(\mathbf{n}, \mathbf{n}')$ at a single impurity. We also introduced the (energy dependent) "out" scattering rate

$$\frac{1}{\tau} = n_i v \int d\mathbf{n}'\sigma_\epsilon(\mathbf{n}, \mathbf{n}'). \tag{6}$$

It is useful to introduce the resolvent of the Boltzmann operator $\mathcal{K} \equiv \mathcal{B}_\epsilon^{-1}$. It satisfies the inhomogeneous equation

$$\left(-i\omega + v\mathbf{n}\cdot\nabla_\mathbf{r} + \frac{1}{\tau}\right)\mathcal{K}_\epsilon(\mathbf{r}, \mathbf{n}; \mathbf{r}', \mathbf{n}') - n_i v \int d\tilde{\mathbf{n}}\sigma_\epsilon(\mathbf{n}, \tilde{\mathbf{n}})\mathcal{K}_\epsilon(\mathbf{r}, \tilde{\mathbf{n}}; \mathbf{r}', \mathbf{n}') = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{n} - \mathbf{n}'). \tag{7}$$

The boundary conditions at insulating interfaces correspond to the condition of zero current across the interface. We will see that the resolvent $\mathcal{K}_\epsilon$ corresponds to the series of "ladder" diagrams in the quantum diagrammatic treatment of electron transport. With its aid the solution of Eq. (4) may be formally expressed as

$$\delta f(\epsilon, \mathbf{r}, \mathbf{n}) = ev\frac{df_0(\epsilon)}{d\epsilon}\int d\mathbf{n}'d\mathbf{r}'\mathcal{K}_\epsilon(\mathbf{r}, \mathbf{n}; \mathbf{r}', \mathbf{n}')\,\mathbf{E}(\mathbf{r}')\cdot\mathbf{n}'. \tag{8}$$

*Resolvent for $s$-wave scatterers.* Let us consider an instructive example of purely $s$-wave scattering. This approximation of point-like impurities captures much of the essential physics and is frequently used in the theory of disordered metals to simplify the treatment. In this case the scattering differential cross-section $\sigma_\epsilon(\mathbf{n}, \tilde{\mathbf{n}})$ is independent of the initial and final momenta. As a result the second term in the left hand side of (7) becomes independent of $\mathbf{n}$. Let us specialize to a bulk conductor, for which the resolvent $\mathcal{K}_\epsilon$ depends only on the coordinate difference, $\mathbf{r} - \mathbf{r}'$. Upon a Fourier transformation with respect to this variable Eq. (7) reduces to

$$\left(-i\omega + iv\mathbf{q}\cdot\mathbf{n} + \frac{1}{\tau}\right)\mathcal{K}_\epsilon(\mathbf{n}, \mathbf{n}') - \frac{1}{\tau}\langle\mathcal{K}_\epsilon(\mathbf{n}, \mathbf{n}')\rangle_\mathbf{n} = \delta(\mathbf{n} - \mathbf{n}'), \tag{9}$$

where $\langle \ldots \rangle_{\mathbf{n}}$ denotes the average over the directions, $\mathbf{n}$. The solution of this equation is

$$\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}') = \frac{\tau\delta(\mathbf{n}-\mathbf{n}')}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} + \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}}\frac{\tau}{\left(1-\left\langle\frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}}\right\rangle_{\mathbf{n}}\right)}\frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}'}, \tag{10}$$

where $l = v\tau$ is the mean free path.

The average over the directions is readily evaluated in two ($d = 2$) and three ($d = 3$) dimensions,

$$\left\langle \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} \right\rangle_{\mathbf{n}} = \begin{cases} \frac{1}{\sqrt{(1-i\omega\tau)^2+q^2l^2}}, & d=2, \\ 1-\frac{1}{2iql}\ln\frac{1-i\omega\tau+iql}{1-i\omega\tau-iql}, & d=3. \end{cases} \tag{11}$$

The expression (9) for the resolvent apply in both the ballistic ($\omega\tau, ql \gg 1$) and diffusive ($\omega\tau, ql \ll 1$) regimes. In the ballistic case the resolvent $\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}')$ has a sensitive dependence on the directions $\mathbf{n},\mathbf{n}'$.

In some cases, for example in the study of propagation of waves through random media both the source (injection) and the observable (detector) may have a high selectivity with respect to direction, and the detailed dependence of $\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}')$ on $\mathbf{n},\mathbf{n}'$ is essential. In contrast, in disordered conductors we are often interested in the diffusive regime, where both the observables and the external perturbations correspond only to the zeroth (density) or first (current) spherical harmonics in $\mathbf{n},\mathbf{n}'$.

