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Diagrammatic and field-theoretic techniques for disordered conductors
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1 Preface

The purpose of these lectures is to give a pedagogical introduction into the diagrammatic and field theoretic techniques for disordered conductors. I will try to expose the main physical ideas behind the formalism and introduce a language that has been developed in the theory of disordered systems. Hopefully these lectures will serve as a basis for the more advanced courses in this school.

When studying theoretical techniques it is useful to have a clear qualitative picture of the physics they describe. I think this is especially important in the theory of disordered conductors, where a compact and intuitive language based on the diagrammatic technique has been developed. This language was developed in informal personal discussions between researchers advancing the field. It is ideally suited for personal communications but is difficult to present in a format of a monograph, where a certain degree of rigor is expected. Lectures at a summer school seem like an ideal setting to present this informal and intuitive language behind the formalism. [1] I would be happy even with a partial success in this regard.

2 Classical treatment of electron transport

In this section I will review the classical theory of electron transport.

2.1 Boltzmann kinetic equation

Let us begin with a brief survey of the Boltzmann kinetic equation. It will help us develop intuition and introduce terminology relevant for the subsequent discussion. More importantly it will help us place the quantum effects that will be the focus of this school in a broader context of general theory of transport phenomena in disordered conductors. I will assume familiarity with the Boltzmann equation and will be somewhat sloppy with notations hoping that some of them will be obvious. A more detailed exposition of the Boltzmann equation be found in many textbooks, for example Ref. [1]. The approach was introduced by Boltzmann, and was originally applied to classical gased. Peierls realized that it also applied to electrons [2] in metals as long their motion between collisions was classical. This condition is expressed by the inequality $k_F l \gg 1$, where k_F is the Fermi momentum (in units of $\hbar$) and l is the mean free path of electrons.

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[1] Anatoly Larkin liked to say that different branches of human knowledge can be classified into "oral" and "written" sciences. Written sciences are those that can be formalized and yield themselves to a rigorous exposition in a texbook (calculus is a good example). The oral sciences are best passed on in informal personal discussions. For example geometry in ancient Greece was taught in informal settings with pictures being drawn in the sand. This tradition is largely lost because of the difficulty of putting it into a written form.

[2] Strictly speaking it describes the motion of quasiparticles in the Landau theory of Fermi liquid, see e.g. Ref. [2]. They obey Fermi statistics and bare many similarities with non-interacting fermions.

The Boltzmann equation describes time evolution of the electron distribution function $f(\mathbf{r},\mathbf{p},t)$ in the presence of an external force $\mathbf{F}$, and is usually written in the form

$$(\partial_t + \mathbf{v}\cdot\nabla + \mathbf{F}\cdot\partial_{\mathbf{p}})\, f = \mathrm{St}[f]. \tag{1}$$

The left hand side describes the time evolution of f due to the free motion of electrons, whereas the collision integral $\mathrm{St}[f]$ in the right hand side describes the evolution due to collisions of electrons with impurities, phonons, and other electrons. Collisions are treated as instantaneous and local (occurring at a single point in space). The collision integral should in general be expressed in terms of higher order correlation functions. Let us take electron-electron collisions as an example. In this case the collision integral is expressed in terms of the two-point distribution function for the colliding electrons. The two point function satisfies its own kinetic equation, where the collision integral is expressed in terms of higher order correlation functions. This procedure leads to a chain of coupled equation for all higher order correlation functions, known as the BBGKY (for Bogoliubov, Born, Green, Kirkwood, and Yvon) hierarchy.

A crucial assumption underlying the Boltzmann description is that of relaxation of correlations. The basic idea is that many-body correlations, which are necessarily induced by collisions between particles, decay during the long stretches of subsequent free motion. Therefore the distribution functions of the colliding electrons may be assumed uncorrelated. This enables us to express the higher order distribution functions appearing in the collision integral in terms of the one-point function f itself. This procedure decouples the different levels in the hierarchy and makes Eq. (1) a closed equation for the one particle distribution function f. According to the principle of relaxation of correlations collisions of different types (electron-electron, electron-phonon, electron-impurity) are also treated as uncorrelated and are described by independent collision integrals. An important outcome of this treatment is that the resistivity turns out to be proportional to the sum of momentum relaxation rates — the result known as the Matthiessen's rule.

In semiconductors and many metals at low temperatures the umklapp processes are frozen out. Therefore electron-electron collisions conserve (quasi-)momentum, and the resistivity is controlled by electron-impurity and electron-phonon collisions. The latter are also suppressed at low temperatures, and the low temperature (so called residual) resistivity is determined by electron-impurity scattering. The corresponding collision integral has the form

$$\mathrm{St}[f] = \int (dp')W(\mathbf{p},\mathbf{p}')[f(\mathbf{r},\mathbf{p}',t) - f(\mathbf{r},\mathbf{p},t)]. \tag{2}$$

Here $W(\mathbf{p},\mathbf{p}')$ denotes the scattering probability per unit time form state $\mathbf{p}'$ to state $\mathbf{p}$, $(dp') = \frac{d^3p'}{(2\pi\hbar)^3}$, and we used the Stückelberg transformation of the collision integral.[3]

In linear response we typically consider monochromatic electric fields $\mathbf{E} = -\mathbf{F}/e = \mathbf{E}(\mathbf{r})e^{-i\omega t}$ (e is the absolute value of the electron charge) and write the distribution function as $f = f_0(\epsilon) + \delta f$, where $f_0(\epsilon)$ is the equilibrium distribution function which depends only on the energy ϵ. Recalling that the equilibrium distribution nullifies the collision integral

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[3] This transformation relies on the identity $\int(dp')W(\mathbf{p}',\mathbf{p}) = \int(dp')W(\mathbf{p},\mathbf{p}')$, which follows from unitarity of scattering, see §§ 2, 3 of Ref. [1]

and passing to the Fourier representation we obtain to linear order in $\mathbf{E}$,

$$(-i\omega + \mathbf{v}\cdot\nabla)\,\delta f - \mathrm{St}[\delta f] = e\mathbf{E}(\mathbf{r})\cdot\mathbf{v}\frac{df_0(\epsilon)}{d\epsilon}. \tag{3}$$

Let us assume the simplest spectrum $\epsilon = p^2/2m$ and write the nonequilibrium part of the electron distribution in the form $\delta f(\epsilon, \mathbf{r}, \mathbf{n})$ where $\mathbf{n}$ is the momentum direction. Keeping in mind that the electron energy is conserved in collisions with impurities we observe the distribution functions at each different energies obey independent equations,

$$\mathcal{B}_\epsilon \delta f(\epsilon, \mathbf{r}, \mathbf{n}) = ev\mathbf{E}(\mathbf{r})\cdot\mathbf{n}\,\frac{df_0(\epsilon)}{d\epsilon}, \tag{4}$$

where $\mathcal{B}_\epsilon$ denotes the Boltzmann operator defined by

$$\mathcal{B}_\epsilon \delta f(\epsilon, \mathbf{r}, \mathbf{n}) = \left(-i\omega + v\mathbf{n}\cdot\nabla + \frac{1}{\tau}\right)\delta f(\epsilon, \mathbf{r}, \mathbf{n}) - n_i v \int d\mathbf{n}' \sigma_\epsilon(\mathbf{n}, \mathbf{n}')\delta f(\epsilon, \mathbf{r}, \mathbf{n}'). \tag{5}$$

Here we expressed the scattering probability in terms of the electron velocity $v = p/m$, impurity concentration n_i and the (energy dependent) differential scattering cross-section $\sigma_\epsilon(\mathbf{n}, \mathbf{n}')$ at a single impurity. We also introduced the (energy dependent) "out" scattering rate

$$\frac{1}{\tau} = n_i v \int d\mathbf{n}' \sigma_\epsilon(\mathbf{n}, \mathbf{n}'). \tag{6}$$

It is useful to introduce the resolvent of the Boltzmann operator $\mathcal{K} \equiv \mathcal{B}_\epsilon^{-1}$. It satisfies the inhomogeneous equation

$$\left(-i\omega + v\mathbf{n}\cdot\nabla_{\mathbf{r}} + \frac{1}{\tau}\right)\mathcal{K}_\epsilon(\mathbf{r}, \mathbf{n}; \mathbf{r}', \mathbf{n}') - n_i v \int d\tilde{\mathbf{n}} \sigma_\epsilon(\mathbf{n}, \tilde{\mathbf{n}})\mathcal{K}_\epsilon(\mathbf{r}, \tilde{\mathbf{n}}; \mathbf{r}', \mathbf{n}') = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{n} - \mathbf{n}'). \tag{7}$$

The boundary conditions at insulating interfaces correspond to the condition of zero current across the interface. We will see that the resolvent $\mathcal{K}_\epsilon$ corresponds to the series of "ladder" diagrams in the quantum diagrammatic treatment of electron transport. With its aid the solution of Eq. (4) may be formally expressed as

$$\delta f(\epsilon, \mathbf{r}, \mathbf{n}) = ev\frac{df_0(\epsilon)}{d\epsilon}\int d\mathbf{n}' d\mathbf{r}' \mathcal{K}_\epsilon(\mathbf{r}, \mathbf{n}; \mathbf{r}', \mathbf{n}')\,\mathbf{E}(\mathbf{r}')\cdot\mathbf{n}'. \tag{8}$$

Resolvent for s-wave scatterers. Let us consider an instructive example of purely s-wave scattering. This approximation of point-like impurities captures much of the essential physics and is frequently used in the theory of disordered metals to simplify the treatment. In this case the scattering differential cross-section $\sigma_\epsilon(\mathbf{n}, \tilde{\mathbf{n}})$ is independent of the initial and final momenta. As a result the second term in the left hand side of (7) becomes independent of $\mathbf{n}$. Let us specialize to a bulk conductor, for which the resolvent $\mathcal{K}_\epsilon$ depends only on the coordinate difference, $\mathbf{r} - \mathbf{r}'$. Upon a Fourier transformation with respect to this variable Eq. (7) reduces to

$$\left(-i\omega + iv\mathbf{q}\cdot\mathbf{n} + \frac{1}{\tau}\right)\mathcal{K}_\epsilon(\mathbf{n}, \mathbf{n}') - \frac{1}{\tau}\langle\mathcal{K}_\epsilon(\mathbf{n}, \mathbf{n}')\rangle_{\mathbf{n}} = \delta(\mathbf{n} - \mathbf{n}'), \tag{9}$$

where $\langle \dots \rangle_{\mathbf{n}}$ denotes the average over the directions, $\mathbf{n}$. The solution of this equation is

$$\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}') = \frac{\tau\delta(\mathbf{n}-\mathbf{n}')}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} + \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} \frac{\tau}{\left(1-\left\langle \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}}\right\rangle_{\mathbf{n}}\right)} \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}'}, \tag{10}$$

where $l = v\tau$ is the mean free path.

The average over the directions is readily evaluated in two ($d = 2$) and three ($d = 3$) dimensions,

$$\left\langle \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}}\right\rangle_{\mathbf{n}} = \begin{cases} \frac{1}{\sqrt{(1-i\omega\tau)^2+q^2l^2}}, & d=2, \\ 1-\frac{1}{2iql}\ln\frac{1-i\omega\tau+iql}{1-i\omega\tau-iql}, & d=3. \end{cases} \tag{11}$$

The expression (9) for the resolvent apply in both the ballistic ($\omega\tau, ql \gg 1$) and diffusive ($\omega\tau, ql \ll 1$) regimes. In the ballistic case the resolvent $\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}')$ has a sensitive dependence on the directions $\mathbf{n},\mathbf{n}'$.

In some cases, for example in the study of propagation of waves through random media both the source (injection) and the observable (detector) may have a high selectivity with respect to direction, and the detailed dependence of $\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}')$ on $\mathbf{n},\mathbf{n}'$ is essential. In contrast, in disordered conductors we are often interested in the diffusive regime, where both the observables and the external perturbations correspond only to the zeroth (density) or first (current) spherical harmonics in $\mathbf{n},\mathbf{n}'$.

Diffusive regime. For $\omega\tau, ql \ll 1$ the expressions in Eq. (11) simplify to

$$\left\langle \frac{1}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}}\right\rangle_{\mathbf{n}} = 1-\tau(-i\omega+Dq^2), \tag{12}$$

where $D = vl/d$ is the diffusion coefficient. Therefore in the diffusive regime the resolvent $\mathcal{K}_\epsilon$ is given by

$$\mathcal{K}_\epsilon^{diff}(\mathbf{n},\mathbf{n}') = \frac{\tau\delta(\mathbf{n}-\mathbf{n}')}{1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}} + \frac{1}{(1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n})(-i\omega+Dq^2)(1-i\omega\tau+il\mathbf{q}\cdot\mathbf{n}')}. \tag{13}$$

When transformed to position space the first term above decays exponentially on the scale of the mean free path and represents the probability for the electron to remain unscattered from the original momentum state. The second term corresponds to a convolution of three kernels, two of which are short range, while the other, $(-i\omega+Dq^2)^{-1}$ represents the diffusion propagator and has a long range character. Keeping only the leading terms in the zeroth and first spherical harmonics in $\mathbf{n},\mathbf{n}'$ we may further approximate the resolvent as

$$\mathcal{K}_\epsilon^{diff}(\mathbf{n},\mathbf{n}') = \tau d\,\mathbf{n}\cdot\mathbf{n}' + \frac{(1-il\mathbf{q}\cdot\mathbf{n})(1-il\mathbf{q}\cdot\mathbf{n}')}{(-i\omega+Dq^2)}. \tag{14}$$

Diffusive regime for general scatterers: transport relaxation time. Let us now turn to the case of non-s-wave scatterers. In the Fourier representation Eq. (7) becomes

$$\left(-i\omega+iv\mathbf{q}\cdot\mathbf{n}+\frac{1}{\tau}\right)\mathcal{K}_\epsilon(\mathbf{n},\mathbf{n}') - n_i v\int d\tilde{\mathbf{n}}\sigma_\epsilon(\mathbf{n},\tilde{\mathbf{n}})\mathcal{K}_\epsilon(\tilde{\mathbf{n}},\mathbf{n}') = \delta(\mathbf{n}-\mathbf{n}'). \tag{15}$$

In the diffusive regime it is convenient to use the multipole expansion in $\mathbf{n}$,

$$\mathcal{K}^{diff}_{\epsilon}(\mathbf{n},\mathbf{n}') = \mathcal{N}_{\epsilon}(\mathbf{n}') + \mathbf{n}\cdot\boldsymbol{\mathcal{J}}_{\epsilon}(\mathbf{n}') + \ldots . \tag{16}$$

Projecting (15) onto the zeroth spherical harmonic in $\mathbf{n}$ and we get

$$-i\omega\mathcal{N}_{\epsilon}(\mathbf{n}') + i\frac{v}{d}\mathbf{q}\cdot\boldsymbol{\mathcal{J}}_{\epsilon}(\mathbf{n}') = 1. \tag{17}$$

Note that the scattering rate disappeared from this relation. This is a consequence of conservation of the number of particles at each energy by collisions with impurities (the left hand side represents the "continuity equation").

Projecting Eq. (15) onto the first spherical harmonic we get to leading order in $\omega\tau$

$$\boldsymbol{\mathcal{J}}_{\epsilon}(\mathbf{n}') = \tau_{tr} d\,\mathbf{n}' - i l_{tr}\mathbf{q}\mathcal{N}_{\epsilon}(\mathbf{n}'), \tag{18}$$

where $l_{tr} = v\tau_{tr}$ is the transport mean free path. The transport mean free time, τ_{tr}, characterizes relaxation of momentum (see e.g. § 11 of Ref. [1]) and is defined by the relation

$$\frac{1}{\tau_{tr}} = n_i v \int d\mathbf{n}'\sigma_{\epsilon}(\mathbf{n},\mathbf{n}')(1-\mathbf{n}\cdot\mathbf{n}'). \tag{19}$$

The difference between the transport and "out" relaxation rates is described by the second term in the bracket in the right hand side. For purely s-wave scattering this term vanishes, and the transport mean free time coincides with the "out" mean free time defined in Eq. (6). In this regime the electron momenta before and after the collision are uncorrelated, electrons "lose" memory of their momentum upon a single collision. In the opposite case of small angle scattering the electron momentum electrons is randomized only after a large number of collisions, $\tau_{tr} \gg \tau$.

Combining Eqs. (16), (17), and (18) we get

$$\mathcal{K}^{diff}_{\epsilon}(\mathbf{n},\mathbf{n}') = \tau_{tr} d\,\mathbf{n}\cdot\mathbf{n}' + \frac{(1 - i l_{tr}\mathbf{q}\cdot\mathbf{n})(1 - i l_{tr}\mathbf{q}\cdot\mathbf{n}')}{(-i\omega + Dq^2)}, \tag{20}$$

where the diffusion constant is expressed in terms of the *transport* mean free path in the usual way, $D = v l_{tr}/d$. The only difference with Eq. (13) is the replacement $\tau \to \tau_{tr}$.

Diffusive response to a longitudinal electric field. Let us consider the electric field of the form $\mathbf{E}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t)$, where the electric potential is of the form $\phi(\mathbf{r},t) = \phi_0\, e^{i\mathbf{q}\cdot\mathbf{E}(\mathbf{r}) - i\omega t}$. In this case Eq. (8) gives in the Fourier representation

$$\delta f(\epsilon,\mathbf{n}) = -ie\phi_0 v\frac{df_0(\epsilon)}{d\epsilon}\int d\mathbf{n}'\mathcal{K}^{diff}_{\epsilon}(\mathbf{n},\mathbf{n}')\,\mathbf{q}\cdot\mathbf{n}'. \tag{21}$$

We expand the nonequilibrium part of the distribution function δf to first order in the spherical harmonics,

$$\delta f(\epsilon,\mathbf{n}) = N_{\epsilon} + \mathbf{n}\cdot\mathbf{J}_{\epsilon}. \tag{22}$$

The nonequilibrium part of density and the current density (at a given energy) are described by the zeroth (N_{ϵ}) and the first ($\mathbf{J}_{\epsilon}$) spherical harmonics respectively.

$$\begin{aligned} n &= 2\int(dp)N_{\epsilon} = 2\int\nu(\epsilon)N_{\epsilon}d\epsilon, && (23)\\ \mathbf{j} &= -2e\int(dp)\mathbf{v}\,(\mathbf{J}_{\epsilon}\cdot\mathbf{n}) = -2\frac{e}{d}\int\nu(\epsilon)v\mathbf{J}_{\epsilon}d\epsilon, && (24)\end{aligned}$$

where $\nu(\epsilon)$ is the density of states per unit energy for a single spin component.

Substituting Eqs. (22) and (20) into (21) we obtain

$$\mathbf{J}_\epsilon = -iel_{tr}\mathbf{q}\phi_0 \frac{df_0(\epsilon)}{d\epsilon} - il_{tr}\mathbf{q}N_\epsilon, \tag{25}$$

where the density response is given by

$$N_\epsilon = -\frac{eDq^2\phi_0}{-i\omega + Dq^2} \frac{df_0(\epsilon)}{d\epsilon} = -\left(1 + \frac{i\omega}{-i\omega + Dq^2}\right) e\phi_0 \frac{df_0(\epsilon)}{d\epsilon}. \tag{26}$$

In the last expression we wrote the answer in a suggestive form that will be useful for comparison with the quantum diagrammatic treatment. The first term in the bracket describes the static response. The second term corresponds to the density induced by the longitudinal *ac* currents. Since it contains the diffusion constant, with the aid of the Einstein relation, we can extract the conductivity from the *ac* density response to an external scalar potential.

The first term in Eq. (25) represents the current induced by the electric field. The second term represents the diffusion current created by the density gradient. For a uniform electric field it vanishes. In this case, combining Eqs. (24) and (25) and substituting the Fermi function for $f_0(\epsilon)$ we obtain the Drude result,

$$\mathbf{j} = \sigma\mathbf{E_0}, \quad \sigma = 2e^2 D\nu_0, \tag{27}$$

where ν_0 is the density of states per spin at the Fermi energy.

The second expression above represents the Einstein relation between the conductivity and the diffusion coefficient. It follows form the fact than in thermodynamic equilibrium in the presence of an external electrostatic potential ϕ the gradiant of the electron density is given by $\nabla n = 2e\nu_0\nabla\phi$ and the current, $\mathbf{j} = -\sigma\nabla\phi + eD\nabla n$, must vanish. The Einstein relation enables us to extract the conductivity from the density response (26). This method is often used in diagrammatic treatment of electron transport.

2.2 Classical transport effects beyond the Boltzmann equation

Although the Boltzmann equation successfully describes the vast majority of transport phenomena in disordered metals in some physical regimes the basic underlying assumptions of the Boltzmann approach are violated. Before considering quantum effects in electron transport I would like to mention some classical transport effects that are not described by the standard approach based on the Boltzmann equation.

Transport in high mobility nanostructures with strongly correlated carriers. Recently it became possible, see Ref. [3], to produce high mobility two-dimensional semiconductor systems in which the interaction parameter $r_s \sim U/\epsilon_F$ (with ϵ_F and U being respectively the Fermi energy and the Coulomb energy at average inter-particle spacing) is large. Charge carriers in such systems form strongly correlated liquids. Their theoretical treatment greatly simplifies in the low temperature limit, where one can use the Fermi-liquid (or Luttinger-liquid in 1D) theory. The latter can be viewed as low temperature fixed points of the system in the renormalization group sense and apply at sufficiently low temperatures. As a result, most theoretical treatments of electron transport focused on this low temperature regime.

At $r_s \gg 1$ the Fermi liquid description applies below the renormalized Fermi energy $\varepsilon_F^* \ll \varepsilon_F$ defined by the renormalized mass. On the other hand, there is a wide and practically unexplored interval of temperatures $\varepsilon_F^* < T < U$ in which electron system remains strongly correlated, but may not be described by the Fermi-liquid theory. Note that the strongly correlated electron liquid may be treated classically only at $\omega_D < T < U$, where $\omega_D \sim \sqrt{\epsilon_F U}$ is the characteristic Debye frequency of the liquid, whereas at $\varepsilon_F^* < T < \omega_D$ the electrons form a semi-quantum fluid. The semi-quantum regime for strongly interacting fluids was first identified and studied by A. F. Andreev [5] in the context of liquid ^{4}He and ^{3}He. It is characterized by power-law temperature dependence of specific heat [5], viscosity and thermal conductivity [6].