*Diffusive regime.* For $\omega\tau, ql \ll 1$ the expressions in Eq. (11) simplify to

$$\left\langle \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} \right\rangle_{\mathbf{n}} = 1-\tau(-i\omega+Dq^2), \tag{12}$$

where $D = vl/d$ is the diffusion coefficient. Therefore in the diffusive regime the resolvent $\mathcal{K}_\epsilon$ is given by

$$\mathcal{K}_\epsilon^{diff}(\mathbf{n},\mathbf{n}') = \frac{\tau\delta(\mathbf{n}-\mathbf{n}')}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} + \frac{1}{(1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n})(-i\omega+Dq^2)(1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}')}. \tag{13}$$

When transformed to position space the first term above decays exponentially on the scale of the mean free path and represents the probability for the electron to remain unscattered from the original momentum state. The second term corresponds to a convolution of three kernels, two of which are short range, while the other, $(-i\omega+Dq^2)^{-1}$ represents the diffusion propagator and has a long range character. Keeping only the leading terms in the zeroth and first spherical harmonics in $\mathbf{n},\mathbf{n}'$ we may further approximate the resolvent as

$$\mathcal{K}_\epsilon^{diff}(\mathbf{n},\mathbf{n}') = \tau d\,\mathbf{n}\cdot\mathbf{n}' + \frac{(1-il\mathbf{q}\cdot\mathbf{n})(1-il\mathbf{q}\cdot\mathbf{n}')}{(-i\omega+Dq^2)}. \tag{14}$$

*Diffusive regime for general scatterers: transport relaxation time.* Let us now turn to the case of non-$s$-wave scatterers. In the Fourier representation Eq. (7) becomes

$$\left(-i\omega+iv\mathbf{q}\cdot\mathbf{n}+\frac{1}{\tau}\right)\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}') - n_i v\int d\tilde{\mathbf{n}}\sigma_\epsilon(\mathbf{n},\tilde{\mathbf{n}})\mathcal{K}_\epsilon(\tilde{\mathbf{n}},\mathbf{n}') = \delta(\mathbf{n}-\mathbf{n}'). \tag{15}$$

In the diffusive regime it is convenient to use the multipole expansion in $\mathbf{n}$,

$$\mathcal{K}_{\epsilon}^{diff}(\mathbf{n},\mathbf{n}') = \mathcal{N}_{\epsilon}(\mathbf{n}') + \mathbf{n}\cdot\boldsymbol{\mathcal{J}}_{\epsilon}(\mathbf{n}') + \dots . \tag{16}$$

Projecting (15) onto the zeroth spherical harmonic in $\mathbf{n}$ and we get

$$-i\omega\mathcal{N}_{\epsilon}(\mathbf{n}') + i\frac{v}{d}\mathbf{q}\cdot\boldsymbol{\mathcal{J}}_{\epsilon}(\mathbf{n}') = 1. \tag{17}$$

Note that the scattering rate disappeared from this relation. This is a consequence of conservation of the number of particles at each energy by collisions with impurities (the left hand side represents the "continuity equation").

Projecting Eq. (15) onto the first spherical harmonic we get to leading order in $\omega\tau$

$$\boldsymbol{\mathcal{J}}_{\epsilon}(\mathbf{n}') = \tau_{tr} d\,\mathbf{n}' - il_{tr}\mathbf{q}\mathcal{N}_{\epsilon}(\mathbf{n}'), \tag{18}$$

where $l_{tr} = v\tau_{tr}$ is the transport mean free path. The transport mean free time, $\tau_{tr}$, characterizes relaxation of momentum (see e.g. § 11 of Ref. [1]) and is defined by the relation

$$\frac{1}{\tau_{tr}} = n_i v\int d\mathbf{n}'\sigma_{\epsilon}(\mathbf{n},\mathbf{n}')(1-\mathbf{n}\cdot\mathbf{n}'). \tag{19}$$

The difference between the transport and "out" relaxation rates is described by the second term in the bracket in the right hand side. For purely $s$-wave scattering this term vanishes, and the transport mean free time coincides with the "out" mean free time defined in Eq. (6). In this regime the electron momenta before and after the collision are uncorrelated, electrons "lose" memory of their momentum upon a single collision. In the opposite case of small angle scattering the electron momentum electrons is randomized only after a large number of collisions, $\tau_{tr} \gg \tau$.