Strongly correlated liquids are realized in semiconductor quantum wells with low carrier densities ($n \sim 10^{10}$ cm^{-2}, $r_s \sim 10 - 36$. The device schematics (hole-doped GaAs quantum well) are shown in Fig. 1 a). Transport measurements on these systems show strong dependence of the resistivity on the temperature and magnetic field, see Ref. [3] for a review. A characteristic temperature dependence of the resistivity of a high mobility GaAs quantum well with strongly correlated carriers ($\varepsilon_F \approx 0.2K$, $r_s \approx 28$) is shown in Fig. 1 b) (adapted from Ref. [4]). The resistivity shows strong nonmonotonic temperature dependence at $T < 2$ K, where the calculated phonon contribution to the resistivity (green dashed line) is already negligible. This unusual decrease in the resistivity with increasing temperature cannot be explained by weak localization, as it occurs at relatively high temperatures, $\varepsilon_F < T < U$. In this wide temperature interval the conventional approaches to electron transport based on either the Fermi liquid picture or the Boltzmann equation are inapplicable.

On the other hand, because of the strong interactions, the electron mean free path due to electron-electron scattering becomes very short in this high temperature regime. As a result, starting from very short distances (on the order of inter-electron spacing) the dynamics of such strongly correlated liquids may be described by classical hydrodynamics.

Hydrodynamic description of resistivity. Because of the large separation between the doping layer and the quantum well there is a temperature interval in which on the one hand the hydrodynamic description is valid at length scales of the order of the correlation radius of the external potential, and on the other hand the phonon contribution to resistivity (dashed green line in Fig. 1 b)) is small. In this regime, denoted by a green frame in Fig. 1 b), the residual (disorder-induced) resistivity may be analyzed using the hydrodynamic approach.

Hydrodynamic theory of resistivity in high mobility semiconductors was developed in Ref. [7]. It applies to both two- and one-dimensional systems and any liquid type (Fermi, Luttinger, spin-incoherent etc). The resistivity in the hydrodynamic regime is governed by the dissipative processes arising in the electron fluid and can be expressed in terms of the intrinsic kinetic coefficients (thermal conductivity and viscosity) of the electron liquid, which are determined by electron-electron collisions. As was mentioned earlier, within the Boltzmann approach electron-electron collisions do not affect the electrical resistivity. This conclusions rests on the assumption of absence of correlations between electron-electron collisions and the disorder potential. Inclusion of such correlations gives rise to dependence of resistivity on the rate of electron electron collisions. In this respect the hydrodynamic results correspond to accounting for these correlation in the limiting case of long range disorder.

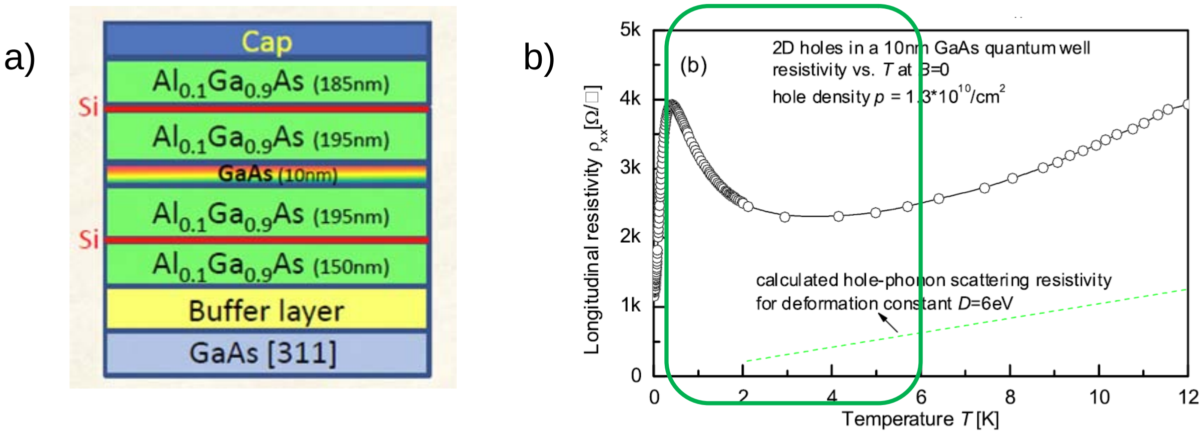

Figure 1: a) Schematic representation of a symmetric hole-doped GaAs/AlGaAs quantum well (curtesy of Xuan Gao). A large separation between the GaAs quantum well and the Si dopants produces a smooth disorder potential at the quantum well and results in high carrier mobility. b) Resistivity $\rho_x x$ over an extended temperature range for the 2D holes at zero magnetic field (adapted from Ref. [4]). The dotted green line depicts the phonon contribution to resistivity based on the deformation potential constant $D = 6eV$.

In the hydrodynamic regime electric current should be regarded as a flow of electron liquid in the presence of a (long range) disorder potential. Electrical resistivity may be obtained by solving equations of hydrodynamics in the presence of the external electric field. On the other hand, if the flow pattern is known the resistivity ρ may be obtained by considering entropy production in the liquid. The latter arises from gradients of temperature , ∇T, and hydrodynamic velocity, $\nabla_j v_k$, and is given by, see Ref. [8],

$$\dot{S} = \int d\mathbf{r} \left(\frac{\kappa (\nabla T)^2}{T^2} + \frac{\sigma'_{ik} \nabla_k v_i}{T} \right). \tag{28}$$

Here κ is the thermal conductivity of the fluid, and σ'_{ik} is the viscous stress tensor, which may be expressed in terms of the shear, η, and bulk, ζ, viscosities as

$$\sigma'_{ik} = \eta \left(\partial_k v_i + \partial_i v_k - \frac{2}{d} \delta_{ik}\, \partial_l v_l \right) + \delta_{ik} \zeta \partial_l v_l. \tag{29}$$

Relating the entropy production in the fluid to Joule heat we obtain

$$\rho \left\langle \mathbf{j} \right\rangle^2 = \left\langle - \frac{\kappa}{T} \left(\boldsymbol{\nabla} T \right)^2 + \sigma'_{ik} \partial_k v_i \right\rangle, \tag{30}$$

where $\langle \ldots \rangle$ denotes spatial average and $\mathbf{j}$ is the electrical current density.

The results are most conveniently expressed in terms of the equilibrium density of electrons, $n^{(0)}(\mathbf{r})$, and entropy per electron in equilibrium, $s^{(0)}(\mathbf{r})$. Both acquire position dependence in the presence of external potential $U(\mathbf{r})$.

Let us start with a one-dimensional case, where the results can be readily obtained based on conservation laws. In this case the current j is uniform along the wire, and to linear order in the current the hydrodynamic velocity is

$$v(x) = -\frac{j}{e} \frac{1}{n^{(0)}(x)}. \tag{31}$$

Since the dissipation of mechanical energy arises only in second order in the current, in linear response the thermal energy flux is also uniform. It may be expressed as

$$q = -\frac{j}{e} T s^{(0)}(x) - \kappa \nabla T. \tag{32}$$

The first term represents the convective heat flux, and the second – the thermal conductivity heat flux arising in the fluid in the presence of temperature gradients. The inhomogeneous part of the convective term (arising from inhomogeneity of equilibrium entropy per electron, $s^{(0)}(x)$) must be balanced by thermal conductivity heat flux. We thus see that passage of the electric current produces temperature gradients that are *linear* in the current and thus not related to heating, $\nabla T = -jT\delta s^{(0)}(x)/e\kappa$, where $\delta s^{(0)}(x) = s^{(0)}(x) - \langle s^{(0)}(x)\rangle$ (we assumed weak disorder). Substituting this expression for the temperature gradient and Eqs. (31) and (29) (without the shear viscosity term) into Eq. (30) we obtain the resistivity in the form,

$$\rho_{1D} = \frac{1}{e^2}\left\langle \frac{T}{\kappa}\left(\delta s^{(0)}\right)^2 + \zeta\left(\partial_x \frac{1}{n^{(0)}}\right)^2 \right\rangle. \tag{33}$$

Note that the first term turns out to be proportional to the variance of the external potential rather than the variance of its gradient. Therefore in the limiting case of long range disorder potential it provides the leading contribution to resistivity. The force impeding the motion of the liquid that corresponds to this term arises from thermal expansion of the liquid. One can show that (regardless of the sign of the thermal expansion coefficient) the current-induced temperature variations cause the density of the liquid on the upstream slope of a potential bump to be greater than that on the downstream slope. This produces a net force by the bump onto the liquid, which is directed against the flow.

The physical origin of the resistivity in the two-dimensional case is the same and the answer is

$$\rho_{2D} = \frac{1}{2e^2}\left\langle \frac{T}{\kappa}\left(\delta s^{(0)}\right)^2 + (\eta+\zeta)\left(\nabla \frac{1}{n^{(0)}}\right)^2 \right\rangle. \tag{34}$$

I would also like to mention that in the spin-rotation invariant liquid placed in a magnetic field the projection of the spin onto the external field is conserved and must be included in the hydrodynamic equations. Similarly to the heat flux the conserved spin current consists of convective part and spin-diffusion current relative to the fluid. This leads to hydrodynamic mechanism of magnetoresistance, which is described by a generalization of Eq. (34), see Ref. [7].

Beyond the hydrodynamic regime. In the case where the correlation radius of the disorder potential is shorter than the equilibration length of the electron liquid correlations between disorder and electron-electron collisions also affect the resistivity. Much progress has been made recently in in understanding the interplay between equilibration of one-dimensional electron liquids and conductance of quantum wires, see Refs. [9, 10, 11].

Kinetics of strongly correlated fluids. Comparison of the hydrodynamic predictions with experiment requires a microscopic theory of viscosity and thermal conductivity of strongly correlated liquids beyond the Fermi- and Luttinger- liquid regimes. Recently significant theoretical progress has been and is being made in theory of equilibration of one-dimensional liquids, see a recent review [12] and references therein. As far as two-dimensional strongly correlated liquids are concerned, no microscopic theory exists. I would like to note that in contrast to liquid ^{4}He and ^{3}He [5, 6] the microscopic Hamiltonian describing the strongly-correlated electron liquids is known, kinetic energy plus Coulomb interaction. Therefore the problem is very well posed and is awaiting solution. I hope that some of the participants of this school will be able to make progress in this area.

3 Diagrammatic approach

In this section I will present a brief summary of the diagram technique for quantum disordered systems, starting with applications to linear response theory.

3.1 Linear response.

The electric field is usually introduced via the (spatially uniform) vector potential

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i\omega}{c}\mathbf{A}_0 e^{-i\omega t}. \tag{35}$$

In linear response theory the current is described by

$$\mathbf{j} = -\frac{1}{c}\hat{Q}(\omega)\mathbf{A}_0 \tag{36}$$

where the kernel $\hat{Q}$ is given by the Kubo formula,

$$Q_{\alpha\beta}(\omega) = -\left[\frac{i}{\hbar}\int_0^\infty dt e^{i\omega t}\int d\mathbf{r}\langle[j_\alpha(\mathbf{r},t), j_\beta(\mathbf{0},0)]\rangle - \frac{ne^2}{m}\delta_{\alpha\beta}\right]. \tag{37}$$

Here n is the electron density and

$$j_\alpha(\mathbf{r},t) = -\frac{ie\hbar}{2m}\left\{[\nabla_\alpha\psi^\dagger(\mathbf{r},t)]\psi(\mathbf{r},t) - \psi^\dagger(\mathbf{r},t)\nabla_\alpha\psi(\mathbf{r},t)\right\} \tag{38}$$

is the second quantized current operator.

The conductivity is expressed in terms of $\hat{Q}(\omega)$ as

$$\sigma_{\alpha\beta}(\omega) = \frac{i}{\omega}Q_{\alpha\beta}(\omega). \tag{39}$$

The commutator of the currents in Eq. (37) may be evaluated with the aid of the diagram technique. The basic objects of the diagram technique are electron Green's functions. They carry information about both propagation of electrons and occupancy of the available states. It is instructive to separate these two aspects. This is especially easy to do for noninteracting electrons. In this case all interesting physics is associated with the retarded and advanced Green's functions that describe electron propagation in a disordered potential, whereas the occupancy of the electron states is given by the usual Fermi distribution. Although this is physically obvious it is instructive to see how this comes about in conventional diagram technique. I will follow Ref. [2].

At zero temperature the kernel $Q_{\alpha\beta}(\omega)$ may be expressed as (henceforth we set $\hbar$ to unity)

$$Q_{\alpha\beta}(\omega) = \frac{ne^2}{m}\delta_{\alpha\beta} - \frac{i2e^2}{m^2}\int\frac{d\varepsilon}{2\pi}e^{i\epsilon\eta}\int(d\mathbf{p})p_\alpha\Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-), \tag{40}$$

where $(d\mathbf{p}) = \frac{d^dp}{(2\pi)^d}$, $\epsilon_\pm = \epsilon \pm \omega/2$, and the vector quantity $\mathbf{\Pi}(\mathbf{p};\varepsilon_+,\varepsilon_-)$ is defined in terms of Green's functions as

$$\mathbf{\Pi}(\mathbf{p};\varepsilon_+,\varepsilon_-) = \int(d\mathbf{p}')G(\mathbf{p},\mathbf{p}';\varepsilon_+)\mathbf{p}'G(\mathbf{p}',\mathbf{p};\varepsilon_-). \tag{41}$$

The exponential factor in (40) ensures the correct ordering of the fermion operators in the observable (current) and makes the integral convergent.

The time-ordered Green's functions that enter Eq. (41) may be represented in terms of the retarded, G^R, and advanced, G^A, Green's functions

$$G^{R/A}_\varepsilon = (\varepsilon \pm i\eta - \hat{H})^{-1}, \tag{42}$$

as

$$G(\varepsilon) = \begin{cases} G^R_\varepsilon, & \varepsilon > 0, \\ G^A_\varepsilon, & \varepsilon < 0. \end{cases} \tag{43}$$

For convenience we include the chemical potential μ into the single particle Hamiltonian,

$$H = H_0 + V(\mathbf{r}), \quad H_0 = \frac{p^2}{2m} - \mu. \tag{44}$$

The disorder potential $V(\mathbf{r})$ may be written in terms of the potential of a single impurity, $u(\mathbf{r})$, as

$$V(\mathbf{r}) = \sum_a u(\mathbf{r}-\mathbf{r}_a), \quad V(\mathbf{q}) = \sum_a u(q)e^{i\mathbf{q}\cdot\mathbf{r}_a}, \tag{45}$$

where $\mathbf{r}_a$ is the location of the a-th impurity (the second expression above corresponds to the Fourier representation).

In disordered systems we need to average the response kernel over the realizations of disorder potential. Before doing so let us rewrite the linear response expression in a form more convenient for disorder averaging.

Cancelation of the diamagnetic term. Note that at $\omega = 0$ the perturbation of the Hamiltonian by the external vector potential amounts to a gauge transformation and should not produce any current. This means that the gradient contribution to the kernel Q (second term in Eq. (40)) must cancel the diamagnetic term (first term in (40)). Using this cancelation we may rewrite the disorder-averaged response kernel Q as

$$Q_{\alpha\beta}(\omega) = -\frac{i2e^2}{m^2}\int\frac{d\varepsilon}{2\pi}\int(d\mathbf{p})p_\alpha\left\langle\Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-) - \Pi_\beta(\mathbf{p};\varepsilon,\varepsilon)\right\rangle, \tag{46}$$

Where $\langle\ldots\rangle$ denotes average over disorder. The advantage of this form over (40), (41) lies in the improved convergence of integrals. In Eq. (40) the integration converges only because of the presence of the exponential factor $e^{i\varepsilon\eta}$. Therefore the integration over energy in it must be done first. In contrast, in Eq. (46) the exponential factor is not needed for convergence and has been dropped. More importantly, we may interchange the order of integrations, and integrate over momenta first. The resulting integrals are dominated by momenta near the Fermi surface and may be evaluated by residues, which is very convenient in applications.

When this procedure is followed, the second term in (46) formally vanishes, and the results look as if we ignore the diamagnetic term and evaluate the gradient contribution by integrating over momenta by residues. In applying this mnemonic rule one should bare in mind that in reality the diamagnetic term is not ignored, and its subtraction is necessary to justify residue evaluation of momentum integrals, which applies only to the difference, $\langle \Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-) - \Pi_\beta(\mathbf{p};\varepsilon,\varepsilon)\rangle$.

We will see that $\langle \Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-) - \Pi_\beta(\mathbf{p};\varepsilon,\varepsilon)\rangle$ evaluated in this way does not vanish only for $|\varepsilon| < \omega/2$. In this narrow energy interval the result is practically energy-independent, and the integral over ε in (46) simply gives a factor $\omega/2\pi$ (we assume $\omega > 0$). Therefore using Eq. (39) we may express the conductivity in the form,

$$\sigma_{\alpha\beta}(\omega) = \frac{e^2}{\pi m^2}\int (d\mathbf{p}) p_\alpha \left.\langle \Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-) - \Pi_\beta(\mathbf{p};\varepsilon,\varepsilon)\rangle\right|_{\varepsilon=0}. \tag{47}$$

Finally, using the connection between the time-ordered and retarded/advanced Green's functions we can express the conductivity as

$$\sigma_{\alpha\beta}(\omega) = \frac{e^2}{\pi m^2}\int (d\mathbf{p})\int (d\mathbf{p}') \left.\langle p_\alpha [G^R(\mathbf{p},\mathbf{p}';\varepsilon+\omega) - G^A(\mathbf{p},\mathbf{p}';\varepsilon)] p'_\beta G^A(\mathbf{p}',\mathbf{p};\varepsilon)\rangle\right|_{\varepsilon=0}. \tag{48}$$

This formula reduces evaluation of the linear response conductivity to averaging products of retarded and advanced Green's functions over disorder and is very convenient for applications.

3.2 Impurity diagram technique.

Disorder averaging is accomplished by expanding the Green's function in perturbation series in the disorder potential (45) and averaging the result over the impurity positions. This procedure corresponds to the impurity (or "cross") diagram technique. Let us briefly summarize its basic points. It is easier to think about retarded and advanced Green's. The perturbation expansion of $G^{R/A}_\varepsilon$ in Eq. (42) corresponds to a simple geometric series

$$G^{R/A}_\varepsilon = \sum_{n=0}^{\infty} G^{R/A}_0(\varepsilon)\left(\hat{V} G^{R/A}_0(\varepsilon)\right)^n. \tag{49}$$

where the unperturbed Green's function is most conveniently expressed in the momentum representation,

$$G^{R/A}_0(\varepsilon,\mathbf{p}) = \frac{1}{\varepsilon - \xi_p \pm i\eta}, \tag{50}$$

where $\xi_p = \frac{p^2}{2m} - \mu$ is the particle energy measured with respect to the chemical potential. In the position representation it oscillates and falls off as a power of the distance. For example in three dimensions it is given by,

$$G^{R/A}_0(\varepsilon,\mathbf{r}) = \frac{m}{2\pi}\frac{e^{\pm i k_\varepsilon r}}{r}, \tag{51}$$

where $k_\varepsilon = \sqrt{2m(\mu + \varepsilon)}$.

In order to average the resulting series we note that the spatial average of the potential may be removed by re-defining the chemical potential. We will therefore set $\langle V \rangle = 0$. For uncorrelated impurities we get for second moment of the random potential,

$$\langle V(\mathbf{q})V(\mathbf{q}')\rangle = (2\pi)^3\delta(\mathbf{q} + \mathbf{q}')w(q), \tag{52}$$

where the $w(q)$ is expressed in terms of the impurity concentration n_i as

$$w(q) = n_i|u(q)|^2. \tag{53}$$

To simplify the treatment we will assume that the impurity potential is weak. We therefore neglect its higher order cumulants and treat the impurity scattering within the Born approximation. Denoting the scattering amplitude with momentum transfer q by $f(q)$ and using the Born formula $f(q) = -\frac{m}{2\pi}u(q)$ we express the variance of the potential in terms of the scattering cross-section as

$$w(q) = \left(\frac{2\pi}{m}\right)^2 n_i|f(q)|^2. \tag{54}$$

Neglecting the higher order cumulants corresponds to a Gaussian random potential. Diagrammatically, each occurrence of the impurity potential is represented by a cross, and each unperturbed Green's function $G_0^{R/A}(\varepsilon, \mathbf{r} - \mathbf{r}')$ by a thin line with an arrow pointing from $\mathbf{r}'$ to $\mathbf{r}$. In a Gaussian disorder potential the average product of all impurity crosses in a given diagram is given by a sum of all pairwise contractions between different impurity crosses. Such pairwise contractions are represented by dashed lines and in the Fourier representation are given by $w(q)$. For example, the lowest order diagram for the self energy shown in Fig. 2 corresponds to the expression,

$$\Sigma_\varepsilon^{R/A}(p) = \left(\frac{2\pi}{m}\right)^2 n_i \int (d\mathbf{p}')|f(\mathbf{p} - \mathbf{p}')|^2 \frac{1}{\varepsilon - \xi_{p'} \pm i\delta}, \tag{55}$$

where

$$\xi_p = \frac{p^2}{2m} - \mu \tag{56}$$

is the electron energy measured relative to the chemical potential. The qualitatively new effect of impurities is the appearance of imaginary part of self energy. Note that the imaginary part arises from on-shell ($\xi_p = \varepsilon$) part of the green's function in the integrand. This corresponds to real scattering processes which conserve electron energy (as opposed to virtual processes in which the energy ξ_p in the virtual state differs from the electron energy ε). Being small the imaginary part of self energy is important only near the pole $\xi_p \approx \varepsilon$. We therefore may set $\mathbf{p} = k_\varepsilon \mathbf{n}$, $\mathbf{p}' = k_\varepsilon \mathbf{n}'$, and replace $|f(\mathbf{p} - \mathbf{p}')|^2 \to \sigma_\varepsilon(\mathbf{n}, \mathbf{n}')$. In the essential integration region we may replace $\int (d\mathbf{p}') \to \frac{mk_\varepsilon}{(2\pi)^3} \int d\mathbf{n}' \int_{-\infty}^{\infty} d\xi_{p'}$, where $\int d\mathbf{n}'$ denotes the integral over the solid angle. Evaluating the integral over $\xi_{p'}$ by residues we obtain

$$\Sigma_\varepsilon^{R/A} = \mp i n_i \frac{k_\varepsilon}{2m} \int d\mathbf{n}' \sigma_\varepsilon(\mathbf{n} - \mathbf{n}') = \mp \frac{i}{2\tau}. \tag{57}$$

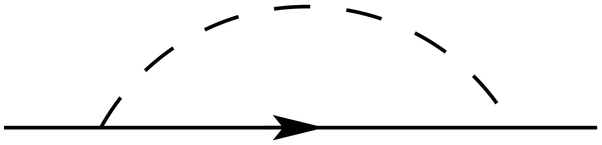

Figure 2: Lowest order impurity correction to self-energy.