Combining Eqs. (16), (17), and (18) we get

$$\mathcal{K}_{\epsilon}^{diff}(\mathbf{n},\mathbf{n}') = \tau_{tr} d\,\mathbf{n}\cdot\mathbf{n}' + \frac{(1-il_{tr}\mathbf{q}\cdot\mathbf{n})(1-il_{tr}\mathbf{q}\cdot\mathbf{n}')}{(-i\omega + Dq^2)}, \tag{20}$$

where the diffusion constant is expressed in terms of the *transport* mean free path in the usual way, $D = vl_{tr}/d$. The only difference with Eq. (13) is the replacement $\tau \to \tau_{tr}$.

*Diffusive response to a longitudinal electric field.* Let us consider the electric field of the form $\mathbf{E}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t)$, where the electric potential is of the form $\phi(\mathbf{r},t) = \phi_0\, e^{i\mathbf{q}\cdot\mathbf{E}(\mathbf{r})-i\omega t}$. In this case Eq. (8) gives in the Fourier representation

$$\delta f(\epsilon,\mathbf{n}) = -ie\phi_0 v\frac{df_0(\epsilon)}{d\epsilon}\int d\mathbf{n}'\mathcal{K}_{\epsilon}^{diff}(\mathbf{n},\mathbf{n}')\,\mathbf{q}\cdot\mathbf{n}'. \tag{21}$$

We expand the nonequilibrium part of the distribution function $\delta f$ to first order in the spherical harmonics,

$$\delta f(\epsilon,\mathbf{n}) = N_{\epsilon} + \mathbf{n}\cdot\mathbf{J}_{\epsilon}. \tag{22}$$

The nonequilibrium part of density and the current density (at a given energy) are described by the zeroth ($N_{\epsilon}$) and the first ($\mathbf{J}_{\epsilon}$) spherical harmonics respectively.

$$\begin{aligned} n &= 2\int(dp)N_{\epsilon} = 2\int\nu(\epsilon)N_{\epsilon}d\epsilon, && (23)\\ \mathbf{j} &= -2e\int(dp)\mathbf{v}\,(\mathbf{J}_{\epsilon}\cdot\mathbf{n}) = -2\frac{e}{d}\int\nu(\epsilon)v\mathbf{J}_{\epsilon}d\epsilon, && (24)\end{aligned}$$

where $\nu(\epsilon)$ is the density of states per unit energy for a single spin component.

Substituting Eqs. (22) and (20) into (21) we obtain

$$\mathbf{J}_\epsilon = -iel_{tr}\mathbf{q}\phi_0 \frac{df_0(\epsilon)}{d\epsilon} - il_{tr}\mathbf{q}N_\epsilon, \tag{25}$$

where the density response is given by

$$N_\epsilon = -\frac{eDq^2\phi_0}{-i\omega + Dq^2}\frac{df_0(\epsilon)}{d\epsilon} = -\left(1 + \frac{i\omega}{-i\omega + Dq^2}\right)e\phi_0\frac{df_0(\epsilon)}{d\epsilon}. \tag{26}$$

In the last expression we wrote the answer in a suggestive form that will be useful for comparison with the quantum diagrammatic treatment. The first term in the bracket describes the static response. The second term corresponds to the density induced by the longitudinal *ac* currents. Since it contains the diffusion constant, with the aid of the Einstein relation, we can extract the conductivity from the *ac* density response to an external scalar potential.

The first term in Eq. (25) represents the current induced by the electric field. The second term represents the diffusion current created by the density gradient. For a uniform electric field it vanishes. In this case, combining Eqs. (24) and (25) and substituting the Fermi function for $f_0(\epsilon)$ we obtain the Drude result,

$$\mathbf{j} = \sigma\mathbf{E_0}, \quad \sigma = 2e^2D\nu_0, \tag{27}$$

where $\nu_0$ is the density of states per spin at the Fermi energy.

The second expression above represents the Einstein relation between the conductivity and the diffusion coefficient. It follows form the fact than in thermodynamic equilibrium in the presence of an external electrostatic potential $\phi$ the gradiant of the electron density is given by $\nabla n = 2e\nu_0\nabla\phi$ and the current, $\mathbf{j} = -\sigma\nabla\phi + eD\nabla n$, must vanish. The Einstein relation enables us to extract the conductivity from the density response (26). This method is often used in diagrammatic treatment of electron transport.

# References

[1] E. M. Lifshitz, and L. P. Pitaevskii, *Physical kinetics.*

[2] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of quantum field theory in statistical physics.*