To express the self energy in terms of the "out" scattering time we used Eq. (6).

To make further progress we note that we work in the semiclassical regime. The semiclassical parameter $k_\varepsilon l \gg 1$ will play a key role in all our approximations. It is not difficult to convince oneself that diagrams with crossing impurity lines are small in $1/k_F l$. Therefore to leading order in $1/k_F l$ the self energy is given by a series of "rainbow" diagrams. In this approximation (frequently called the self-consistent Born approximation, or SCBA) the self-energy satisfies the equation,

$$\Sigma_\varepsilon = n_i \frac{k_\varepsilon}{2\pi m}\int d\mathbf{n}' \sigma_\varepsilon(\mathbf{n}-\mathbf{n}')\int_{-\infty}^{\infty} d\xi \frac{1}{\varepsilon-\xi-\Sigma_\varepsilon} = \frac{1}{2\pi\tau}\int_{-\infty}^{\infty} d\xi \frac{1}{\varepsilon-\xi-\Sigma_\varepsilon}. \tag{58}$$

It has two solutions $\Sigma_\varepsilon = \pm\frac{i}{2\tau}$. The sign should be chosen consistently with the lowest order approximation (57) therefore we see that as a result of disorder averaging the infinitesimal imaginary part in the unperturbed Green's function determines the sign of the finite disorder-induced self-energy, $\pm i\eta \to \pm\frac{i}{2\tau}$ (this is reminiscent of spontaneous symmetry breaking, which indeed takes place in a field-theoretic formulation of the problem). Thus, the disorder-averaged Green's function (denoted in diagrams by a thick solid line with an arrow) corresponds to the expression,

$$\bar{G}(\varepsilon, p) = \frac{1}{\varepsilon+\mu-\frac{p^2}{2m}+\frac{i}{2\tau}\mathrm{sgn}(\varepsilon)}. \tag{59}$$

In coordinate space the disorder averaged Green's function has the form

$$\bar{G}(\varepsilon, \mathbf{r}) = \frac{m}{2\pi}\frac{e^{\left[-\frac{1}{2l}+ik_\varepsilon \mathrm{sgn}(\varepsilon)\right]r}}{r}. \tag{60}$$

It is useful to note that the exponential decay arises from randomization of the phase of the oscillations. In a given sample the Green's function does not decay exponentially. It may be loosely thought of as $\sim \frac{e^{\pm i[k_\varepsilon r+\phi(r)]}}{r}$, where $\phi(r)$ is a random phase. Averaging of the phase results in an exponentially decaying average, whereas the typical Green's function decays as a power of the distance. In this sense the average Green's function is a poor representation for the typical one. This leads to many important consequences. For example, the average product of retarded and advanced Green's functions decays as a power of the distance because the disorder induced-phases nearly cancel each other. In order to see this in diagram technique we need to identify the relevant class of diagrams describing Green's function products.

In evaluating the self-energy we have neglected all diagrams with crossing impurity lines. To preserve gauge invariance within our approximation we must make the same approximation for the vertices. Therefore in the average product of Green's functions in Eq. (41) only "ladder" diagrams should be retained. In this approximation $\langle \mathbf{\Pi} \rangle$ in Eq. (41) satisfies the equation

$$\langle \mathbf{\Pi}(\mathbf{p};\varepsilon_+,\varepsilon_-)\rangle = \bar{G}(\varepsilon_+,\mathbf{p}_+)\bar{G}(\varepsilon_-,\mathbf{p}_-)\left[\mathbf{p} + n_i \int (d\mathbf{p}')|u(\mathbf{p}-\mathbf{p}')|^2 \langle \mathbf{\Pi}(\mathbf{p}';\varepsilon_+,\varepsilon_-)\rangle\right]. \tag{61}$$

Gauge invariance check. Let us now check by an explicit calculation in the cross diagram technique that, as required by gauge invariance, within the conserving approximation the diamagnetic and the gradient terms in Eq. (40) cancel each other at zero frequency. This exercise will help us build intuition with diagrammatic calculations. For simplicity we consider point-like impurities. In this case the second term in Eq. (61) vanishes, and we obtain for the kernel Q,

$$Q_{\alpha\beta}(0) \;=\; \frac{ne^2}{m}\delta_{\alpha\beta} - \frac{i2e^2}{m^2}\int (d\mathbf{p}) p_\alpha p_\beta \int \frac{d\varepsilon}{2\pi} e^{i\epsilon\eta} \frac{1}{\left(\varepsilon + \mu - \xi_p + \frac{i}{2\tau}\mathrm{sgn}(\varepsilon)\right)^2}. \tag{62}$$

It is tempting to perform the integral over momenta in the second term first (and evaluate it by residues, as is common in impurity diagram technique). Such a procedure nullifies the second term and thus violates gauge invariance. The mathematical explanation of the mistake made in this way is the following. Without the exponential factor the integral over both energy and momenta diverges, as can be seen from simple power counting. Therefore the energy integration must be done first. To get a convergent result let us transform the second term as follows,

$$-\frac{i2e^2}{m^2}\int (d\mathbf{p}) p_\alpha p_\beta \frac{d}{d\xi_p} \int \frac{d\varepsilon}{2\pi} e^{i\epsilon\eta} \frac{1}{\varepsilon + \mu - \xi_p + \frac{i}{2\tau}\mathrm{sgn}(\varepsilon)}.$$

Since the integral over energy gives $in_\mathbf{p}$ ($n_\mathbf{p}$ being the average number of particles with momentum $\mathbf{p}$ per spin), the integral over momentum converges, and we may evaluate it integrating by parts. This gives

$$Q_{\alpha\beta}(0) \;=\; \frac{ne^2}{m}\delta_{\alpha\beta} + \frac{i2e^2}{m}\delta_{\alpha\beta}\int (d\mathbf{p}) \int \frac{d\varepsilon}{2\pi} e^{i\epsilon\eta} \frac{1}{\varepsilon + \mu - \xi_p + \frac{i}{2\tau}\mathrm{sgn}(\varepsilon)} = 0. \tag{63}$$

The lesson of this exercise is the we can rewrite the diamagnetic term in the form

$$\frac{ne^2}{m}\delta_{\alpha\beta} = \frac{i2e^2}{m^2}\int \frac{d\varepsilon}{2\pi} e^{i\epsilon\eta} (d\mathbf{p}) p_\alpha \Pi_\beta(\mathbf{p};\varepsilon,\varepsilon). \tag{64}$$

This relation was used to arrive at Eqs. (46) and (48).

Conductivity for point-like impurities. Let us now evaluate the conductivity. For point-like impurities Eq. (48) reduces to,

$$\sigma_{\alpha\beta}(\omega) = \frac{e^2}{\pi m^2}\int (d\mathbf{p}) p_\alpha p_\beta \left[\bar{G}^R(\varepsilon+\omega,p) - \bar{G}^A(\varepsilon,p)\right] \bar{G}^A(\varepsilon,p)\Big|_{\varepsilon=0}. \tag{65}$$

For $\omega \ll 1/\tau$ this gives the Drude result

$$\sigma_{\alpha\beta}(\omega) = 2e^2\nu_0 D. \tag{66}$$

General scatterers. Current vertex renormalization. Let us now turn to the case of general scatterers. From now on, using the rule formulated below Eq. (46) we will tacitly assume that Eq. (61) is written for the difference, $\langle \Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-) - \Pi_\beta(\mathbf{p};\varepsilon,\varepsilon)\rangle$ rather than $\langle \Pi_\beta(\mathbf{p};\varepsilon_+,\varepsilon_-)\rangle$ itself, and evaluate the momentum integrals by residues. Since the product of the average Green's functions is strongly peaked, in the square bracket in the right hand side of Eq. (61) we may set $\mathbf{p} = k_\epsilon \mathbf{n}$, and express $\langle \mathbf{\Pi} \rangle$ in terms of the vector vertex $\mathbf{n}\Gamma$,

$$\langle \mathbf{\Pi}(\mathbf{p};\varepsilon_+,\varepsilon_-)\rangle = \bar{G}(\varepsilon_+,\mathbf{p})\bar{G}(\varepsilon_-,\mathbf{p})\mathbf{n}\Gamma(\varepsilon_+,\varepsilon_-). \tag{67}$$

Assuming for simplicity $1/\tau \gg \omega > 0$ we obtain the following equation,

$$\Gamma(\varepsilon_+,\varepsilon_-) = k_\varepsilon + \tau n_i \frac{k_\varepsilon}{2m}\left[\mathrm{sgn}(\epsilon_+) - \mathrm{sgn}(\epsilon_-)\right] \int d\mathbf{n}' \mathbf{n}\cdot\mathbf{n}' \sigma_\varepsilon(\mathbf{n}-\mathbf{n}')\Gamma(\varepsilon_+,\varepsilon_-). \tag{68}$$

The first term above represents the bare vector vertex. The second (which vanishes for point-like impurities) described its renormalization by disorder. Because of the subsequent momentum integration (by residues) in Eq. (46) we only need the vertex in the interval $|\varepsilon| < \omega/2$, where the solution is given by

$$\Gamma(\varepsilon_+,\varepsilon_-) = \frac{\tau_{tr}}{\tau}\, p_F. \tag{69}$$

Substitution of this expression into Eq. (67) and (46) gives Eq. (66) in which the diffusion constant is expressed in term of τ_{tr} in the usual way.

3.3 Boltzmann equation

Let us now make a connection between the diagram technique and the Boltzmann equation. First of all it is clear that the ladder approximation employed by us corresponds to the classical limit of description of electron propagation. Indeed, the retarded and advanced Green's functions represent the quantum probability amplitudes of electron propagation forward and backward in time,

$$G^{R/A}_\varepsilon = -i \int_0^{\pm\infty} dt e^{i(\varepsilon \pm i\eta - H)t}. \tag{70}$$

In the ladder diagrams the amplitudes of the forward and backward propagation correspond to the same path (go through the same impurities in exactly the same order). In contrast, the full quantum mechanical description of probability of electron propagation is given by a double sum over the amplitudes A_j of different trajectories

$$\sim \sum_{jk} A_j A_k^*.$$

It is useful to separate the above sum into the "diagonal contribution" and the interference contribution

$$\sum_{jk} A_j A_k^* = \sum_j |A_j|^2 + \sum_{j\neq k} A_j A_k^*. \tag{71}$$

The ladder approximation amounts to neglecting the interference contribution. The remaining diagonal contribution represents the classical sum over the probabilities. Therefore the ladder approximation is equivalent to the Boltzmann description. Let us demonstrate this explicitly.

Consider the ladder approximation to the average product $\langle G^R(\varepsilon_+;\mathbf{p}_+,\mathbf{p}'_+)G^A(\varepsilon_-;\mathbf{p}'_-,\mathbf{p}_-)\rangle$, where $\varepsilon_\pm = \varepsilon \pm \omega/2$, $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{q}/2$, and $\mathbf{p}'_\pm = \mathbf{p}' \pm \mathbf{q}/2$. Since it begins and ends with a pair of averaged Green's functions, as a function of modulus of p and p' this object is strongly peaked near $p, p' = k_\varepsilon$. We therefore will not lose information by integrating it over ξ_p. Let us define

$$\mathcal{K}_\varepsilon(\mathbf{n},\mathbf{n}') = \frac{\nu_\varepsilon}{8\pi^2}\int d\xi_p \int d\xi_{p'}\langle G^R(\varepsilon_+;\mathbf{p}_+,\mathbf{p}'_+)G^A(\varepsilon_-;\mathbf{p}'_-,\mathbf{p}_-)\rangle, \tag{72}$$

where the $\nu_\varepsilon = mk_\varepsilon/2\pi^2$ is the energy dependent density of states. Using the identity

$$(2\pi)^3\delta(\mathbf{p}-\mathbf{p}') = \frac{4\pi}{\nu_{\xi_p}}\delta(\xi_p - \xi'_p)\delta(\mathbf{n}-\mathbf{n}') \tag{73}$$

we may express the ladder series in the form

$$\mathcal{K}_\varepsilon(\mathbf{n},\mathbf{n}') = \frac{1}{-i\omega + i\mathbf{q}\cdot\mathbf{n}k_\varepsilon/m + 1/\tau}\left[\delta(\mathbf{n}-\mathbf{n}') + \frac{k_\varepsilon}{m}n_i\int d\tilde{\mathbf{n}}\sigma_\varepsilon(\mathbf{n},\tilde{\mathbf{n}})\mathcal{K}_\varepsilon(\tilde{\mathbf{n}},\mathbf{n}')\right], \tag{74}$$

which is equivalent to Eq. (15). This establishes the correspondence between the impurity diagram technique and the Boltzmann equation. The correlator of Green's functions in Eq. (72) evaluated in the ladder approximation given the resolvent of the Boltzmann equation.

Although for disordered conductors we mostly need the diffusive form of the resolvent, (14), for some applications it is insufficient. A good example is a problem of propagation of coherent directed waves (e.g. light) in random forward-scattering media [14]. In this case the photons undergo mostly small angle scattering and the transport mean free path may be longer than the sample size, while the "out" mean free path may be much shorter than that.

Main diagram blocks in the diffusive regime. Let us conclude with a dictionary of main objects of the impurity diagram technique in the diffusive regime. We will consider the simplest case of point-like impurities and use momentum representation.

- Impurity line (two impurity crosses connected by a dashed line): $\frac{1}{2\pi\nu_0\tau}$
- Average Green's function: $\bar{G}^{R/A}(\varepsilon,\mathbf{p}) = \frac{1}{\varepsilon-\xi_p\pm i/2\tau}$
- $\int(d\mathbf{p})\bar{G}^{R/A}(\varepsilon_+,\mathbf{p}_+)\bar{G}^A(\varepsilon_-,\mathbf{p}_-) = 2\pi\nu_0\tau[1-(-i\omega\tau + Dq^2)]$
- Diffuson (particle-hole ladder starting and ending with an impurity line):

  $\mathcal{D}(q,\omega) = \frac{1}{2\pi\nu_0\tau^2(-i\omega+Dq^2)}$

3.4 Beyond the Boltzmann equation.

Randomization of phase of electron propagation amplitudes, which enables us to talk about classical trajectories of electron or rays of light, occurs during the long stretches of free

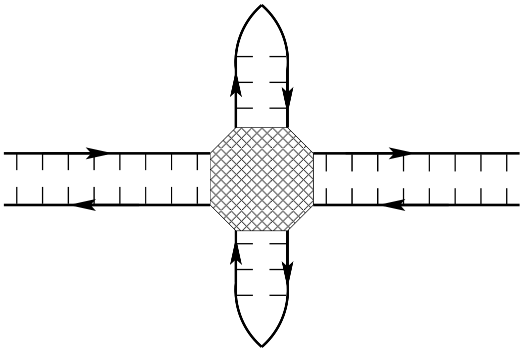

Figure 3: The diagram for the irreducible correlator of light intensities at different points. Two impurity ladders emanating at the radiation source enter the Hikami box, represented by the hashed octagon, from left and right. The ladders going to the observation points leave the Hikami box from the top and the bottom.

motion. On the other hand, impurity scattering is not classical. Accordingly, quantum effects that lie outside the Boltzmann description are associated with scattering at impurities.

Speckle statistics. Speckles [13], arise from interference of probability amplitudes. They correspond to the interference contribution in Eq. (71) and cannot be described by a classical (Boltzmann) theory of transport. Nevertheless they can be understood in the semiclassical regime, $k_F l \gg 1$, as corrections to the leading classical approximation. Considering these corrections we will introduce the last essential object arising in the diagram technique, known as the "Hikami box".

One important nonclassical phenomenon associated with impurity scattering is analogous to beam splitting by a partially reflecting mirror. Two light beams incident at a glass plate at the same angle but from opposite sides get partially transmitted and partially reflected. As a result, even if the incident beams are uncorrelated (incoherent) the intensities of the outgoing beams become correlated. Similarly, when two rays of light, or electrons moving along a classical trajectories (i.e. two ladder series), impinge at the same impurity from different directions, the outgoing rays/trajectories (ladder series) contain a mixture of photons/electrons from different incident paths and become correlated. These correlations are described by the diagram in Fig. 3. In the case of electromagnetic waves they describe speckles of coherent light propagating through a random dielectric medium, twinkling of stars caused by propagation of their radiation through the atmosphere with a fluctuating refraction index, etc. In the case of electrons, they describe mesoscopic fluctuations of electron densities and currents. They fall off slowly, as a power of the distance, in space. I would like to mention that these correlations may be described by introducing Langevin sources in the diffusion equation or, more generally, Boltzmann equation. The Boltzmann-Langevin formalism was introduced in the diffusive regime in the context of mesoscopic phenomena in solids, see Ref. [15]. The generalization to directed waves propagating in forward scattering media was introduced in Ref. [16]. The formalism is similar to the Boltzmann-Langevin method used in the theory of classical fluctuations in gases and disordered conductors, see Refs. [17, 18, 19, 1].

The "beam splitting region" represented by the hashed area in Fig. 3 is known as the Hikami box. It corresponds to the sum of three diagrams shown in Fig. 4. All three diagrams are equally small in the semiclassical parameter $1/k_F l$. The prefactor of the semeclassical

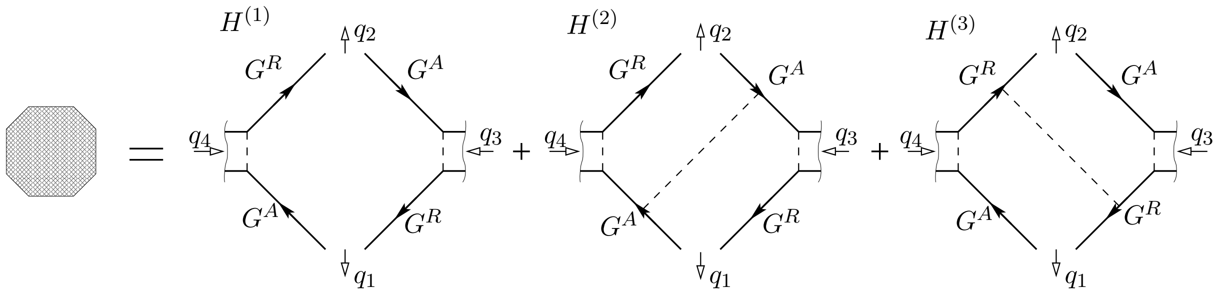

Figure 4: The three diagrams for the Hikami box; $H^{(1)}$, $H^{(2)}$, and $H^{(3)}$. The ladders coming from the radiation sources enter the Hikami box from left and right and are characterized by the four-momenta $q_{3/4} = (\omega_{3/4}, \boldsymbol{q}_{3/4})$ and the ray directions $\boldsymbol{n}_{3/4}$. The ladders going to the observation points exit the Hikami box from the top and the bottom and are characterized by the four-momenta $q_{1/2} = (\omega_{1/2}, \boldsymbol{q}_{1/2})$ and the ray directions $\boldsymbol{n}_{1/2}$.

parameter may be expressed in terms of the purely classical quantities. Evaluation of the Hikami box does not involve conceptually new steps, and we will not discuss it here.

Time-reversal symmetry: Cooperons. In the presence of time reversal symmetry we have $G^{R/A}(\varepsilon; \mathbf{p}, \mathbf{p}') = G^{R/A}(\varepsilon; -\mathbf{p}', -\mathbf{p})$. Therefore in each diagram block we may perform the time reversal operation on some of the Green's function lines without changing the value of the diagram.

Applying time reversal operation to one of the Green's function in the disorder-averaged product we get

$$\langle [G^R(\mathbf{p}, \mathbf{p}'; \varepsilon+\omega) - G^A(\mathbf{p}, \mathbf{p}'; \varepsilon)] G^A(\mathbf{p}', \mathbf{p}; \varepsilon) \rangle = \langle [G^R(\mathbf{p}, \mathbf{p}'; \varepsilon+\omega) - G^A(\mathbf{p}, \mathbf{p}'; \varepsilon)] G^A(-\mathbf{p}, -\mathbf{p}'; \varepsilon) \rangle. \tag{75}$$

In particular, to a diffuson ladder there corresponds a ladder in the the particle-particle channel - the cooperon, described by an identical analytic expression,

$$\mathcal{C}(q, \omega) = \frac{1}{2\pi\nu_0\tau^2(-i\omega + Dq^2)}. \tag{76}$$

Similarly, applying the time reversal operation to two of the Green's functions in the Hikami box connecting four diffusons, see Fig. 4 we obtain a Hikami box connecting two diffusons and two Cooperons.

The cooperon represents a non-classical interference contribution of Feynman amplitudes that does not vanish upon disorder averaging. This has profound consequences for electron transport. Let us consider the contribution of self-crossing trajectories to the sum of Feynman amplitudes in Eq. (71).[4] The interference term contains a contribution in which the amplitudes A_j and A_k correspond to trajectories which traverse the loop portion of the path in opposite directions. This interference contribution does not vanish upon disorder averaging. Its inclusion effectively doubles the diagonal contributions of self-intersecting trajectories in the diagonal part of the sum. This means that the probability for a particle to return to the original point is enhanced by quantum interference with respect to the naive

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[4] This reasoning is due to D. E. Khmel'nitskii.

classical prediction. This enhancement represents a tendency towards Anderson localization. In the semiclassical limit, $k_F l \gg 1$, this tendency is weak (weak localization).

Diagrammatically the interference contribution of self-intersecting trajectories is represented by the cooperon loop connected by a Hikami box either to a diffuson (in the case of density-density correlator) or two current vertices. The corresponding weak localization correction to the conductivity was obtained in Ref. [20]. It is negative, and proportional to the value of the cooperon at coinciding points,

$$\Delta\sigma_{WL} = \begin{cases} -\frac{e^2}{(2\pi)^2}\left(\frac{\sim 1}{l} - \frac{1}{L_\omega}\right), & 3D \\ -\frac{e^2}{(2\pi)^2}\ln\frac{L_\omega}{l}, & 2D \\ -\frac{e^2}{(2\pi)^2}L_\omega, & \mathrm{quasi}-1D \end{cases} \tag{77}$$

where $L_\omega = \sqrt{D/(-i\omega)}$. Note that although in three dimensions the correction is dominated by short trajectories (of the order of the mean free path) its dependence on the frequency (or temperature) comes from long trajectories (second term in the brackets). Analogously its dependence on the magnetic field (which suppresses the cooperon) is also dominated by long trajectories. This sensitive dependence of the interference contribution on the magnetic field leads to negative magnetoresistance. Although, as expected, the weak localization correction is small in $k_F l$ (with resect to the classical Drude result) it diverges in $2D$ and quasi-$1D$. This divergence signals the onset of Anderson localization. In $2D$ and quasi-$1D$ even weak disorder ($k_F l \gg 1$) leads to localization of single particle states.

Summary. In the diffusive regime the impurity diagram technique simplifies significantly. The relevant physics is described by blocks of diagrams with diffusive poles $1/(-i\omega + Dq^2)$ - diffusons and cooperons. They describe interference contributions of Feynmann amplitudes which do not vanish upon disorder-averaging and have a long range character. Connections between these diffusion modes are described by Hikami boxes. The nonlinear σ-model that we will discuss next may be viewed as an effective theory that describes the low energy physics in the diffusive regime in terms of soft modes (diffusons and cooperons).

This technique describes a host of quantum interference phenomena in the semiclassical regime. To name a few: mesoscopic (sample-to-sample) fluctuations of transport quantities, weak localization, negative magnetoresistance, speckles statistics, sensitivity of speckles and electron transport to changes in energy, disorder potential etc. Some of these phenomena will be discussed in more advanced courses in this school.

4 nonlinear σ-model

The problem of Anderson localization is usually approached from the metallic phase, where the presence of the semiclassical parameter $1/k_F l$ leads to dramatic simplifications. The σ-model formalism applies to this regime. The diffusive degrees of freedom (diffusons and cooperons) correspond to the Goldstone modes of the theory. The weak localization corrections correspond to interactions between the Goldstone modes. In the metallic phase the coupling between the Goldstone modes is weak. In two and one dimensions it grows under renormalization, and eventually leads to confinement. The confined phase, where the Goldstone modes disappear altogether corresponds to the localized regime.

Let us sketch the derivation of the σ-model. It was originally proposed by Wegner in the bosonic replica formulation [21]. Subsequently, the fermionic replica [22] formulation was proposed. The supersymmetric version of the was developed by Efetov, see Ref. [23] and references therein. We will present a cartoon derivation of the supersymmetric σ-model for the simplest case of broken time-reversal symmetry (so-called unitary ensemble). A detailed derivation can be found in Ref. [23].

It is convenient to introduce the generating function, $Z[\hat{J}]$, of electron Green's functions. It may be expressed in terms of a functional integral over a supervector field Φ with bosonic, S, and fermionic, χ, components,

$$\Phi = \begin{pmatrix} \chi^R \\ S^R \\ \chi^A \\ \chi^A \end{pmatrix}, \tag{78}$$

in the following form

$$Z[\hat{J}] = \int D[\Phi, \Phi^\dagger](\mathbf{r}) e^{-F[\hat{J}]}. \tag{79}$$

The "free energy" in the exponent is given by

$$F[\hat{J}] = -i \int d\mathbf{r} \Phi^\dagger(\mathbf{r}) \left[(\varepsilon - H_0)\Lambda + \frac{\omega + i\eta}{2} - V(\mathbf{r})\Lambda + \hat{J} \right] \Phi(\mathbf{r}). \tag{80}$$

Here Λ is a 4×4 diagonal matrix which breaks the symmetry between the retarded (R) and advanced (A) spaces, $\Lambda = \text{diag}\,\{1, 1, -1, -1\}$, and $\hat{J}$ is the source matrix whose specific form is not important for the present purposes. The product of advanced and retarded Green's functions for a specific realization of disorder may be obtained by differentiating $Z[\hat{J}]$ with respect to the appropriate components of the source matrix $\hat{J}$.

In the absence of the source the generating function is normalized to unity, $Z[\hat{J} = 0] = 1$ for any Hamiltonian. Indeed, the integral over the bosonic components ($\propto 1/\det(\varepsilon - H)$) is canceled by the integral over the anticommuting components, ($\propto \det(\varepsilon - H)$). The absence of disorder-dependent denominator in (79) is very convenient for the subsequent averaging over disorder and is the essence of the supersymmetric method.

Note that the integral over the bosonic variables converges only because of the positive infinitesimal η in the "free energy" (80). This forces the Hamiltonian to be multiplied by the matrix Λ (at least in the bosonic block).[5]

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[5] In the fermion block this requirement is absent because Gaussian integrals over the fermion variable converge regardless of the sign in the exponent.

Disorder averaging. For a Gaussian short-range correlated disorder, $\langle V(\mathbf{r})V(\mathbf{r}')\rangle = \frac{1}{2\pi\nu_0\tau}$, the disorder averaging of the generating function in Eq. (79) is easily accomplished with the aid of the relation

$$\left\langle e^{-i\int d\mathbf{r} V(\mathbf{r})\Phi^\dagger(\mathbf{r})\Lambda\Phi(\mathbf{r})}\right\rangle = e^{-\frac{1}{4\pi\nu_0\tau}\int d\mathbf{r}\left[\Phi^\dagger(\mathbf{r})\Lambda\Phi(\mathbf{r})\right]^2}. \tag{81}$$

Therefore upon disorder averaging we have

$$\langle e^{-F[\hat{J}]}\rangle = e^{\int d\mathbf{r}\left\{i\Phi^\dagger(\mathbf{r})\Lambda\left[(\varepsilon - H_0) + \left(\frac{\omega+i\eta}{2}+\hat{J}\right)\Lambda\right]\Phi(\mathbf{r}) - \frac{1}{4\pi\nu_0\tau}\left[\Phi^\dagger(\mathbf{r})\Lambda\Phi(\mathbf{r})\right]^2\right\}}. \tag{82}$$

Symmetry. It is important to note that with the exception of the source $\hat{J}$ and the term $\frac{\omega+i\eta}{2}\Phi^\dagger\Phi$ (which is small in the diffusive regime, $\omega \ll 1/\tau$) the disorder-averaged action above is invariant with respect to the field transformations preserving the scalar product $\Phi^\dagger\Lambda\Phi$, namely $\Phi \to U\Phi$, $U^\dagger\Lambda U = \Lambda$. The noncompactness of the symmetry group follows from the requirement of convergence of the bosonic integrals in (79). For the anticommuting variables this requirement is absent and the true "metric tensor" preserved by the symmetry group turns out to be $M = \mathrm{diag}\{1, 1, 1, -1\}$,[6]

$$\Phi \to U\Phi, \quad U^\dagger M U = M. \tag{83}$$

This symmetry group is compact, $U(2)$, in the fermion-fermion (FF) sector and noncompact $U(1, 1)$ in the boson-boson (BB) sector. The full supergroup also contains nilpotent (anticommuting) elements and is denoted by $U(1, 1|2)$. It is this symmetry that is spontaneously broken in the σ-model formalism.

Hubbard-Stratonovich transformation. The quartic interaction of the fields Φ may be decoupled by a matrix Hubbard-Stratonovich transformation. The advantage of the matrix decoupling over the original scalar decoupling (for which the Hubbard-Stratonovich field corresponds to the disorder potential) is that the matrix field is not invariant with respect to the symmetry group of the problem and enables us to look for spontaneous symmetry breaking. Upon the introduction of the supermatrix field $Q(\mathbf{r})$ the expression in (82) may be written as

$$\langle e^{-F[\hat{J}]}\rangle = \int D[Q] e^{\int d\mathbf{r}\left\{i\Phi^\dagger(\mathbf{r})\Lambda\left[(\varepsilon - H_0) + \frac{i}{2\tau}Q(\mathbf{r}) + \left(\frac{\omega+i\eta}{2}+\hat{J}\right)\Lambda\right]\Phi(\mathbf{r}) - \frac{\pi\nu_0}{4\tau}\mathrm{str}Q^2(\mathbf{r})\right\}}. \tag{84}$$

Here $\mathrm{str}A = \mathrm{tr}A_{FF} - \mathrm{tr}A_{BB}$ denotes the supertrace (generalization the usual trace to supermatrices). Integrating over the fields Φ we obtain the following expression for the generating function

$$Z[\hat{J}] = \int D[Q] e^{\mathrm{str}\ln\left[(\varepsilon - H_0) + \frac{i}{2\tau}Q(\mathbf{r}) + \left(\frac{\omega+i\eta}{2}+\hat{J}\right)\Lambda\right] - \frac{\pi\nu_0}{4\tau}\int d\mathbf{r}\,\mathrm{str}Q^2(\mathbf{r})}. \tag{85}$$

Saddle point. Let us find the saddle point of the action (84). The source $\hat{J}$ and the frequency ω may be ignored for this purpose. Minimizing action with respect to Q we obtain the saddle point equation

$$\frac{i}{2\tau}\left[\varepsilon - H_0 + \frac{i}{2\tau}Q(\mathbf{r})\right]^{-1}\Bigg|_{\mathbf{r}=\mathbf{r}'} = \frac{\pi\nu_0}{2\tau}Q(\mathbf{r}). \tag{86}$$

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[6] Although it may appear that there is an ambiguity in the choice of compact vs noncompact symmetry in the fermion block this is not so. For a detailed discussion of this issue see Refs. [24], [25].

We begin by looking at the diagonal and spatially independent solutions, which obey

$$\int (d\mathbf{p}) \frac{1}{\varepsilon - \xi_p + \frac{i}{2\tau} Q} = -i\pi\nu_0 Q. \tag{87}$$

The relevant solution is given by [7]

$$Q = \Lambda. \tag{88}$$

The coupling between the Q matrices and the fields Φ in (84) induces transformations of Q under the action of the symmetry group (83). This symmetry of the global action ensures the existence of the degenerate (at $\omega, \hat{J} \to 0$) manifold of uniform saddle point. The Q-matrices at the saddle-point manifold may be obtained by acting on the matrix Λ with the symmetry transformation. Some of the symmetry transformations do not change the saddle points. They correspond to rotations between retarded or advanced components only and form the group $U(1|1) \times U(1|1)$. Therefore the saddle point manifold corresponds to a coset space $U(1,1|2)/U(1|1) \times U(1|1)$ and may be parameterized by rotation matrices T,

$$Q = T^{-1} \Lambda T, \quad T \in U(1,1|2)/U(1|1) \times U(1|1). \tag{89}$$

The resulting Q-matrices satisfy a nonlinear constraint

$$Q^2 = 1. \tag{90}$$

The low energy degrees of freedom correspond to slowly varying in space matrices from the saddle point manifold. They correspond to diffusons (and cooperons for time-reversal invariant systems) in the language of diagrams. The other modes are gapped (have gaps $\sim 1/\tau$), and may be ignored in the diffusive regime.

Effective action. To derive the effective low energy theory let us substitute the form (89) with slowly varying in space rotation matrices $T(\mathbf{r})$ into Eq. (85).

$$Z[\hat{J}] = \int D[T] e^{\mathrm{str}\ln\left[(\varepsilon - H_0) + \frac{i}{2\tau} T^{-1}(\mathbf{r}) \Lambda T(\mathbf{r}) + \left(\frac{\omega + i\eta}{2} + \hat{J}\right)\Lambda\right]}, \tag{91}$$

here $D[T]$ is the invariant measure on the coset space $U(1,1|2)/U(1|1) \times U(1|1)$.

Next let us perform the cyclic permutation of the matrices under str ln, c.f. Ref. [26]. Using invariance of the trace under cyclic permutation we obtain,

$$Z[\hat{J}] = \int D[T] e^{-\mathcal{S}} = \int D[T] e^{\mathrm{str}\ln\left[(\varepsilon - H_0) + \frac{i}{2\tau}(\mathbf{r})\Lambda + B[T] + T(\mathbf{r})\left(\frac{\omega + i\eta}{2} + + \hat{J}\right)\Lambda T^{-1}\right]}, \tag{92}$$

where

$$B[T] = T H_0 T^{-1} - H_0. \tag{93}$$

Following Pruisken [26] we write the free Hamiltonian in the form,

$$H_0 = \frac{1}{2m} \overleftarrow{\pi} \cdot \overrightarrow{\pi} - \mu + U(\mathbf{r}). \tag{94}$$

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[7] This selection from the possible set $Q = \mathrm{diag}\{\pm 1, \pm 1, \pm 1, \pm 1\}$ is dictated by convergence consideration of the bosonic integrals and the value of the action at the saddle point.

Here we consider a system bounded by the confining potential by $U(\mathbf{r})$, and denote by $\overleftarrow{\pi}$, and $\overrightarrow{\pi}$ covariant derivatives acting on the left and right respectively: $\overleftarrow{\pi} = (i\hbar\overleftarrow{\nabla} - \frac{e}{c}\mathbf{A})$, $\overrightarrow{\pi} = (-i\hbar\overrightarrow{\nabla} - \frac{e}{c}\mathbf{A})$. This gives

$$B[T] = \frac{1}{2m}\left[\mathbf{D}\cdot\mathbf{D} + \mathbf{D}\cdot\overrightarrow{\pi} + \overleftarrow{\pi}\cdot\mathbf{D}\right] \approx \frac{1}{2m}\left[\mathbf{D}\cdot\overrightarrow{\pi} + \overleftarrow{\pi}\cdot\mathbf{D}\right], \tag{95}$$

where

$$\mathbf{D} = -iT\left(\boldsymbol{\nabla}T^{-1}\right). \tag{96}$$

In the last expression in (95) we neglected the small gradient of T in comparison to the Fermi momentum ($\sim \overrightarrow{\pi}$).

In the diffusive regime the last two terms in the action of (92) represent small perturbations, and we may expand the action in them. Noting that the first three terms under the logarithm in (92) represent the inverse of the disorder-averaged (matrix) Green's function $\bar{G}^{-1} = (\varepsilon - H_0) + \frac{i}{2\tau}$ we may write the expansion up to second order in the form (omitting the source)

$$\mathcal{S} = -\mathrm{str}\left[\bar{G}\left(B[T] + \frac{\omega_+}{2}T\Lambda T^{-1}\right) - \frac{1}{2}\bar{G}\left(B[T]\bar{G}B[T]\right)\right]. \tag{97}$$

Here the trace includes the position space and we retained only the lowest order terms in ω.

Let us now specialize to two-dimensional systems and obtain Pruisken's topological term. To do so one needs to carefully handle the boundaries. I will only sketch the main idea and refer the reader to Ref. [26] for the detailed derivation. The first order term in (97) may be written as

$$\mathcal{S}_1 = i\frac{\pi\nu_0}{2}(\omega + i\eta)\int d\mathbf{r}\,\mathrm{str}(\Lambda Q) + \int d\mathbf{r}\,\mathrm{str}[\mathbf{D}(\mathbf{r})\mathbf{J}(\mathbf{r})], \tag{98}$$

where $\mathbf{J}(\mathbf{r}) = \frac{1}{2}\left(\bar{G}(\mathbf{r},\mathbf{r}')\overleftarrow{\pi}_{\mathbf{r}'} + \overrightarrow{\pi}_{\mathbf{r}}\bar{G}(\mathbf{r},\mathbf{r}')\right)\Big|_{\mathbf{r}=\mathbf{r}'}$ is the matrix "current density" (at a given energy). In the presence of a magnetic field the latter does not vanish at the boundary and may be expressed in in terms of the matrix "magnetization", $J_\alpha = -\epsilon_{\alpha\beta}\nabla_\beta M$. Integrating by parts, using Eq. (96), and keeping in mind that the matrix magnetization is proportional to the Λ-matrix we may rewrite the second term above in the form,

$$\sim \int d\mathbf{r}\,\epsilon_{\alpha\beta}\mathrm{str}[\Lambda\nabla_\alpha T(\mathbf{r})\nabla_\beta T^{-1}(\mathbf{r})] = \frac{1}{2}\int d\mathbf{r}\epsilon_{\alpha\beta}\mathrm{str}\left[Q(\mathbf{r})\nabla_\alpha Q(\mathbf{r})\nabla_\beta Q(\mathbf{r})\right].$$

To arrive at the last expression we used Eq. (89). Another contribution of this type arises from a careful consideration of the second order term. We will not do it here and refer the reader to Ref. [26] for details. The sum of the two contributions may be expressed in terms of the Hall conductivity σ_{xy} as

$$\frac{\pi}{4e^2}\sigma_{xy}\int d\mathbf{r}\epsilon_{\alpha\beta}\mathrm{str}\left[Q(\mathbf{r})\nabla_\alpha Q(\mathbf{r})\nabla_\beta Q(\mathbf{r})\right].$$

To evaluate the second order contribution corresponding to the diagonal conductivity, σ_{xx}, we may neglect the magnetic field. The resulting calculation is straightforward and gives

$$\frac{\pi}{4e^2}\sigma_{xx}\int d\mathbf{r}\,\mathrm{str}\left[\boldsymbol{\nabla}Q(\mathbf{r})\right]^2.$$

We thus obtain the σ-model action in the form

$$\mathcal{S}[Q]=\frac{\pi\nu_0}{4}\int d\mathbf{r}\,\mathrm{str}\left[(-2i\omega+\eta)\left[\Lambda Q(\mathbf{r})\right]-D\left[\boldsymbol{\nabla}Q(\mathbf{r})\right]^2+\frac{\sigma_{xy}}{\nu_0 e^2}\epsilon_{\alpha\beta}Q(\mathbf{r})\nabla_\alpha Q(\mathbf{r})\nabla_\beta Q(\mathbf{r})\right]. \tag{99}$$

Various correlators of Green's functions are thus expressed in terms of the functional integrals over the nonlinear supermatrix field Q. For example, the "density-density" correlator

$$\Pi(\mathbf{r},\mathbf{r}';\omega)=\langle G^R_{\epsilon_+}(\mathbf{r},\mathbf{r}')G^A_{\epsilon_-}(\mathbf{r}',\mathbf{r})\rangle. \tag{100}$$

may be expressed in terms of the Q-matrix as

$$\Pi(\mathbf{r},\mathbf{r}';\omega)=\int D[Q]Q^{RA}_{BB}(\mathbf{r})Q^{AR}_{BB}(\mathbf{r}')e^{-\mathcal{S}[Q]}. \tag{101}$$

The boundary conditions at insulating boundaries correspond to vanishing "current" across the boundary, $\mathbf{n}\cdot\boldsymbol{\nabla}Q=0$, where $\mathbf{n}$ is a vector normal to the boundary.

Comparison with perturbation theory. Perturbation theory corresponds to expanding the action in small fluctuations of the Q-matrix from $Q=\Lambda$,

$$T=e^{W/2}\approx 1+\frac{W}{2}-\frac{W^2}{2}. \tag{102}$$

Let us find the criterion of applicability of perturbation theory. We expand the action to second order in W and express $W(\mathbf{r})$ in terms of the Fourier amplitudes, $W_\mathbf{q}$ (note that the latter are dimensionless) $W(\mathbf{r})=\sum_\mathbf{q}W_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{r}}$. Doing so we obtain,

$$\mathcal{S}\sim\nu_0L^d\sum_\mathbf{q}\left(-i\omega+Dq^2\right)\mathrm{str}\left(W_\mathbf{q}W_{-\mathbf{q}}\right), \tag{103}$$

where L is the system size. In a finite system the wave vectors are quantized, $\mathbf{q}\sim 1/L\mathbf{n}$. Therefore the magnitude of Gaussian fluctuations of $W_\mathbf{q}$ may be estimated as

$$\langle W_\mathbf{q}W_{-\mathbf{q}}\rangle\sim\frac{\delta}{-i\omega+Dq^2}\sim\left\{\begin{array}{ll}\frac{i\delta}{\omega}, & q=0,\\ \frac{\delta}{-i\omega+E_T\mathbf{n}^2}, & \mathbf{q}\neq 0.\end{array}\right. . \tag{104}$$

Here $\delta=1/(\nu_0L^d)$, is the single particle mean level spacing of the system at the Fermi level, and $E_T\sim D/L^2$ is the Thouless energy. The latter corresponds to the inverse diffusion time across the system.

The fluctuations of the zero mode ($\mathbf{q}=0$) are small only for $\omega\gg\delta$. For the non-uniform modes the perturbative expansion is justified at any frequencies provided the Thouless energy exceeds the mean level spacing the the system,

$$E_T/\delta\sim\nu_0DL^{d-2}\gg 1. \tag{105}$$

Note that the quantity above is the dimensionless conductance of the system, $g=\frac{2\pi\hbar}{e^2}\sigma L^{d-2}$. Thus, the dimensionless coupling constant in the σ=model is $1/g$. In a good metal, $g\gg 1$, the gaussian approximation is justified and coupling between different modes may be neglected. As the conductance decreases the fluctuations of the Q matrix grow, their nonlinear coupling (corresponding to weak localization corrections) becomes important, and we enter the strongly coupled (insulating regime).

Nonperturbative effects. Since in deriving the σ-model we started from the metallic phase, $k_F l \gg 1$ it is natural to ask whether the σ-model is equivalent to perturbation theory. The answer is no.

One celebrated example is Efetov's proof that level statistics in metal grains are described by the random matrix theory. At frequencies of the order of the mean level spacing, $\omega \sim \delta$, the fluctuations of the zero mode may not be treated perturbatively. On the other hand, in a good metal, $g \gg 1$, one may neglect inhomogeneous fluctuations of the Q-matrix. In this (zero-mode) approximation the functional integral in (101) reduces to a definite integral over a super-manifold. This integral can be evaluated, see Ref. [23], and reproduces the result of random matrix theory.

Nonperturbative effects are associated with the topology of the Q-matrix manifold. For example, the two-point correlation function of energy level densities for the unitary ensemble is

$$\langle \nu(\varepsilon)\nu(\varepsilon+\omega)\rangle \sim -\frac{\sin^2(\pi\omega/\delta)}{(\pi\omega)^2} = -\frac{1-\cos(2\pi\omega/\delta)}{2(\pi\omega)^2}.$$

The oscillatory contribution, being non-analytic in $1/\omega$ is non-perturbative. It is intimately related to the compactness of the FF block of the manifold. For the unitary ensemble the latter is a sphere. For high frequencies, $\omega \gg \delta$, the integral is dominated by two saddle points. The $1/\omega^2$ term comes from , $Q \approx \Lambda$, while the oscillatory term, comes from the "opposite pole" of the sphere, $Q \approx \tilde{\Lambda} = \mathrm{diag}(-1, 1, 1, -1)$. This observations enables one to obtain non-perturbative results for energy level statistics beyond random matrix theory [27].

The topology of the Q-matrix manifold leads to the existence of instantons and non-perturbative effects in the RG-flow diagram for the quantum Hall effect [26] and for localization in quasi-one dimension. I think these topics will be discussed in the course by A. Mirlin.

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